Chapter 1: Methods of mathematical reasoning

A. Djehiche

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I will start by asking some quick questions:

Q1 : What do we want to do in mathematics?

A: Most of the time we are proving something.

Q2: Someone asks: What are we proving?

A: In mathematics we prove in order to make sure that the statement related to this thing is true or false.

1 Mathematical Logic

Logic is a method of reasoning based on several axioms, relationships and facts that we believe are correct and that we use them in a right way to decide whether a certain **logical** *statement* is correct or incorrect.

1.1 What does it mean a logical statement?

In logic, A statement presents new information, which can be used to uncover additional facts. A logical statement is made up of two components: the hypothesis and the conclusion, and it can be either true or false.

Example:

1. If I work hard, I will succeed in the exam.
2. If the unknown number x is divisible by 2, then x is an even number.

The first statement in this example it can be decomposed as follows:

P: "I work hard " is the hypothesis.

Q: "I will succeed in the exam" is the conclusion.

If P is considered to be true, then the sentence "I do not work hard" is false. Thus, the false statement of P and Q is given as "if I dont work hard, I will not scceed in the exam", In this case P and Q are regarded as individual or specific statements. Also P and Q are called intput and the compsition of them represents the ouput.

1.2 Mthematical logical tools (operators)

Conjunction (and)

From the two individual statements P and Q, we can obtain another statement written as $(P \wedge Q)$ that means ($P$and Q). $(P \wedge Q)$ is true only if P is true and Q is true, as shown in the following truth table:

P	Q	$(P \wedge Q)$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction (OR)

Also P and Q can be joined in $(P \vee Q)$ which means (P or Q) that gives the following trurh value

P	Q	$(P \wedge Q)$
T	T	T
T	F	T
F	T	T
F	F	F

The statement $(P \vee Q)$ is false only if P is false and Q is false

Negation (NOT)

The $notP$ or $\overline{P}$ is the oposite statement of P. If P true, then $\overline{P}$ is false and if P is false, then $\overline{P}$ will be true.

P	$\overline{P}$
F	T
T	F

Example : The oposite of $P : (x < 3)$ is $\overline{P} : (x \geq 3)$

Implication

If P then Q denotes as $(P \Rightarrow Q)$ is a statement means the validity of P implies the validity of Q or Q is a necessary condition for P as follows

P	Q	$(P \Rightarrow Q)$
T	T	T
T	F	F
F	T	T
F	F	T

The statement $(P \Rightarrow Q)$ will be false only if the hypothesis P is true and the conclusion Q is false. For that reason, the statements $(P \Rightarrow Q)$ and $(Q \Rightarrow P)$ are not the same. The $(Q \Rightarrow P)$ is the converse of $(P \Rightarrow Q)$.

You can show that the statements $(P \Rightarrow Q)$ and $(\overline{P} \vee Q)$ are equivalent.

Biconditional

The statement P if and only if Q is denoted by $(P \Leftrightarrow Q)$ and it means that $(P \Rightarrow Q)$ $\wedge$ $(Q \Rightarrow P)$ which is true only if P and Q are both true or both false as shown in the following truth table

P	Q	$(P \Rightarrow Q)$	$(Q \Rightarrow P)$	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Examples

- $(\frac{10}{2} = 5) \wedge (10 < 5)$ is false
- $(\sqrt{120} = 10) \Longleftrightarrow (5^2 = 10)$ is true
- (5 is even number)$\Rightarrow$($\sqrt{4} = 4$) is false

Some properties

If P and Q are two statements, the following properties are correct:

- $P \wedge P \Leftrightarrow P$
- $P \vee P \Leftrightarrow P$
- $(P \wedge Q) \Leftrightarrow (Q \wedge P)$
- $(P \vee Q) \Leftrightarrow (Q \vee P)$
- $\overline{P \wedge Q} \Leftrightarrow \overline{P} \vee \overline{Q}$
- $\overline{P \vee Q} \Leftrightarrow \overline{P} \wedge \overline{Q}$
- $P \Rightarrow Q \Leftrightarrow \overline{Q} \Rightarrow \overline{P}$.

Universal and the existential quantifiers

In logic a quantifier refers to "how many". The universal quantifier $\forall$ stands for "for all" and the formula $\forall x \in E \mid P(x)$ express that for every element in the domain E the propertie $P(x)$ is hold true.
The existential quantifier $\exists$ stands for "there exist at least" and the formula $\exists x \in E \mid P(x)$ express that there exist at least an element in the domain E such that the propertie $P(x)$ is hold true.

The expression $\forall x \in D \mid R(x)$ where $D = \{1, 2, 3\}$is equivalent to $R(1) \wedge R(2) \wedge R(3)$.
The expression $\exists x \in D \mid R(x)$ where $D = \{1, 2, 3\}$is equivalent to $R(1) \vee R(2) \vee R(3)$.
The negation of a quantified statement it can be given as
$\overline{\forall x \in E \mid P(x)} \Leftrightarrow \exists x \in E \mid \overline{P(x)}$
$\overline{\exists x \in E \mid P(x)} \Leftrightarrow \forall x \in E \mid \overline{P(x)}$

Example:

The quantified statement	The negation of a quantified statement
$\forall x \in \mathbb{R}_+^* \mid x + 1 < 0$	$\exists x \in \mathbb{R}_+^* \mid x + 1 \geq 0$
F	T

Note: in the formula $\forall x \in D \mid R(x)$, where D is a set, we should note that the property denoted by $R(x)$ becomes a statement if x is substituted by an element of the set D

2 Methods of mathematical reasoning

2.1 Direct reasoning

In direct reasoning or direct proof, to prove if the statement or the conclusion Q is true, we have to assum that some hypothesis P is true, and then if $(P \Rightarrow Q)$ is true, then Q is true. In short, if P is true and $(P \Rightarrow Q)$ is true, then Q is true.

