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Introduction

A **dynamical system** refers to any system that evolves through at least one real parameter (which can play the role of time) and is described using differential equations (ordinary, partial differential, delay differential, etc.), integro-differential equations, iterations, or a combination of these. More generally, it is described by one or more relations that connect the state of the system at one time (or condition) to the state of the system at another time (or condition).

Therefore, almost any description of an evolving phenomenon can be considered as a dynamical system. In this course, we focus on continuous-time dynamical systems described by a system of differential equations of the form :

$$\frac{dx}{dt} = F(x,t) \tag{1}$$

Where $x \in U \subset \mathbb{R}^n$ is the state vector, and $F : U \times \mathbb{R} \to \mathbb{R}^n$ is called the **vector field** on the open set U. When F explicitly depends on time, the system (1) is said to be **non-autonomous** (or forced) ; otherwise, it is considered **autonomous**.

Chapitre 1

Global Properties of Nonlinear Systems

In this course, we are interested in the study of autonomous systems of the form :

$$\frac{dx}{dt} = f(x). \tag{1.1}$$

When f is a nonlinear function, the system is said to be nonlinear. Nonlinear systems are very complex, and generally, explicit solutions cannot be found. Therefore, we focus primarily on qualitative analysis. On the other hand, the range of possible dynamic behaviors for these systems is greater than that of linear systems (which is why nonlinear systems are particularly interesting).

1.1 Flow

According to the Cauchy-Lipschitz (Picard-Lindelöf) existence and uniqueness theorem, if f is a C^1 class function, then there exists a unique maximal solution $x(t)$ to the system (1.1) such that $x(0) = x_0$.

Définition 1.1.1. *The mapping $\phi_t : x_0 \longmapsto x(t)$, which associates the initial condition x_0 with the value of the maximal solution $x(t)$ at time t corresponding to this initial condition, is called the flow at time t of the vector field f.*
The flow of the vector field is the mapping that associates to (t, x) the maximal solution $x(t)$ at time t corresponding to the initial condition x :

$$(t, x) \longmapsto \phi(t, x) = \phi_t(x) = x(t).$$

The flow is said to be complete if this mapping is defined for all $t \in]-\infty, +\infty[$.

FIGURE 1.1 – Representation of the flow

Remarque 1.1.
1- (Regularity of the flow)
If f is of class C^k, the flow is itself of class C^k.

2- (Transitivity of the flow)
The flow satisfies, for all t and $s \in \mathbb{R}^+$,

$$\phi_t \circ \phi_s = \phi_{s+t}.$$

3- For a linear system $\dot{x} = Ax$, the flow is given by

$$\phi_t(x) = e^{tA}x.$$

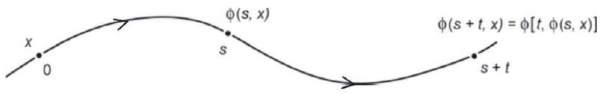


FIGURE 1.2 – Transitivity of the flow

1.2 Trajectory (Orbit), Phase Portrait

Let $x_0 \in U$ be an initial condition, and let $x(t, x_0)$ be the solution of (1.1) passing through x_0. The set of points $\{x(t, x_0), t \in \mathbb{R}\}$ is called the trajectory (or orbit) in the state space, passing through the point x_0 at the initial time $t = 0$. We denote it by

$$\gamma_{x_0} = \{x(t, x_0), t \in \mathbb{R}\}.$$

Remarque 1.2. *The trajectory of an autonomous dynamical system depends only on the initial state.*

We can distinguish between the positive orbit $\gamma^+_{x_0} = \{x(t), t \geq 0\}$ and the negative orbit $\gamma^-_{x_0} = \{x(t), t \leq 0\}$ passing through the point $x(0) = x_0$.
The phase portrait of the vector field f is the partition of the open set U into orbits (see Figure 1.3). The purpose of qualitative analysis is to study the geometric characteristics of the phase portrait.

