Series N°5: Direct Methods for Solving Systems of Linear Equations

Exercise 1

Consider the following system.

$$\begin{cases} \alpha x_1 + x_3 = 2 \\ \alpha x_1 + x_2 + 2\alpha x_3 = 1 \\ 2\alpha x_1 + 2x_2 - x_3 = 3 \end{cases}$$

1- Write the matrices A and b relating to the system.

2- Find the expression for the determinant of A and deduce the condition for the system to admit a unique solution.

3- Triangularize the system using the Gauss method.

4- Find the system solution.

Exercise 2

1- Show that matrix $A = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 10 & 6 \\ 6 & 6 & 26 \end{bmatrix}$ is definite positive.

If $b = \begin{bmatrix} 72 \\ 144 \\ 48 \end{bmatrix}$, use the Ckolesky method to resolve the system $AX = b$.

Exercise 3 (additional)

Let the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$, use the Gauss method to calculate:

- The determinant of A, and
- The inverse of matrix A (A^{-1}).

Solution of the Series n°5

Exercise 1

1-

$$A=\begin{bmatrix}\alpha & 0 & 2\\ \alpha & 1 & 2\alpha\\ 2\alpha & 2 & -1\end{bmatrix},\quad x=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}\quad et\quad b=\begin{bmatrix}2\\ 1\\ 3\end{bmatrix}$$

2-

$$Det(A))=\alpha\begin{bmatrix}1 & 2\alpha\\ 2 & -1\end{bmatrix}-0\begin{bmatrix}\alpha & 2\alpha\\ 2\alpha & -1\end{bmatrix}+2\begin{bmatrix}\alpha & 1\\ 2\alpha & 2\end{bmatrix}=\alpha(1+4\alpha)$$

3- The system accepts a single solution only if the determinant of A is different from zero.

$$Det(A))\neq 0\Rightarrow\begin{cases}\alpha\neq 0\\ \alpha\neq -\dfrac{1}{4}\end{cases}.$$

4- Triangularization of the system:

The expanded matrix $\left|\begin{matrix}\alpha & 0 & 1\\ \alpha & 1 & 2\alpha\\ 2\alpha & 2 & -1\end{matrix}\middle|\begin{matrix}2\\ 1\\ 3\end{matrix}\right|$

$$\text{Step 1}:\left|\begin{matrix}\alpha & 0 & 1\\ \alpha & 1 & 2\alpha\\ 2\alpha & 2 & -1\end{matrix}\middle|\begin{matrix}2\\ 1\\ 3\end{matrix}\right|\quad\begin{matrix}\rightarrow L_1\\ \rightarrow L_2\ \ldots\\ \rightarrow L_3\end{matrix}$$

$$\text{Step 2}:\left|\begin{matrix}\alpha & 0 & 1\\ 0 & 1 & 2\alpha\\ 0 & 2 & -1\end{matrix}\middle|\begin{matrix}2\\ -1\\ -1\end{matrix}\right|\quad\begin{matrix}\rightarrow L_1=L_1\\ \rightarrow L_2=L_2-L_1\ \ ..\\ \rightarrow L_3=L_2-2L_1\end{matrix}$$

$$\text{Step 3}:\left|\begin{matrix}\alpha & 0 & 1\\ 0 & 1 & (2\alpha-1)\\ 0 & 0 & -(4\alpha+1)\end{matrix}\middle|\begin{matrix}2\\ -1\\ 1\end{matrix}\right|\quad\begin{matrix}\rightarrow L_1=L_1\\ \rightarrow L_2=L_2\quad\quad ....\\ \rightarrow L_3=L_3-2L_2\end{matrix}$$

The system becomes,

$$\begin{cases} \alpha x_1 + x_2 = 2 \\ x_2 + (2\alpha - 1)x_3 = -1 \\ -(4\alpha + 1)x_3 = 1 \end{cases} \Rightarrow \begin{cases} x_1 = -\dfrac{8\alpha + 3}{\alpha(4\alpha + 1)} \\ x_2 = -\dfrac{2\alpha + 2}{4\alpha + 1} \\ x_3 = \dfrac{-1}{4\alpha + 1} \end{cases} \quad \text{.............}$$

Exercise 2 :

1- A^t=A then, A is symmetric.

2- To check if the matrix A is strictly positive, we must check that its three secondary determinants are exactly positive.

$$A = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 10 & 6 \\ 6 & 6 & 26 \end{bmatrix}.$$

$$I_1 = 4 > 0, \quad I_2 = \det\begin{vmatrix} 4 & 2 \\ 2 & 10 \end{vmatrix} = 36 > 0 \text{ et } I_3 = \det\begin{bmatrix} 4 & 2 & 6 \\ 2 & 10 & 6 \\ 6 & 6 & 26 \end{bmatrix},$$

$$I_3 = 4\begin{bmatrix} 10 & 6 \\ 6 & 26 \end{bmatrix} - \begin{bmatrix} 2 & 6 \\ 6 & 26 \end{bmatrix} + 6\begin{bmatrix} 2 & 10 \\ 6 & 6 \end{bmatrix} = 576 > 0$$, therefore A is symmetric positive-definite.

The Cholesky method consists of decomposing the matrix A to a product of a lower triangular matrix and its transpose i.e. $A=LL^t$.

