Solutions of tutorial exercices set 3

EXE 01:

1. $D_f =]-\infty, 0[\cup]0, +\infty[$.

2. $D_f = \bigcup_{k\in\mathbb{Z}}]2k\pi, \pi + 2k\pi[$.

3. $D_f =]-\infty, -1] \cup]1, +\infty[$.

4. $f(x) = (1+\ln x)^{\frac{1}{x}} = e^{\frac{1}{x}\ln(1+\ln x)}$

So we have $D_f = \{x \in \mathbb{R} \mid x > 0 \text{ and } 1 + \ln x > 0$

$\Rightarrow D_f =]e^{-1}, +\infty[$.

5. $f(x) = \frac{1}{[x]}$ so $D_f =]-\infty, 0[\cup [1, +\infty[$

because $[x] = 0 \Leftrightarrow x \in [0, 1[$.

6. $D_f =]-2, 1] \cup [2, +\infty[$.

EXE 02:

1. $l_1 = \lim_{x\to 0} x \sin\frac{1}{x} = 0$, because $\lim_{x\to 0} x = 0$ and $-1 \leq \sin\frac{1}{x}$

2. $l_2 = \lim_{x\to +\infty} x \sin\frac{1}{x} = \lim_{x\to +\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}} = 1$.

3. $l_3 = \lim_{x\to 0} \frac{x - \sin(2x)}{x - \sin(3x)} = \lim_{x\to 0} \frac{2x\left[\frac{1}{2} - \frac{\sin(2x)}{2x}\right]}{3x\left[\frac{1}{3} + \frac{\sin(3x)}{3x}\right]} = -\frac{1}{4}$

4. $l_4 = \lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\sin x}{x\cos x} = 1$.

5. $l_5 = \lim_{x\to \frac{\pi}{4}} \frac{\sin x - \cos x}{1 - \tan x} = \lim_{x\to \frac{\pi}{4}} \frac{\cos x(\tan x - 1)}{1 - \tan x} = -\frac{\sqrt{2}}{2}$

6. $l_6 = \lim_{x \to a} \frac{\sqrt{x} - \sqrt{a} + \sqrt{x-a}}{\sqrt{x^2 - a^2}} = \lim_{x\to a^+} \frac{\sqrt{x} - \sqrt{a} + \sqrt{x-a}}{\sqrt{x^2 - a^2}}$

$$l_7 = \lim_{x \to a^+} \frac{\frac{\sqrt{x}-\sqrt{a}}{\sqrt{x-a}} + 1}{\sqrt{x+a}}$$

$$= \lim_{x \to a^+} \frac{\frac{\sqrt{x-a}}{(\sqrt{x}+\sqrt{a})} + 1}{\sqrt{x+a}}$$

$$= \frac{1}{\sqrt{2a}}$$

8. $l_8 = \lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x \to +\infty} e^{x \ln\left(1+\frac{1}{x}\right)}$

Let us take: $t = \frac{1}{x}$, thus: $l_8 = \lim_{t \to 0} e^{\frac{\ln(1+t)}{t}} = e$

9. $l_9 = \lim_{x \to +\infty} \left(\sin\sqrt{x+1} - \sin\sqrt{x}\right)$

$$= \lim_{x \to +\infty} 2 \sin\left(\frac{\sqrt{x+1}-\sqrt{x}}{2}\right) \cos\left(\frac{\sqrt{x+1}+\sqrt{x}}{2}\right)$$

$$= \lim_{x \to +\infty} 2 \sin\left(\frac{1}{2\left(\sqrt{x+1}+\sqrt{x}\right)}\right) \cos\left(\frac{\sqrt{x+1}+\sqrt{x}}{2}\right) = 0$$

10. $l_{10} = \lim_{x \to 1} (1-x) \tan\left(\frac{\pi x}{2}\right)$

Let us take: $t = x - 1$, so we have: $x = t + 1$ and

Thus, $l_{10} = \lim_{t \to 0} (-t) \tan\left(\frac{\pi t}{2} + \frac{\pi}{2}\right)$

$$= \lim_{t \to 0} \frac{t}{\tan\left(\frac{\pi t}{2}\right)}$$

$$= \lim_{t \to 0} \frac{1}{\frac{\pi}{2} \cdot \frac{\tan\left(\frac{\pi t}{2}\right)}{\frac{\pi t}{2}}} = \frac{2}{\pi}$$

but $\tan\left(\alpha + \frac{\pi}{2}\right) = -\frac{1}{\tan\alpha}$.