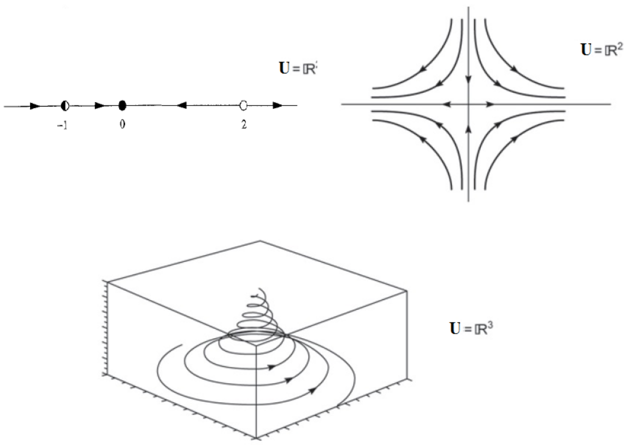


FIGURE 1.3 – Three Phase Portraits

1.3 Equilibrium Points

1. A point a is called an equilibrium point of (1.1) if it satisfies $F(a) = 0$ (or, for all t, $\phi(t, a) = a$) ; otherwise, a is referred to as an ordinary point. From this definition, we deduce that the orbit of an equilibrium point is reduced to the point itself :
$$\gamma_{x_0}^{-} = \{x_0\}.$$
In contrast, the orbit of an ordinary point is a smooth curve that has the vector f as a tangent vector at each of its points. Among ordinary points, we distinguish periodic points and recurrent points.
2. An ordinary point a is said to be periodic if there exists $T > 0$ such that $\phi(T, a) = a$.
3. An ordinary point that is not periodic, a, is said to be recurrent if for every neighborhood V of a and every $T \in \mathbb{R}$, there exists $t > T$ such that $\phi(t, a) \in V$.
4. An orbit γ_{x_0} such that there exist two equilibrium points a and b satisfying :
$$\lim_{t\to+\infty} \phi_t(x_0) = a \quad \text{and} \quad \lim_{t\to-\infty} \phi_t(x_0) = b$$
is called a heteroclinic orbit if $a \neq b$ and a homoclinic orbit if $a = b$.

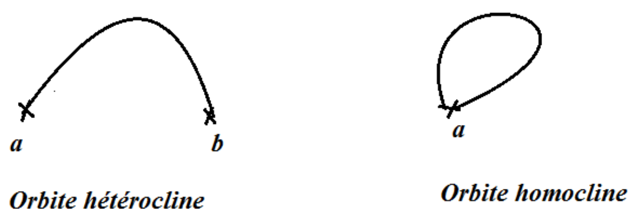


FIGURE 1.4 – Heteroclinic and Homoclinic Orbits

5. A set $S \subset U$ is said to be invariant under the flow ϕ_t on U (or by the corresponding system $\dot{x} = f(x)$) if for every $x \in S$ and every $t \in \mathbb{R}$, we have $\phi_t(x) \in S$. If S satisfies the property that $\phi_t(x) \in S$ for all $x \in S$ and all $t > 0$, then we say that S is positively invariant.

1.4 Limit Sets and Attractors

1.4.1 Limit Sets

Définition 1.4.1. *A point $a \in U$ is called an ω-limit point of a trajectory $\phi(., x_0)$ of (1.1) if there exists a sequence $(t_n) \to +\infty$ as $n \to +\infty$ such that*
$$\lim_{n\to+\infty} \phi(t_n, x_0) = a.$$

— *Similarly, a point $b \in U$ is called an α-limit point of a trajectory $\phi(., x_0)$ of (1.1) if there exists a sequence $(t_n) \to -\infty$ as $n \to +\infty$ such that*
$$\lim_{n\to+\infty} \phi(t_n, x_0) = b.$$

— *The set of all ω-limit points of a trajectory γ_{x_0} is called the ω-limit set of γ_{x_0} and is denoted by $\omega(\gamma_{x_0})$ or $\omega(x_0)$.*

— *The set of all α-limit points of a trajectory γ_{x_0} is called the α-limit set of γ_{x_0} and is denoted by $\alpha(\gamma_{x_0})$ or $\alpha(x_0)$.*

— *The set $\omega(x_0) \cup \alpha(x_0)$ is called the limit set of γ_{x_0}.*

Example 1.4.1. *If the point c belongs to a heteroclinic orbit, then by definition, there exist points $a, b \in U$ such that :*

$$\lim_{t\to+\infty} \phi(t,c) = a \quad \text{and} \quad \lim_{t\to-\infty} \phi(t,c) = b.$$

Thus, a is an ω-limit point of γ_c and b is an α-limit point of γ_c. Furthermore :

$$\omega(c) = \{a\} \quad \text{and} \quad \alpha(c) = \{b\}.$$

Theorem 1.4.1. *The ω-limit and α-limit sets of a trajectory γ_{x_0} of (1.1) are closed subsets of U. If γ_{x_0} is contained within a compact subset of U, then $\omega(x_0)$ and $\alpha(x_0)$ are non-empty, compact, and connected in U.*

Démonstration. By definition, $\omega(x_0), \alpha(x_0) \subset U$.

