Chapter 2

Polynomial interpolation

2.1 Introduction

Interpolation is a process that consists of connecting discrete data points (Figure 2.1a) in some way to obtain an acceptable estimation of the intermediate points (Fig. 2.1b and 2.1c) or to replace a complex function with a simple polynomial where they coincide at a finite number of points (Fig. 2.1d).

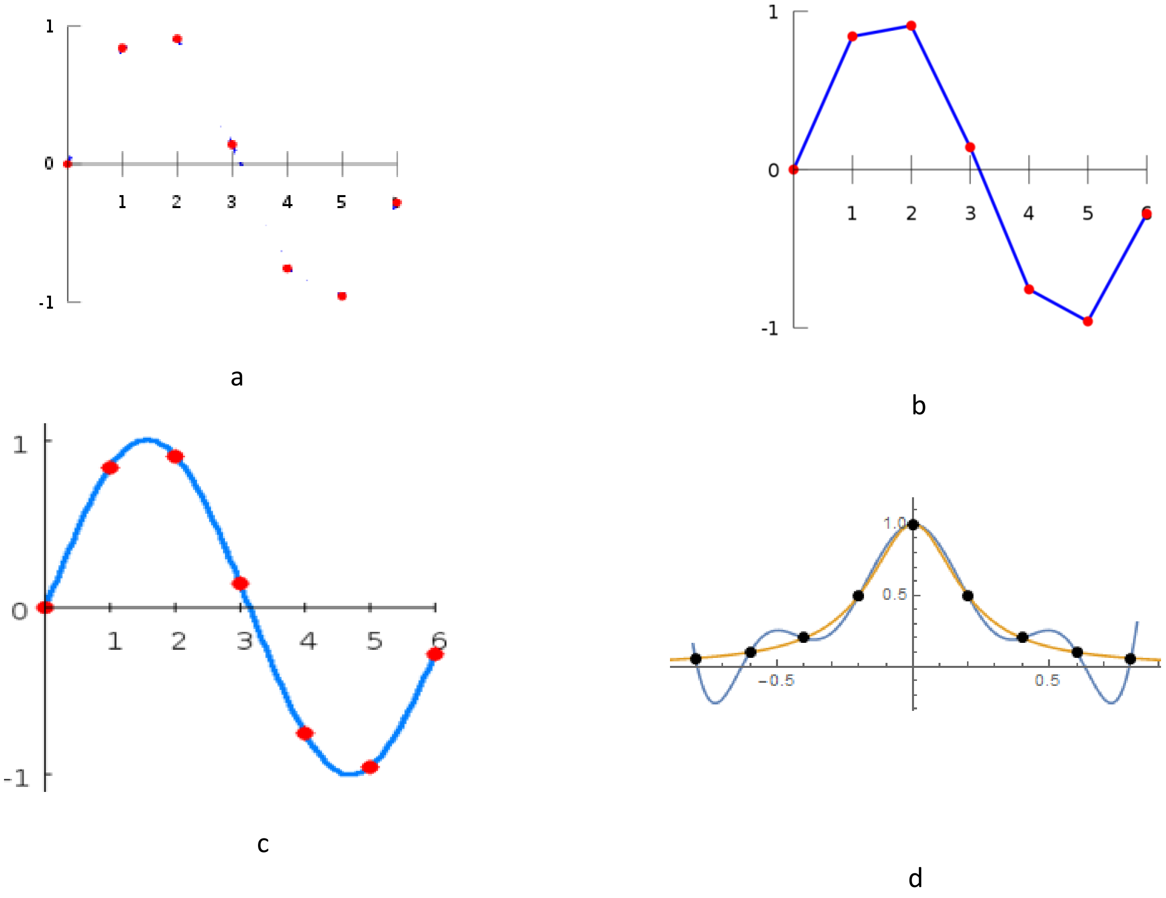

Figure 2.1 Graphical representation of interpolation: a) discrete points, b) linear interpolation, c) polynomial interpolation, d) interpolation of a function by a polynomial.

- A simple line (Figure 2.1b) can connect the discrete points; we then talk of linear interpolation.

- If we connect the discrete points by a polynomial (as shown in Figure 2.1c), the interpolation is called polynomial.

In other words, the goal of interpolation is to establish a relationship between points whose values are known to predict intermediate values. In this chapter, we give a brief explanation of linear interpolation, and we will focus much more on polynomial interpolation.

2. 2 linear Interpolation

Linear interpolation consists of connecting the two adjacent points with a line, as shown in Figure 2.2.

