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First year of Computer Science License 2024/2025

Algebra I, Worksheet 2

Exercise n°1 : Let the set $A = \{1, 2, 3\}$. Are the following assertions true ?

$$3 \in A, \quad 3 \subset A, \quad \phi \in A, \quad \{\{1,2\}, 3\} = A, \quad \{1,2\} \subset A, \quad A \cup \left\{\phi\right\} = A$$

Exercise n°2 : Let A and B be two sets.
1. Prove the following properties :
a. if $A \subset B$, then $\mathcal{P}(A) \subset \mathcal{P}(B)$.
b. $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.
c. $\mathcal{P}(A) \cup \mathcal{P}(B) \subset \mathcal{P}(A \cup B)$.
2. Find two sets A and B such that $\mathcal{P}(A \cup B) \not\subset \mathcal{P}(A) \cup \mathcal{P}(B)$.

Exercise n°3 : Let A, B, and D be three subsets of a set E.
a. Prove the following properties :

1. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. 2. $A \subset B \Longrightarrow C_E^B \subset C_E^A$ 3. $A \backslash (A \cap B) = A \backslash B$

4. $A \backslash (B \cap D) = A \backslash B \cup A \backslash D$ 5. $A \triangle B = (A \cup B) \backslash (A \cap B)$

b. Determine the following sets : $A \triangle A, A \triangle \phi, A \triangle E, A \triangle C_E^A$.

Exercise n°4 : Let A, B and C be three subsets of a non-empty set E. Define $P(A, B)$ as the assertion "$\forall x \in E : (x \in A \Longrightarrow x \notin B)$" and $Q(A, B)$ as the assertion "$\exists x \in E : (x \in A \wedge x \notin B)$".
1. Express $P(A, B)$ in terms of a relationship between sets.
2. Write the negation *non* $(P(A, B))$ and express it in terms of a relationship between sets.
3. What can be concluded about A and B if they satisfy both $P(A, B)$ and $P(C_E^A, C_E^B)$?
4. What can be concluded about A and B if they satisfy both *non* $(Q(A, B))$ and *non* $(Q(B, A))$?

Exercise n°5 : Determine whether $\mathfrak{R}$ is reflexive, symmetric, antisymmetric, or transitive.
1. $\forall x, y \in \mathbb{Z} : x \mathfrak{R}_1 y \Longleftrightarrow x = -y$.
2. $\forall x, y \in \mathbb{R} : x \mathfrak{R}_2 y \Longleftrightarrow \cos^2 x + \sin^2 y = 1$.
3. $\forall x, y \in \mathbb{R} : x \mathfrak{R}_3 y \Longleftrightarrow |x| = \left|y\right|$.

Exercise n°6 : Let $\mathfrak{R}$ be a binary relation defined on the set $\mathbb{Z}$ as follows

$$\forall x, y \in \mathbb{Z} : x \mathfrak{R} y \Longleftrightarrow \exists k \in \mathbb{Z} : x + 2y = 3k.$$

1. Show that $\mathfrak{R}$ is an equivalence relation on the set $\mathbb{Z}$.
2. Let $x \in \mathbb{Z}$. Determine the equivalence class of x, denoted by $\dot{x}$.
3. Determine the quotient set $\mathbb{Z}/\mathfrak{R}$.

Exercise n°7 : We define a relation $\mathfrak{R}$ on $\mathbb{N}^*$ as follows

$$\forall x, y \in \mathbb{N}^* : x \mathfrak{R} y \Longleftrightarrow \exists n \in \mathbb{N}^* : y = x^n.$$

This relation can also be expressed as "y is a non-zero integer power of x".
1. Show that $\mathfrak{R}$ is a partial order relation on $\mathbb{N}^*$.
2. Let $A = \{2, 4, 16\}$ be a subset of $\mathbb{N}^*$. Examine the existence of a greatest element and a least element in A (denoted $max(A)$ and $min(A)$) with respect to the relation $\mathfrak{R}$.

Exercice n°8 : (Supplementary Exercise)
Let E be a set, and let $A \subset E$. We define a binary relation $\mathfrak{R}$ on $\mathcal{P}(E)$ (the power set of E, which is the set of all subsets of E) as follows

$$\forall X, Y \in \mathcal{P}(E) : X \mathfrak{R} Y \Longleftrightarrow X \cap A = Y \cap A.$$

1. Show that $\mathfrak{R}$ is an equivalence relation on $\mathcal{P}(E)$.
2. Let X be a subset of E. Denote by $\dot{X}$ the equivalence class of X for the relation $\mathfrak{R}$. Determine the equivalence classes of $\phi, A, E, \bar{A}$ ($\bar{A}$ the complement of A).