1. To show that $\omega(x_0)$ is a closed set, it suffices to show that the limit P of any sequence (P_m) in $\omega(x_0)$ belongs to $\omega(x_0)$. For each P_m, $m = 1, 2, \ldots$, there exists a sequence
$$\begin{array}{rcl}(t_k^{(m)}) & \to & +\infty \\ k & \to & +\infty\end{array}$$
such that
$$\lim_{k\to+\infty} \phi(t_k^{(m)}, x_0) = P_m \quad (\,*\,).$$
We assume that $t_k^{(m)} > t_k^{(m-1)}$ (otherwise, we choose a subsequence of $t_k^{(m)}$ that satisfies this property).
By the definition of the limit (), there exists a sequence of integers $k(m) > k(m-1)$ such that for all $k \geq k(m)$,
$$|\phi(t_k^{(m)}, x_0) - P_m| < \frac{1}{m}.$$
We define $t_m = t_{k(m)}^{(m)}$. It is clear that $t_m \to +\infty$ as $m \to +\infty$. Thus, we have
$$\begin{array}{rcl}|\phi(t_m, x_0) - P| & = & |\phi(t_m, x_0) - P_m + P_m - P| \\ & \leqslant & |\phi(t_m, x_0) - P_m| + |P_m - P|.\end{array}$$
Therefore,
$$\lim_{m\to+\infty} \phi(t_m, x_0) = P.$$
Hence, $P \in \omega(x_0)$. Similarly, one can show that $\alpha(x_0)$ is a closed set.
2. Suppose that $\gamma_{x_0} \subset K$ is compact in $\mathbb{R}^n$, which is separated. Since K is closed, let t_n be a sequence of reals that tends towards infinity. As the sequence $\phi(t_n, x_0)$ is contained within a compact set, there exists a convergent subsequence. Let q be the limit of this subsequence ; we have $q \in \omega(x_0)$, which implies that $\omega(x_0)$ is non-empty.
Let us show that $\omega(x_0)$ is compact in U. Since $p \in \omega(x_0)$, there exists a sequence t_k such that
$$\phi(t_k, x_0) \to p.$$
Since $\phi(t_k, x_0) \in \gamma_{x_0} \subset K$ is closed, it follows that $p \in K$. Thus,
$$\omega(x_0) \subset K.$$
The set $\omega(x_0)$ is closed and contained within a compact set, hence it is compact.

- We will now show that it is connected. By contradiction, assume that $\omega(x_0)$ consists of two disjoint closed sets A and B, and let $d = d(A, B)$. There exists a sequence t'_n tending towards infinity such that

$$\phi(t'_n, x_0) \to a \in A,$$

and another sequence t''_n such that

$$\phi(t''_n, x_0) \to b \in B.$$

There exists an integer $N > 0$ such that for all $n \geq N$, we have

$$d(\phi(t'_n, x_0), A) < \frac{d}{2} \quad \text{and} \quad d(\phi(t''_n, x_0), A) > \frac{d}{2}.$$

We can thus form a new sequence $t_k = t'_{N+k}$ if k is even and $t_k = t''_{N+k}$ if k is odd. The function

$$f(t) = d(\phi(t, x_0), A)$$

is continuous on the segment (t_k, t_{k+1}), taking values both above and below $\frac{d}{2}$. By the intermediate value theorem, there exists a value $\tau_k \in (t_k, t_{k+1})$ such that

$$d(\phi(\tau_k, x_0), A) = \frac{d}{2}.$$

From the sequence $\phi(\tau_k, x_0)$, we can extract a convergent subsequence $\phi(\tilde{\tau}_n, x_0)$, whose limit we denote as q^*. We have $q^* \in \omega(x_0)$, and moreover,

$$d(q^*, A) = \lim_{n \to +\infty} d(\phi(\tilde{\tau}_n, x_0), A) = \frac{d}{2}$$

and

$$d(q^*, B) \geq d(A, B) - d(q^*, A) = \frac{d}{2}.$$

It follows that q^* belongs to neither A nor B, which is a contradiction. Similarly, the argument holds for $\alpha(x_0)$.