The system becomes:

$$Ax = b \quad \to LL^t x = b \quad on\ met\ L^t x = y \quad \to \begin{cases} Ly = b..........1 \\ L^t x = y..........2 \end{cases}. \text{ Where } b = \begin{bmatrix} 72 \\ -144 \\ 48 \end{bmatrix}$$

- We find the vector y from the equation 1.
- Then we find the vector x from the equation 2

The Lower Triangular matrix L is given by;

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}, \quad L^t = \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}.$$

$$LL^t = \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{21}l_{11} & (l_{21}^2 + l_{22}^2) & (l_{21}l_{31} + l_{22}l_{32}) \\ l_{31}l_{11} & (l_{31}l_{21} + l_{32}l_{22}) & (l_{31}^2 + l_{32}^2 + l_{33}^2) \end{bmatrix} = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 10 & 6 \\ 6 & 6 & 26 \end{bmatrix} = A,$$

- From the first column ;

  $l_{11}^2 = 4 \Rightarrow l_{11} = 2$

  $l_{21}l_{11} = 2 \Rightarrow l_{21} = 1$,

  $l_{31}l_{11} = 6 \Rightarrow l_{31} = 3$

- From the second column ;

  $l_{21}^2 + l_{22}^2 = 10 \Rightarrow l_{22} = 1$

  $l_{31}l_{21} + l_{32}l_{22} = 6 \Rightarrow l_{32} = 1$,

- From the third column ;

  $l_{31}^2 + l_{32}^2 + l_{33}^2 = 26 \Rightarrow l_{33} = 4$,

Then, $L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, $L^t = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$,

1- We find the vector y form Ly=b,

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 72 \\ 144 \\ 48 \end{bmatrix} \Rightarrow \text{y} = \begin{bmatrix} 36 \\ -60 \\ 0 \end{bmatrix},$$

2- And we find the vector x from $L^t x = y$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 36 \\ 36 \\ 12 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 28 \\ -20 \\ 0 \end{bmatrix}.$$

Exercise 3 :

Pour calculer le déterminant d'une matrice A avec la méthode de Gauss :

- Rendre la matrice triangulaire.

- Le déterminant est simplement le produit de tous les éléments diagonaux. C.a.d,

$$\det(A) = \prod_{i=1}^{n} a_{ii} .$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 0 & 1 & 2 \end{vmatrix} \begin{matrix} \rightarrow L_1 = L_1 \\ \rightarrow L_2 = L_2 - L_1 \\ \rightarrow L_3 = L_2 - 2L_1 \end{matrix} \quad \ldots\ldots\ldots$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & -5 & -6 \\ 0 & 1 & 2 \end{vmatrix} \begin{matrix} \rightarrow L_1 = L_1 \\ \rightarrow L_2 = L_2 - 2L_1 \\ \rightarrow L_3 = L_3 \end{matrix}$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & -5 & -6 \\ 0 & 0 & \frac{4}{5} \end{vmatrix} \begin{matrix} \rightarrow L_1 = L_1 \\ \rightarrow L_2 = L_2 \\ \rightarrow L_3 = L_3 + \frac{L_2}{5} \end{matrix}$$

$$\det(A) = 1 \times (-5) \times (\frac{4}{5}) = -4$$

- Pour trouver l'inverse de la matrice A (A^{-1}) par la méthode de Gauss :
  - On doit écrire la matrice élargie constituée de A et la matrice unitaire sous la forme,

$$\left|\begin{matrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 0 & 1 & 2 \end{matrix}\right|\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right| ,$$

  - on va appliquer l'échelonnement de Gauss pour rendre la matrice A une matrice unitaire.

1- $$\left|\begin{matrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 0 & 1 & 2 \end{matrix}\right|\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\left|\begin{matrix} \rightarrow L_1 \\ \rightarrow L_2 \\ \rightarrow L_3 \end{matrix}\right. .$$

2- $$\left|\begin{matrix} 1 & 2 & 3 \\ 0 & -5 & -6 \\ 0 & 1 & 2 \end{matrix}\right|\begin{matrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\left|\begin{matrix} \rightarrow L_1 \\ \rightarrow L_2 = L_2 - 2L_1 \\ \rightarrow L_3 \end{matrix}\right. ,$$

3- $$\left|\begin{matrix} 1 & 2 & 3 \\ 0 & -5 & -6 \\ 0 & 0 & \frac{4}{5} \end{matrix}\right| \begin{matrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -\frac{2}{5} & \frac{1}{5} & 1 \end{matrix}\left| \begin{matrix} \to L_1 = L_1 \\ \to L_2 = L_2 \\ \to L_3 = L_3 + \frac{1}{5}L_2 \end{matrix}\right. ,$$

4- , $$\left|\begin{matrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & \frac{4}{5} \end{matrix}\right| \begin{matrix} \frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\ -5 & \frac{5}{2} & \frac{30}{4} \\ -\frac{2}{5} & \frac{1}{5} & 1 \end{matrix}\left| \begin{matrix} \to L_1 = L_1 \\ \to L_2 = L_2 + \frac{30}{4}L_3 \\ \to L_3 = L_3 \end{matrix}\right. ,$$

5- $$\left|\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right| \begin{matrix} \frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\ 1 & -\frac{1}{2} & -\frac{6}{4} \\ -\frac{1}{2} & \frac{1}{4} & \frac{5}{4} \end{matrix}\left| \begin{matrix} \to L_1 = L_1 \\ \to L_2 = -\frac{1}{5}L_2 \\ \to L_3 = \frac{5}{4}L_3 \end{matrix}\right. ,$$

6- $$\Rightarrow A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\ 1 & -\frac{1}{2} & -\frac{6}{4} \\ -\frac{1}{2} & \frac{1}{4} & \frac{5}{4} \end{bmatrix},$$

Pour confirmer, on va vérifier le produit $AA^{-1} = I$;

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \times \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\ 1 & -\frac{1}{2} & -\frac{6}{4} \\ -\frac{1}{2} & \frac{1}{4} & \frac{5}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$