EXO 3:

1. $\left(\lim_{x\to 4}(2x-1) = 7\right) \Leftrightarrow \left(\forall \varepsilon > 0, \exists \alpha > 0, \forall x \in \mathbb{R};\ |x-4| < \alpha \Rightarrow |2x-8| < \varepsilon\right)$

$|2x-8| < \varepsilon \Leftrightarrow 2|x-4| < \varepsilon \Leftrightarrow |x-4| < \frac{\varepsilon}{2}$

we can take: $\alpha = \frac{\varepsilon}{2}$

2. $\left(\lim_{x\to+\infty} \frac{3x-1}{2x+1} = \frac{3}{2}\right) \Leftrightarrow \left(\forall \varepsilon > 0, \exists \alpha > 0, \forall x \in \mathbb{R};\ x > \alpha \Rightarrow \left|\frac{3x-1}{2x+1} - \frac{3}{2}\right| < \varepsilon\right) \Leftrightarrow \frac{5}{4x+2} < \varepsilon \Leftrightarrow x > \frac{5-2\varepsilon}{4\varepsilon}$

we can take: $\alpha = \left|\frac{5-2\varepsilon}{4\varepsilon}\right|$.

3. $\left(\lim_{x\to+\infty} \ln(x) = +\infty\right) \Leftrightarrow \left(\forall A > 0, \exists \alpha > 0, \forall x \in \mathbb{R};\ x > \alpha \Rightarrow \ln x > A\right)$

$\ln x > A \Leftrightarrow x > e^A$;

we can take $\alpha = e^A$

4. $\left(\lim_{x\to -3} \frac{4}{x+3} = +\infty\right) \Leftrightarrow \left(\forall A > 0, \exists \alpha > 0, \forall x \in \mathbb{R};\ -3 < x < -3 + \alpha \Rightarrow \frac{4}{x+3} > A\right)$

$\frac{4}{x+3} > A \Leftrightarrow x < \frac{4}{A} - 3$,

we can take: $\alpha = \frac{4}{A}$.

EXE 04:

1) $-1 \leq \sin\frac{1}{x} \leq 1 \Rightarrow -x \leq x\sin\frac{1}{x} \leq x$

$$\lim_{x\to 0} x = 0 \Rightarrow \lim_{x\to 0} f(x) = 0.$$

But this limit is not equal to $f(0)=3$, so that $f(x)$ is discontinuous at $x=0$.

2) By redefining $f(x)$ so that $f(0)=0$, the function becomes continuous. Because the function can be made continuous at a point by redefining the function at the point, we call the point a removable discontinuity

EXE 05:

Method 1: $\lim_{x\to 3} f(x) = f(3) = 9$ and so $f(x)$ is continuous at $x=2$.

Method 2:

We must show that given any $\varepsilon > 0$, we can find $\delta > 0$ (depending on ε) such that $|f(x) - f(3)| = |x^2 - 9| < \varepsilon$ when $|x-3| < \delta$.

choose $\delta \leq 1$ so that $0 < |x-3| < 1 \Rightarrow 1 < x < 4$

Then $|x^2 - 9| = |(x-3)(x+3)| < \delta|x+3| < 7\delta$

Take say 1 or $\frac{\varepsilon}{7}$, whichever is smaller. Then we have $|x^2 - 9| < \varepsilon$ whenever $0 < |x-3| < \delta$ and the required result is proved

EXE 06:

Method 1:

suppose that $f(x)$ is uniformly continuous in the given interval. Then for any $\varepsilon > 0$ we should be able to find δ, say between 0 and 1, such that $|f(x) - f(x_0)| < \varepsilon$

When $|x-x_0|<\delta$ for all x and x_0 in the interval.
Let $x=\delta$ and $x_0=\frac{\delta}{1+\varepsilon}$. Then $|x-x_0|=\left|\delta-\frac{\delta}{1+\varepsilon}\right|=\frac{\varepsilon}{1+\varepsilon}$
However, $\left|\frac{1}{x}-\frac{1}{x_0}\right|=\left|\frac{1}{\delta}-\frac{1+\varepsilon}{\delta}\right|=\frac{\varepsilon}{\delta}>\varepsilon$ (since $0<$

Thus we have a contradiction and it follows that
$f(x)=\frac{1}{x}$ cannot be uniformly continuous in $0<x$

Method 2:

Let x_0 and $x_0+\delta$ be any two points in $]0,1[$. Then

$$|f(x_0)-f(x_0+\delta)|=\left|\frac{1}{x_0}-\frac{1}{x_0+\delta}\right|=\frac{\delta}{x_0(x_0+\delta)}$$

can be made larger than any positive number by choos
x_0 sufficiently close to 0. Hence the function cannot be uniformly continuous.

2) $|x-x_0|<\delta \overset{?}{\Rightarrow} \left|\frac{1}{x}-\frac{1}{x_0}\right|=\frac{|x-x_0|}{x\,x_0}<\frac{|x-x_0|}{4}$

We can take $\delta=4\varepsilon$ which is not depend on x_0, so $f(x)$ is uniformly continuous in $]2,+\infty[$.

EXE 07:

$$f(x_0+h)-f(x_0)=\frac{f(x_0+h)-f(x_0)}{h}\cdot h,\quad h\neq 0$$

Then $\lim\limits_{h\to 0} f(x_0+h)-f(x_0)=\lim\limits_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}\cdot h,\ h\neq 0$

Then $\lim\limits_{h\to 0} f(x_0+h)-f(x_0)=\lim\limits_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}\lim\limits_{h\to 0} h$

$$=f'(x_0)\cdot 0=0$$

since $f'(x_0)$ exists by hypothesis. Thus

$\lim\limits_{h\to 0} f(x_0+h)-f(x_0)=0$ or $\lim\limits_{h\to 0} f(x_0+h)=f(x_0)$
showing that $f(x)$ is continuous at $x=x_0$.

EXE 08

1) by EXE 04, $f(x)$ is continuous at $x=0$.

2) $f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$

$= \lim_{h \to 0} \frac{f(h) - f(0)}{h}$

$= \lim_{h \to 0} \frac{h \sin\frac{1}{h} - 0}{h}$

$= \lim_{h \to 0} \sin\frac{1}{h}$

which does not exist.

EXE 09:

1) $f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$

$= \lim_{h \to 0} \frac{h^2 \sin\frac{1}{h} - 0}{h} = \lim_{h \to 0} h \sin\frac{1}{h} = 0$

Then $f(x)$ has a derivative (is differentiable) at $x=0$ and its value is 0.

2) if $x \neq 0$:

$f'(x) = \frac{d}{dx}\left(x^2 \sin\frac{1}{x}\right) = x^2 \frac{d}{dx}\left(\sin\frac{1}{x}\right) + \left(\sin\frac{1}{x}\right)\frac{d}{dx}(x$

$= x^2\left(\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) + \left(\sin\frac{1}{x}\right)(2x)$

$= -\cos\frac{1}{x} + 2x \sin\frac{1}{x}$

Since $\lim_{x \to 0} f'(x) = \lim_{x \to 0}\left(-\cos\frac{1}{x} + 2x \sin\frac{1}{x}\right)$ does not exist (because $\lim_{x \to 0} \cos\frac{1}{x}$ does not exist),

$f'(x)$ cannot be continuous at $x=0$ in spite of the fact that $f'(0)$ exists.