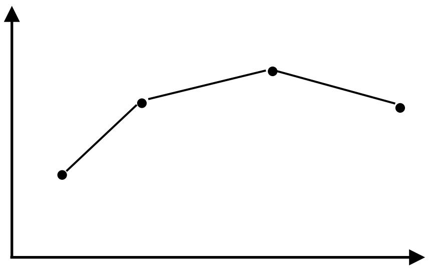

Figure 2.2 Linear interpolation

The equation of the line y 1 is given by

$$Y_1 = ax + b$$

Where the tangent is given by

$$a = \frac{y_1 - y_0}{x_1 - x_0}$$

at $x = x_0, \; y = y_0,$ then $b = y_0 - \frac{y_1 - y_0}{x_1 - x_0} x_0$

$$Y_1 = \frac{y_1 - y_0}{x_1 - x_0} x + y_0 - \frac{y_1 - y_0}{x_1 - x_0} x_0$$

$$Y_1 = \frac{y_1 - y_0}{x_1 - x_0} (x - x_0) + y_0$$

$Y_2 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1) + y_1.$

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$$Y_i = \frac{y_i - y_{1-1}}{x_i - x_{i-1}} (x - x_{i-1}) + y_i$$

2. 3 Polynomial interpolation

2.3.1 Singularity (uniqueness) of the interpolation polynomial

Theorem

The necessary and sufficient condition for the existence of a single polynomial for interpolation is that all points x_i be distinct.

- One point: there is a single polynomial of order zero that passes through this point (fig 2.2 a).
- Two points: there is a single polynomial of order one that passes through both points (fig 2.2 b).
- Three points: there is a single polynomial of order two that passes through the three points (fig 2.2 c).
- n points, there is a single polynomial of order n-1 that passes through the n points.

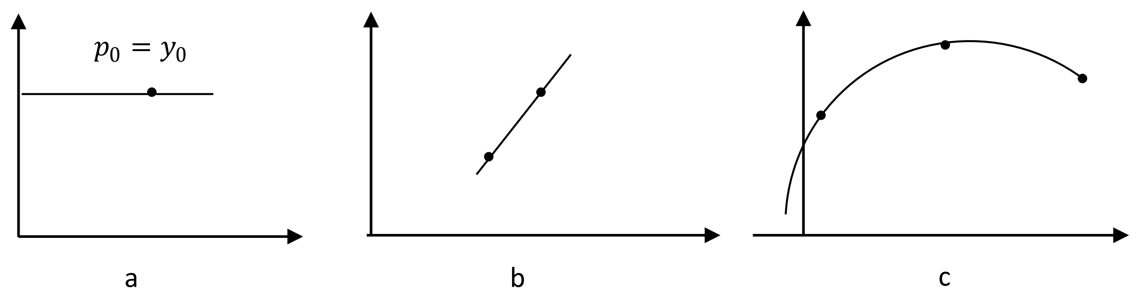

Figure: 2. 3: polynomial interpolation

2.3.2 Lagrange Polynomial

We call the Lagrange polynomial of degree n, based on the interpolation points $(x_i, f(x_i))$, the unique polynomial of order n, which passes exactly through the (n+1) points $(i = 0, \dots n)$. The unique polynomial $P_n(x)$ is defined by

$$P_n(x) = \sum_{k=0}^{n} \mathrm{L}_k(x) \quad f(x_k),$$

$$P_n(x) = L_0(x) \times f(x_0) + L_1(x) \times f(x_1) + \cdots \; L_n(x) \times f(x_n).$$

Where,

$\mathrm{L}_k(x)$ is the elementary Lagrange polynomial, which is given by:

$$\mathrm{L}_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{i=n} \frac{x - x_i}{x_k - x_i}$$

So we can write the Lagrange interpolation polynomial in the form;

$$L_0(x) = \frac{(x - x_0)}{(x_0 - x_0)} \times \frac{(x - x_1)}{(x_0 - x_1)} \times \frac{(x - x_2)}{(x_0 - x_2)} \times \dots \frac{(x - x_n)}{(x_0 - x_n)}.$$

Must be removed, because i must be different from j.

$$L_1(x) = \frac{(x - x_0)}{(x_1 - x_0)} \times \frac{(x - x_1)}{(x_1 - x_1)} \times \frac{(x - x_2)}{(x_1 - x_2)} \times \dots \frac{(x - x_n)}{(x_1 - x_n)}.$$

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$$L_n(x) = \frac{(x - x_0)}{(x_n - x_0)} \times \frac{(x - x_1)}{(x_n - x_1)} \times \frac{(x - x_2)}{(x_n - x_2)} \times \dots \frac{(x - x_n)}{(x_n - x_n)}$$