□

Example 1.4.2. ***(Non-connected ω-limit set)***

Consider the system in $\mathbb{R}^2$ given by :

$$\begin{cases} \dot{x} = x - y \\ \\ \dot{y} = x + y \end{cases} \tag{1.2}$$

The solution to this system is given by :

$$X(t) = e^t R X_0,$$

where

$$R = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}.$$

Thus, the trajectories are spirals centered at the origin $(0, 0)$*.*

If we set $z = \arctan(y)$, the phase space becomes $U = \mathbb{R} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The trajectories of the system, represented in the state space as $\begin{pmatrix} x \\ z \end{pmatrix}$, take the form of spirals that alternately approach the lines $z = -\frac{\pi}{2}$ and $z = \frac{\pi}{2}$. These trajectories are not contained within a compact subset of U. The limit set of any non-zero point is the union of these two lines, which is not connected (see Figure 1.5).

Theorem 1.4.2. *The sets ω-limit and α-limit of a trajectory γ_{x_0} from (1.1) are invariant under the flow ϕ of (1.1).*

Démonstration. Let $p \in \omega(x_0)$. We will show that $\gamma_p \subset \omega(x_0)$. Let $q \in \gamma_p$. Then there exists $\tilde{t} \in \mathbb{R}$ such that $q = \phi(\tilde{t}, p)$.

Since p is an ω-limit point of $\gamma(x_0)$, there exists a sequence $t_k \to +\infty$ such that $\phi(t_k, x_0) \to p$. Consequently, we have :

$$\phi(\tilde{t} + t_k, x_0) = \phi(\tilde{t}, \phi(t_k, x_0)) \to \phi(\tilde{t}, p) = q.$$

As $\tilde{t}_k = \tilde{t} + t_k \to +\infty$, it follows that $q \in \omega(x_0)$. Hence, $\gamma_p \subset \omega(x_0)$.

The proof is analogous for the α-limit set. □

Remarque 1.3. - *If x_0 is an equilibrium point of (1.1), then $\omega(x_0) = \alpha(x_0) = \{x_0\}$.*

- If a trajectory γ_{x_0} of (1.1) has a unique ω-limit point x_0, then by the above theorem, x_0 is an equilibrium point of (1.1).

- A node or a stable focus a is the ω-limit set of every trajectory in some neighborhood of the point a.

- If q is a regular point in $\omega(x_0)$ or $\alpha(x_0)$, then the trajectory passing through q is called the limit orbit of γ_{x_0}. Thus, we see that $\omega(x_0)$ and $\alpha(x_0)$ are composed of equilibrium points and limit orbits of (1.1).

1.4.2 Attractors

In the following definitions, a neighborhood of a set A is an open set U containing A, and we say that $x(t) \to A$ as $t \to +\infty$ if the distance $d(x(t), A) \to 0$ as $t \to +\infty$.

Définition 1.4.2. *A closed invariant set $A \subset U$ is called an attracting set of (1.1) if there exists a neighborhood V of A such that for every $x \in V$, $\phi(t, x) \in V$ for all $t > 0$ and $\phi(t, x) \to A$ as $t \to +\infty$.*

An attractor of (1.1) is an attracting set that contains a dense orbit.

Définition 1.4.3 (Claudine Delcart, Bifurcations and Chaos...)**.** *[**?**], [**?**] Let A be a compact, closed, and invariant set (i.e., $\phi(t, A) = A$ for all t) in the phase space. We say that A is stable for the flow of (1.1) if for every neighborhood U of A, there exists a neighborhood V of A such that every solution $\phi(t, x_0)$ remains in U whenever $x_0 \in V$. Moreover, if :*

$$\bigcap_{t \geq 0} \phi(t, V) = A,$$

we say that A is attracting ; if there exists a dense orbit in A, then A is an attractor.

The set

$$B = \bigcup_{t<0} \phi(t, V)$$

is called the basin of attraction of A. It is the set of points whose asymptotic trajectories converge to A.

