Directed Work Series No.1

October 9, 2024

Exercise 1.

Among the following statements, determine which is true and which is false:

- (3 is odd number)$\Longrightarrow$(3 is a prime number)
- ((3 is odd number)$\Longrightarrow$(3 is a prime number))$\Longrightarrow$(9 is prime number)
- (4 is odd number)$\Longrightarrow$(11 is a prime number)
- $((\forall x \in]-5,-1[\mid \ \mid x+3 \mid < 2)) \wedge (\forall x \in \ [-5,1] \mid x^2+2x-8 \leq 0)$
- $\overline{(x^2=4 \Leftrightarrow x=2)} \Leftrightarrow (x^2=4 \Leftrightarrow x \neq 2)$
- ($\int_2^5 \sqrt{x+2}dx = \int_2^7 ydy$,where $y^2=x+2$) $\quad \vee \quad$ ($\forall k \in \mathbb{N}\ k^2+1$ is odd)

Exercise 2.

Determine wether the following pairs are logically equivalent:

a) $((P \Rightarrow Q) \Rightarrow R) \quad and \quad P$

b) $(P \Leftrightarrow Q) \quad and \quad ((P \Rightarrow Q) \wedge (Q \Rightarrow P))$

c) $(\overline{P} \vee Q) \quad and \quad (\overline{Q} \vee P)$

d) $\overline{(P \Leftrightarrow Q)} \quad and \quad (P \Leftrightarrow \overline{Q})$

e) $(P \vee (Q \Rightarrow R)) \quad and \quad ((P \Rightarrow Q) \Leftrightarrow (P \Rightarrow R))$

f) $(P \Rightarrow (Q \Leftrightarrow R)) \quad and \quad P$

g) $(P \wedge (Q \Leftrightarrow R)) \quad and \quad ((P \wedge Q) \Leftrightarrow (P \wedge R))$

Exercise 3.

1) Are the following statements true or false?

- $\forall k \in \mathbb{N} \mid \ k(k+1)$ is odd
- $\forall x \in \mathbb{R} \mid \ x^2 + 5 > 2x$
- $\forall x \in \mathbb{R} \mid x^3 + x - 1 \geq 0$
- $\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z} \mid \ x^3 + y^2 \geq 1$
- $\exists x \in \mathbb{R}, \forall n \in \mathbb{N} \mid \ x + 3n$ is multiple of 3
- $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} \mid \ y > x(2-x) + 1$

2) Determine the negation of the previous statements.

Exercise 4.

Show that:

1. $\forall a, b \ and \ c \in \mathbb{N}$: if a divides b and a divides c then a also divides $b + c$.
2. If n is an odd integer, then $n^2$is also an odd integer.
3. $\forall a\ , b \in \mathbb{R}_+^*$: if$(a = b)$, then $\left(\frac{a}{b+1} = \frac{b}{b+1}\right)$.
4. $\forall a\ , b \in \mathbb{R}_+^*$: $\left(\frac{a}{b+1} = \frac{b}{a+1}\right)$if and only if $(a = b)$.
5. $\forall a\ , b \in \mathbb{R} : \quad \frac{a+b}{2} \geq \sqrt{ab}$

Exercise 5.

Prove by contrapositive the statements 1, 2 and 3 of the exercise 4.

Exercise 6.

By using proof by contradiction, show that

1. $\forall k \in \mathbb{Z} : \sqrt{k^2+1} \geq k$
2. $\forall a, b, n \in \mathbb{N} \ (a+b)^n \neq a^n + b^n$
3. $\forall n \in \mathbb{N} \ 2^n > n$
4. $\forall a\ , b \in \mathbb{R} \mid \quad \frac{a+b}{2} \geq \sqrt{ab}$

Exercise 7.

Prove by recurrence the following

1. $\forall k > 0 \; 1 + 2 + 2^2 + ... + 2^k = 2^{k+1} - 1$
2. $\forall n \in \mathbb{N}^* \mid 1 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$
3. $\forall n \geq 2 \mid \; n! \leq \left(\frac{n+1}{2}\right)^n$
4. $\forall n \in \mathbb{N}^* \mid \; \frac{1}{1.2} + \frac{1}{2.3} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1}$
5. $\forall k \in \mathbb{N} \mid \; k^3 + 2k$ is a multiple of 3.