Note that $L_k(x)$ has an interesting property, which is

$$L_k(x_i) = \begin{cases} 1 & if \quad i = k \\ 0 & if \quad i \neq k \end{cases}$$

For example,

$$L_0(x_0) = \frac{(x_0 - x_1)}{(x_0 - x_1)} \times \frac{(x_0 - x_2)}{(x_0 - x_2)} \times \dots \frac{(x_0 - x_n)}{(x_0 - x_n)} = 1$$

$$L_0(x_1) = \frac{(x_1 - x_1)}{(x_0 - x_1)} \times \frac{(x_1 - x_2)}{(x_0 - x_2)} \times \dots \frac{(x_1 - x_n)}{(x_0 - x_n)} = 0$$

$$L_0(x_2) = \frac{(x_2 - x_1)}{(x_0 - x_1)} \times \frac{(x_2 - x_2)}{(x_0 - x_2)} \times \dots \frac{(x_2 - x_n)}{(x_0 - x_n)} = 0$$

$$L_1(x_1) = \frac{(x_1 - x_0)}{(x_1 - x_0)} \times \frac{(x_1 - x_2)}{(x_1 - x_2)} \times \dots \frac{(x_1 - x_n)}{(x_1 - x_n)} = 1$$

$$L_1(x_2) = \frac{(x_2 - x_0)}{(x_1 - x_0)} \times \frac{(x_2 - x_2)}{(x_1 - x_2)} \times \dots \frac{(x_2 - x_n)}{(x_1 - x_n)} = 0$$

$$L_1(x_n) = \frac{(x_n - x_0)}{(x_1 - x_0)} \times \frac{(x_n - x_2)}{(x_1 - x_2)} \times \dots \frac{(x_n - x_n)}{(x_1 - x_n)} = 0$$

Example 1

We want to find the polynomial passing through the points (x_i, y_i) resulting from a physical experiment, which are recorded in the following table.

x_i	0	1	2	3
y_i	1	4	8	14

We have four points, so the interpolation polynomial will be of order 3.

$$P_3(x) = L_0 \times f(x_0) + L_1 \times f(x_1) + L_2 \times f(x_2) + L_3 \times f(x_3)$$

$$P_3(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} \times f(x_0)$$

$$+ \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} \times f(x_1)$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} \times f(x_2)$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_1)} \times f(x_3)$$

Then,

$$P_3(x) = \frac{(x - 1)(x - 2)(x - 3)}{(0 - 1)(0 - 2)(0 - 3)} \times 1 = -\frac{1}{6}x^3 + x^2 - \frac{11}{6}x + 1$$

$$+ \frac{(x - 0)(x - 2)(x - 3)}{(1 - 0)(1 - 2)(1 - 3)} \times 4 = 2x^3 - 10x^2 + 12x$$

$$+ \frac{(x - 0)(x - 1)(x - 3)}{(2 - 0)(2 - 1)(2 - 3)} \times 8 = -4x^3 + 16x^2 - 12x$$

$$+ \frac{(x - 0)(x - 1)(x - 2)}{(3 - 0)(3 - 1)(3 - 2)} \times 14 = \frac{7}{3}x^3 - 7x^2 + \frac{14}{3}x$$

$$P_3(x) = \frac{1}{6}x^3 + \frac{17}{6}x + 1$$

Notes 1: We need to check that the resulting polynomial passes exactly through all these points.

Notes 2: It is necessary to verify that each elementary polynomial satisfies the point that corresponds and is equal to zero at all other interpolated points.

Notes 3: Lagrange polynomial is the sum of all elementary polynomials.

Example 2

As we said earlier, we can also approximate a function to a polynomial to simplify its study. Let us take $f(x) = \frac{1}{x}$ as an example and look at what the polynomial that corresponds to this function looks like on the interval $[2, 4]$ interpolating the function at the points represented in the table below.

x_i	2	2.5	4
y_i	0.5	0.4	0.25

Solution :

$$P_2(x) = \frac{1}{20}x^2 - \frac{51}{120}x + \frac{23}{20}$$

The resulting polynomial is only an approximation, so an error was made during the approximation of a value belong to $[a, b]$. To evaluate this error made on calculating x, we can use Taylor's development.

$$|f(x) - p_n(x)| = \frac{(x - x_0) \times (x - x_1) \times \dots (x - x_n)}{(n + 1)!} f^{[n+1]}(\delta) = |R(x)|$$

$$f(x) = f(c) + f'(c) \times (x - c) + f''(c) \times \frac{(x - c)^2}{2!} + \cdots \dots .. f^{[n]}(c) \times \frac{(x - c)^n}{n!} + f^{[n+1]}(c) \times \frac{(x - c)^{n+1}}{(n + 1)!}$$