Remarque 1.4.
- A node or a stable focus of (1.1) is an attractor. However, not every ω-limit set of a trajectory is an attracting set; for example, a saddle point x_0 of a planar system is the ω-limit set of three trajectories in a neighborhood $N(x_0)$, but no other trajectory passing through points of $N(x_0)$ approaches x_0 as $t \to +\infty$.

There are two types of attractors : regular attractors and strange (or chaotic) attractors.

a) Regular Attractors

Regular attractors characterize the evolution of non-chaotic systems and can be of three types :

- **Fixed Point :**

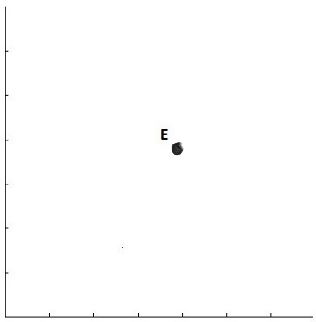
This is the simplest attractor, represented by a point in the phase space.

- **Limit Cycle (ω-Limit Cycle, Periodic Attractor) :**

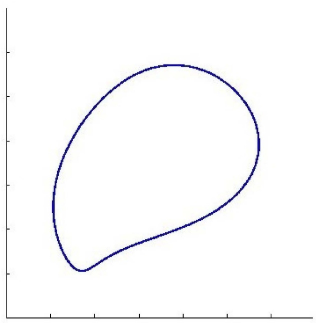
This is a closed trajectory that attracts all nearby trajectories.

- **Quasiperiodic Attractor (Torus) :**

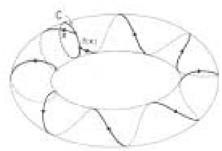
This is a trajectory that winds around a torus, densely filling its surface and eventually closing on itself after an infinite amount of time.



point fixe



cycle ω-limite



Tore

Example 1.4.3. *[Claudine Delcart, Bifurcations and Chaos...] [?] Consider the differential system :*

$$\begin{cases} \dfrac{dx}{dt} = -y + x(1 - x^2 - y^2) \\ \\ \dfrac{dy}{dt} = x + y(1 - x^2 - y^2) \end{cases} \tag{1.3}$$

In polar coordinates $x = r\cos(\theta), y = r\sin(\theta)$, the system (1.3) can be written as :

$$\begin{cases} \dfrac{dr}{dt} = r(1 - r^2) \\ \\ \dfrac{d\theta}{dt} = 1 \end{cases} \tag{1.4}$$

The general solution of (1.4) is given by :

$$r(t) = \frac{r_0}{[r_0^2 + (1 - r_0^2)e^{-2t}]^{\frac{1}{2}}}, \quad \theta(t) = t + \theta_0$$

Thus, we have

$$x(t) = \frac{r_0}{[r_0^2 + (1 - r_0^2)e^{-2t}]^{\frac{1}{2}}} \cos(t + \theta_0), \quad y(t) = \frac{r_0}{[r_0^2 + (1 - r_0^2)e^{-2t}]^{\frac{1}{2}}} \sin(t + \theta_0)$$

It can be observed that $(0, 0)$ is the only equilibrium point. Moreover, the solution corresponding to $r_0 = 1$ is

$$x(t) = \cos(t + \theta_0), \quad y(t) = \sin(t + \theta_0)$$

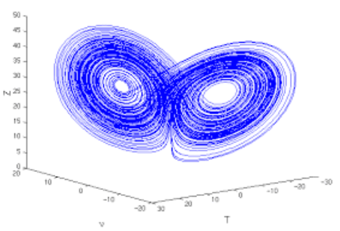
which is periodic with a period of 2π ; its orbit is the unit circle $x^2 + y^2 = 1$.

It is clear that all orbits of (1.3), except for the origin, spiral towards the unit circle, which is a limit cycle, and its basin of attraction is $\mathbb{R}^2 - \{(0, 0)\}$, as illustrated in Figure (1.6).

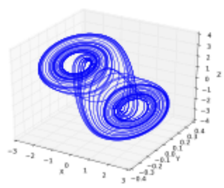
b) Strange Attractors

A strange attractor is a complex geometric shape that characterizes the evolution of chaotic systems; it was introduced by Ruelle and Takens [?] [Claudine Delcart, Bifurcations and Chaos...].

The characteristics of a strange attractor are :
1- In the phase space, the attractor has zero volume.
2- The dimension of the strange attractor is fractal (non-integer) for a continuous autonomous system, where $2 < d < n$, with n being the dimension of the phase space.
3- Sensitivity to initial conditions (two initially close trajectories will eventually diverge from each other).