Example:

- **The Statement**: Prove that the sum of two odd integers is even
- **The Hypothesis**: Let a and b be two odd integers, and by definition an odd integer it can be written as $a = 2k + 1$and $b = 2m + 1$ where $k, m \in \mathbb{N}$. Besides that the definition of an even integer c is that $c = 2n/n \in \mathbb{N}$.
- **The LogicalRelation**: we deduce that $a + b = 2k + 1 + 2m + 1 = 2(k + m + 1)$
- **The Conclusion**: Since $n = k + m + 1$ is an integer, the sum of $a + b$ is divisible by 2 meaning it is even.

2.2 Proof by Contraposition

The two statements $(P \Rightarrow Q)$ and $(\overline{Q} \Rightarrow \overline{P})$ have the same truth value, means that to prove if $(P \Rightarrow Q)$ is true, it is sufficient to prove that $(\overline{Q} \Rightarrow \overline{P})$ is true. we say $(\overline{Q} \Rightarrow \overline{P})$ is the cotrapositive of $(P \Rightarrow Q)$.

Example :

To prove if this $\forall x \in \mathbb{R} : (x^2 + x - 5 = 0) \Rightarrow (x \neq 1)$ is true it is sufficient to prove its contraposition $\forall x \in \mathbb{R} : (x = 1) \Rightarrow (x^2 + x - 5 \neq 0)$ is true. we have $1^2 + 1 - 5 = -3 \neq 0$.Since the contraposition is true, then $\forall x \in \mathbb{R} : (x^2 + x - 5 \neq 0) \Rightarrow (x = 1)$ is true.

2.3 Proof by Contradiction

To prove if the statement P is true, we need to prove that assuming $\overline{P}$ is true leads to a contradiction.

Example:

If a is odd, then $2a$ is even.
$\textbf{Proof}$: The negation of conclusion: Assume that a is odd, but $2a$ is not even. a is odd means$\forall k \in \mathbb{N} a = 2k + 1$ and then $2a = 2(2k + 1)$. It clearly shows that $2(2k + 1)$ is even. We deduce that the assumption if a is odd, then $2a$ is not even leads to a contradiction and this is a sufficient reason to conclude that the statement If a is odd, then $2a$ is even.

2.4 Proof by recurrence

Proof by recurrence, also known as proof by mathematical induction, is one of the most familiar forms of proof. We use it to assert that the statement $P(n)$ is true for all all natural numbers n (or integers starting from some base case). To prove that the statement $P(n)$ is true for all $n \geq n_0$, we will follow these steps

- showing that $P(n)$ is true for $n = n_0$
- Assuming that $P(n)$ is true and we prove that $P(n + 1)$ is also true.

Example 1.

Let $\alpha > 0$. Prove that $\forall n \in \mathbb{N}(1 + \alpha)^n \geq 1 + n\alpha$
$\textbf{Proof by recurrence}$:

- For $n = 1$, we found that $(1 + \alpha)^1 \geq 1 + \alpha$, which mean the statement is true for $n = 1$.

- keeping in mind that $\alpha > 0$ and by assuming that the statement is true for n we can write

$$\begin{aligned}
&\forall n \in \mathbb{N}(1+\alpha)^n(1+\alpha) \geq (1+n\alpha)(1+\alpha)\\
&(1+\alpha)^n(1+\alpha) \geq 1+\alpha+n\alpha+n\alpha^2\\
&(1+\alpha)^n(1+\alpha) \geq 1+\alpha+n\alpha+n\alpha^2\\
&(1+\alpha)^{n+1} \geq 1+(1+n)\alpha+n\alpha^2\\
&\Longrightarrow (1+\alpha)^{n+1} \geq 1+(1+n)\alpha
\end{aligned}$$

Example 2.

Prove that

$$S(n) = 1+2+3+......+n = \frac{n(n+1)}{2}$$

Proof by recurrence:

- For $n=1$, $S(1) = 1 = \frac{1(1+1)}{2}$ and it is holds.
- Assuming that $S(n)$ is true, leads to

$$S(n+1) = 1+2+3+...+n+(n+1) = \frac{(n+1)(n+2)}{2}$$

We know that $S(n+1) = S(n)+(n+1)$ which gives

$$\begin{aligned}
S(n+1) &= \tfrac{n(n+1)}{2}+(n+1)\\
&= \tfrac{n(n+1)}{2}+\tfrac{2(n+1)}{2}\\
&= \tfrac{(n+1)(n+2)}{2}
\end{aligned}$$

$$S(n+1) = \frac{(n+1)(n+2)}{2}$$

which is exactly what we wanted to prove.