We can write,

$$f(x) = p_n(x) + R(x)$$

$R(x)$: Represents the error can be made for approximating the function $f(x)$ to the polynomial $p_n(x)$.

$$R(x) = f^{[n+1]}(c) \times \frac{(x - c)^{n+1}}{(n + 1)!}$$

We should maximize the error,

- $f^{[n+1]}(c) \rightarrow \max_{\delta \in [a;b]} \left[(f^{[n+1]}(\delta) \right]$
- $\max \frac{(x-c)^{n+1}}{(n+1)!} \rightarrow \frac{(b-a)^{n+1}}{(n+1)!}$.

The maximum error can be made in approximating is given by

$$|R(x)| \leq max\left[(f^{[n+1]}(\delta) \right] \times \frac{(b-a)^{n+1}}{(n+1)!}.$$

$max\left[(f^{[n+1]}(\delta) \right]$ is the maximum value of the nth +1 derivative of the function f on the interval $[a.b]$.

For the example before

$$P_2(3) = \frac{1}{20}3^2 - \frac{51}{120}3 + \frac{23}{20} = 0.325 \qquad (1)$$

$$f(3) = \frac{1}{3} = 0.333 \qquad (2)$$

$E = |f(3) - P_2(3)| = 0.008$

$$f(x) = \frac{1}{x} \rightarrow f'(x) = \frac{-1}{x^2} \rightarrow f''(x) = \frac{2}{x^3} \longrightarrow f'''(x) = \frac{-6}{x^4}$$

$$f'''(2) = \frac{-6}{2^4} = -0.3750$$

$$f'''(4) = \frac{-6}{4^4} = -0.0234$$

$M = 0.3750$

$$\frac{1}{(n+1)!}\prod_{i=0}^{n}(\mathrm{x} - \mathrm{x}_i) = \frac{(3-2)(3-2.5)(3-4)}{4!} = -0.0208$$

$|E_n(x)| = 0.375 \times 0.083 = 0.078$

2.3.3 Newton polynomial

As we saw in the previous paragraph:

- The polynomial that passes through one point (x_0, y_0) is a polynomial of order 0 given by:

$$P_0(x) = y_0 = a_0$$

- The polynomial that passes through the two points (x_0, y_0) and (x_1, y_1) is a polynomial of order 1 given by:

$$P_1(x) = a_0 + a_1(x - x_0)$$

- The polynomial that passes through the three points (x_0, y_0), (x_1, y_1) and (x_2, y_2) is a polynomial of order 2 given by:

$$P_2(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_2)$$

In the general case, if we have n+1 points, Newton's polynomial will be of order n, given by:

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) \dots + a_n(x - x_0)(x - x_1) \dots (x - x_{n-1})$$

Split difference method is used to find the coefficients a_i:

We know that $a_0 = y_0$. We would like to know the value of a_1.At $x = x_1$, we have

$P_1(x_1) = y_0 + a_1(x_1 - x_0) = y_1$, then

$a_1 = \frac{y_1 - y_0}{x_1 - x_0} = f[x_0 - x_1] = \Delta y_1$ is the first derivative difference of order 1.

The divided differences of order 1 are given by:

$$f[x_0 - x_1] = \Delta y_1 = \frac{y_1 - y_0}{x_1 - x_0}$$

$$f[x_1 - x_2] = \Delta y_2 = \frac{y_2 - y_1}{x_2 - x_1}$$

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$$f[x_{i-1} - x_i] = \Delta y_i = \frac{y_i - y_{i-1}}{x_i - x_{i-1}}$$

The divided differences of order 2 are given by:

$$f[x_{i-1}, x_i, x_{i+1}] = \Delta y_i^2 = \frac{\Delta y_{i+1} - \Delta y_i}{x_{i+1} - x_{i-1}} = \frac{\frac{y_{i+1} - y_i}{x_{i+1} - x_i} - \frac{y_i - y_{i-1}}{x_i - x_{i-1}}}{x_{i+1} - x_{i-1}}$$

a_2 is the first divided difference term of order 2,

$$a_2 = f[x_0, x_1, x_2] = \Delta y_0^2 = \frac{\Delta y_1 - \Delta y_0}{x_2 - x_0} = \frac{\frac{y_2 - y_1}{x_2 - x_0} - \frac{y_1 - y_0}{x_1 - x_0}}{x_2 - x_0}$$