Attracteur de Lorenz



Attracteur de Chua

The following theorem often demonstrates that the volume of the phase space tends towards zero without needing to solve the system.

Theorem 1.4.3. *(Divergence Theorem) Let ϕ_t be the flow of (1.1), and let V be a volume in the phase space at time $t = 0$, with $V(t) = \phi_t(V)$ as the image of V under ϕ_t. Then, we have :*

$$\frac{dV(t)}{dt}|_{t=0} = \int_V Div f \, dx_1 \ldots dx_n, \quad Div f = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}.$$

In particular, if

$$\text{Div} f = \lambda = \text{Const},$$

then we have :

$$\frac{dV}{dt} = \lambda \int_V dx = \lambda V,$$

from which it follows that

$$\frac{1}{V}\frac{dV}{dt} = \lambda.$$

Integrating, we obtain :

$$V(t) = V_0 e^{\lambda t}$$

which tends towards zero as $t \to +\infty$ and $\lambda < 0$.

Remarque 1.5. *The system is dissipative (resp. conservative) if $dV/dt < 0$ (resp. $dV/dt = 0$).*

Example 1.4.4. *Consider the simplified model of natural convection between two infinite horizontal plates. The upper plate is at temperature T_1 and the lower plate is at temperature $T_2 = T_1 + \delta T$. The dynamics of this model is given by the Lorenz system :*

$$\left\{ \begin{array}{c} \dot{x} = \sigma(y - x), \\ \dot{y} = -xz + rx - y, \\ \dot{z} = xy - bz, \end{array} \right. \tag{1.5}$$

where $\sigma = 10$ and $b = \frac{8}{3}$.

The divergence of the vector field f is negative everywhere :

$$Divf = -\sigma - 1 - b = -\frac{41}{3}.$$

Thus, volume elements contract. After one unit of time, this contraction reduces a given volume V_0 by a factor of

$$e^{-(\sigma+b+1)} = e^{-41/3}.$$

1.5 Periodic Orbits

In this section, we will present the necessary theory for the qualitative study of periodic orbits (cycles) of an autonomous system of the form (1.1).

Définition 1.5.1. *A periodic orbit (cycle) is defined as any closed trajectory of (1.1) that is not a fixed point.*

- *A periodic orbit γ of (1.1) is said to be stable if for every $\epsilon > 0$, there exists a neighborhood V of γ such that for all $x \in V$ and $t > 0$, $d(\phi(t, x), \gamma) < \epsilon$.*
- *The periodic orbit γ is said to be asymptotically stable if it is stable and for every point x in a neighborhood V of γ, we have $\lim\limits_{t \to +\infty} d(\phi(t, x), \gamma) = 0$.*

Example 1.5.1. *The center is an equilibrium point surrounded by a continuous band of cycles. Every cycle within the band is stable but not asymptotically stable.*

Définition 1.5.2. *1. The limit cycle γ of a system (in the plane $\mathbb{R}^2$) is a cycle of (1.1) that is an ω-limit or α-limit set of some trajectory of (1.1) other than γ.*

2. If a cycle γ is an ω-limit set of every trajectory in a certain neighborhood of γ, then γ is called an ω-limit cycle or stable limit cycle.