The divided differences of order n are given by:

$$f[x_0, x_1, \dots x_n] = \Delta y_i^n = \frac{\Delta y_{i+1}^n - \Delta y_i^n}{x_{i+1} - x_{i-1}}$$

a_i is given by the first term of the divided difference of order i,

$$a_i = \Delta y_1^n = \frac{\Delta y_2^n - \Delta y_1^n}{x_n - x_0}$$

To better illustrate the idea, we give the following table:

x_0 y_0

$$\Delta y_1^1 = \frac{y_1 - y_0}{x_1 - x_0}$$

x_1 y_1

$$\Delta y_1^2 = \frac{\Delta y_1 - \Delta y_0}{x_2 - x_1}$$

$$\Delta y_2^1 = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\Delta y_1^3 = \frac{\Delta y_1^2 - \Delta y_0^2}{x_4 - x_0}$$

x_2 y_2

$$\Delta y_2^2 = \frac{\Delta y_2 - \Delta y_1}{x_3 - x_0}$$

.

$$\Delta y_3^1 = \frac{y_3 - y_2}{x_3 - x_2}$$

. .

x_3 y_3 .

$$\Delta y_3^2 = \frac{\Delta y_3 - \Delta y_2}{x_4 - x_0}$$

. . . . .

$$\Delta y_1^{n-1} = \frac{\Delta y_2 - \Delta y_1}{x_n - x_0}$$

. .

$$\Delta y_{n-2}^3 = \frac{\Delta y_{n-2} - \Delta y_n}{x_n - x_0}.$$

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$$\Delta y_{n-1}^2 = \frac{\Delta y_{n-1} - \Delta y_n}{x_n - x_0}$$

$$\Delta y_n^1 = \frac{y_n - y_{n-1}}{x_n - x_{n-1}}$$

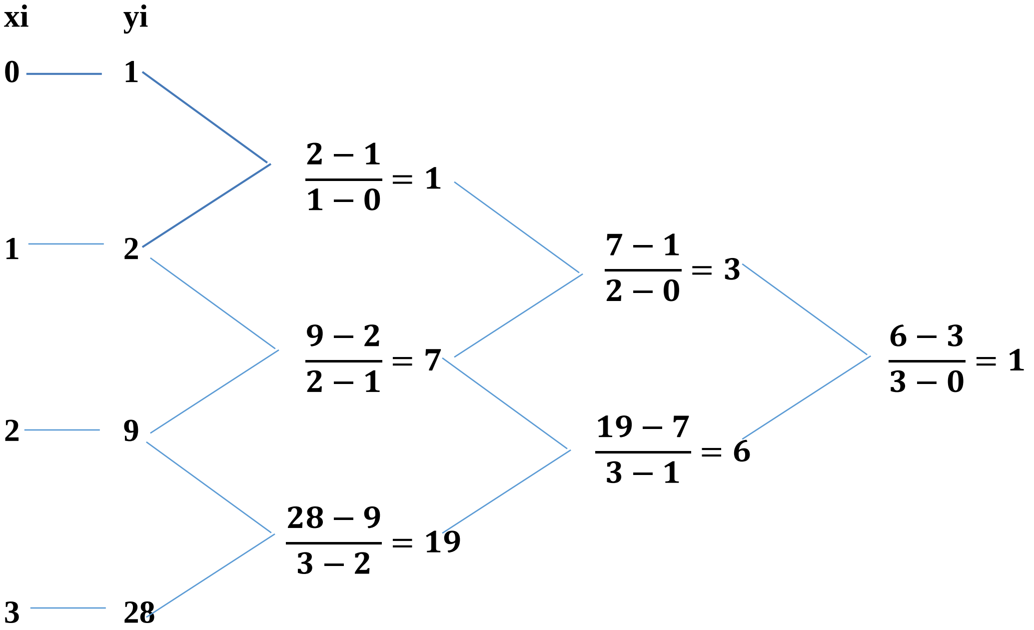
x_n y_n

Example 1

We would like to find the newton polynomial that interpolates values issued from a physical experiment, which are represented in the table.

x	0	1	2	3
y	1	2	9	28

First, we have to write the data in vertical way. We have four point, and then the polynomial should be of orders three.



$$P_n(x) = 1 + 1 \times (x-0) + 3 \times (x-0)(x-1) + 1 \times (x-0)(x-1)(x-2)$$

$$P_n(x) = x^3 + 1$$

Example 2

Find the newton polynomial that interpolates values in the following table.

x	0	1	2	3
y	1	2	1	10

Solution

$P_n(x) = 2x^3 - 7x^2 + 6x + 1$

Note : we could evaluate the error made through the approximation by the same formula used in Lagrangian interpolation.