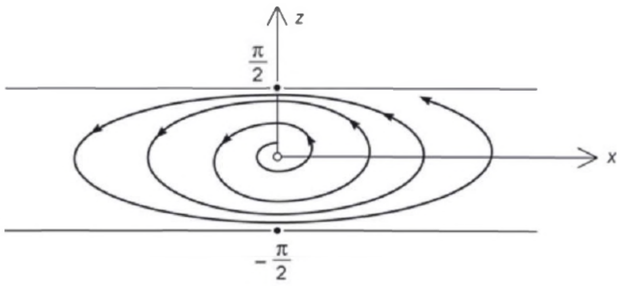


FIGURE 1.5 – A non-connected ω-limit set

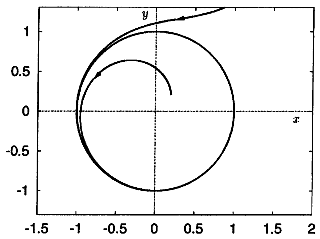


FIGURE 1.6 – Limit cycle of Example (1.4.3) and its basin of attraction

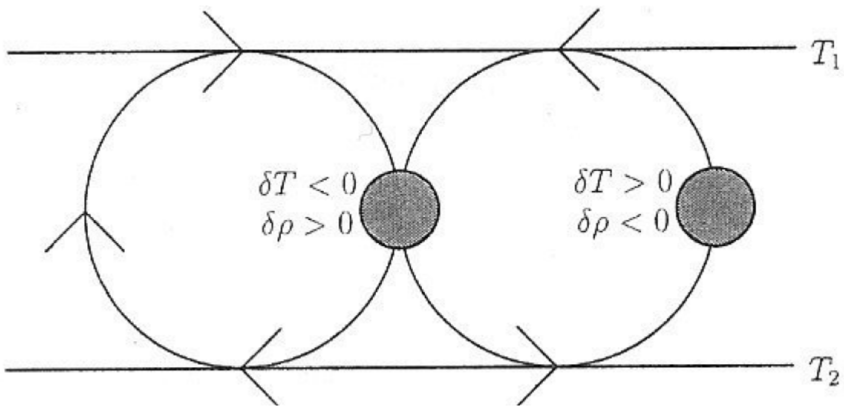


FIGURE 1.7 – Natural convection phenomenon between two horizontal plates

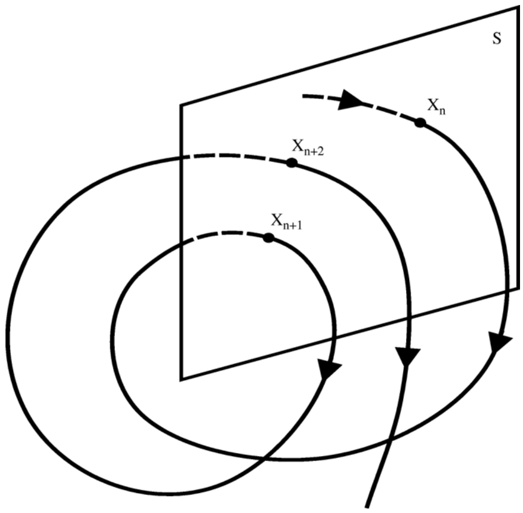


FIGURE 1.8 – Poincaré Section

3. *If γ is the α-limit set of every trajectory in a certain neighborhood of γ, then γ is called an α-limit cycle or unstable limit cycle.*

4. *If γ is an ω-limit set of a trajectory other than γ and the α-limit set of another trajectory other than γ, then γ is called a semi-stable limit cycle.*

Theorem 1.5.1. *If a trajectory outside a limit cycle γ of a C^1 system in the plane (1.1) has γ as its ω-limit set, then every trajectory in a certain external neighborhood V of γ also has γ as its ω-limit set. Moreover, every trajectory in V converges to γ in a spiral manner. The same result holds for the interior of γ and also when γ is the α-limit set of a certain trajectory.*

Theorem 1.5.2 (Dulac 1923)**.** *[Proof Correction 1988] In any bounded region of the plane, an analytic system on $\mathbb{R}^2$ admits at most a finite number of limit cycles.*

1.6 Poincaré map

The Poincaré map is an important tool for the analysis of periodic orbits, allowing us to reduce the study of a continuous dynamical system of dimension n in the vicinity of the periodic orbit to that of a mapping (iteration or discrete dynamical system) of a transverse section of dimension $n-1$ into itself.
If γ_{x_0} is a periodic orbit of the system (1.1) ($\dot{x} = f(x)$) and Σ is the hyperplane orthogonal to γ_{x_0} at x_0, then for any point $x \in \Sigma$ sufficiently close to x_0, the solution of (1.1) passing through x at time $t = 0$, denoted $\phi(t, x)$, intersects Σ at a point $P(x)$. The mapping

$$x \mapsto P(x)$$

is called the Poincaré map (first return map).

Theorem 1.6.1. *Let U be an open set in $\mathbb{R}^n$ and let $f \in C^k(U)$. Suppose that $\phi(t, x_0)$ is a periodic solution of (1.1) with period T, and that the cycle*

$$\gamma_{x_0} = \{x \in \mathbb{R}^n \,|\, x = \phi(t, x_0),\, 0 \leq t \leq T\}$$

is contained in U $(\gamma_{x_0} \subset U)$.
Let Σ be the hyperplane orthogonal to γ_{x_0} at x_0 :

$$\Sigma = \{x \in \mathbb{R}^n \mid (x - x_0) \cdot f(x_0) = 0\}.$$

Then there exists $\delta > 0$ and a unique function $\tau(x)$ of class C^k on the neighborhood $N_\delta(x_0) \cap \Sigma$ such that :

$$\tau(x_0) = T \quad \text{and} \quad \phi(\tau(x), x) \in \Sigma \quad \text{for all } x \in N_\delta(x_0) \cap \Sigma.$$

Démonstration. This theorem is a direct application of the implicit function theorem.
- We define the function :

$$F(t, x) = (\phi(t, x) - x_0) \cdot f(x_0).$$

It is clear that F is of class C^k, and we have :

1. $F(T, x_0) = (\phi(T, x_0) - x_0) = 0$ (from the periodicity of γ_{x_0}).

2. $\frac{\partial F}{\partial t}(T, x_0) = \left.\frac{\partial \phi(t,x_0)}{\partial t}\right|_{t=T} \cdot f(x_0) = f(x_0) \cdot f(x_0) = |f(x_0)|^2 \neq 0$
   because x_0 cannot be a fixed point. Thus, according to the implicit function theorem, there exists a $\delta > 0$ and a unique function $\tau(x)$ of class C^k on $N_\delta(x_0) \cap \Sigma$ such that :

$$\begin{cases} \tau(x_0) = T \\ \wedge \\ (\phi(\tau(x), x) - x_0) \cdot f(x_0) = 0 \end{cases}$$

   From which it follows that $\phi(\tau(x), x) \in \Sigma$.

□

Définition 1.6.1. *The mapping :*

$$\begin{array}{rcl} P : N_\delta(x_0) \cap \Sigma & \longrightarrow & \Sigma \\ x & \longrightarrow & P(x) = \phi(\tau(x), x) \end{array}$$

is called the first return map of Poincaré for the periodic orbit γ_{x_0}.

Remarque 1.6. ⋆ *The mapping P is of class C^k on $N_\delta(x_0) \cap \Sigma$.*

⋆ *The fixed points of P (the points $x \in N_\delta(x_0) \cap \Sigma$ such that $P(x) = x$) correspond to the periodic orbits $\phi(., x)$ of (1.1).*

⋆ *The Poincaré map has an inverse $P^{-1}(x) = \phi(-\tau(x), x)$ of class C^k, thus P is a diffeomorphism.*

For a planar system, we assume that the origin is transformed to $x_0 \in \gamma \cap \Sigma$.
The hyperplane Σ will be a line passing through the origin, where the point $x_0 = 0 \in \gamma \cap \Sigma$ divides the line Σ into two open segments Σ^+ and Σ^-, with Σ^+ being the line outside of γ.
Let s be the algebraic distance between the points of Σ and $x_0 = 0$, with $s > 0$ for Σ^+ and $s < 0$ for Σ^-.
Thus, the mapping P is defined for $|s| < \delta$. And we have :

$$P(0) = 0.$$

Introducing the displacement function $d(s) = P(s) - s$, we then have $d(0) = 0$ and $d'(s) = P'(s) - 1$.
By the mean value theorem, for $|s| < \delta$, we have :

$$d(s) = d'(c) \cdot s$$

for some c between 0 and s.
Since $d'(s)$ is continuous, its sign will match the sign of $d'(0)$ for $|s|$ sufficiently small and such that $d'(0) \neq 0$.
Therefore :

- If $d'(0) < 0$, then $d(s) > 0$ for $s < 0$ and $d(s) < 0$ for $s > 0$ (i.e., the cycle γ is an ω-limit cycle or stable).
- Similarly, if $d'(0) > 0$, then $d(s) > 0$ for $s > 0$ and $d(s) < 0$ for $s < 0$ (i.e., the cycle γ is an unstable limit cycle (α-limit)).

We have the following results for P :

- If $P(0) = 0$ and $P'(0) < 1$, then γ is a stable limit cycle.

- If $P(0) = 0$ and $P'(0) > 1$, then γ is an unstable limit cycle.

Thus, the stability of γ is determined by the derivative of the Poincaré map P.
The following theorem provides a formula for $P'(0)$ in terms of $f(x)$ and $\gamma(t)$.