Chapter 1

Numerical series

Contents

1.1 Series with real terms

Definition 1.1.1. *Let* $(u_n)_{n\in\mathbb{N}}$ *be a sequence with real numbers. The infinite sum:*

$$u_0 + u_1 + \cdots + u_n + \cdots = \sum_{n\geq 0} u_n$$

is called a numerical series.

where $u_0, u_1, \cdots, u_n, \cdots$ *the terms of the series and* u_n *is called general term of the series.*

Considering now the following partial sums

$$\begin{cases} S_0 = u_0 \\ S_1 = u_0 + u_1 \\ \vdots \\ S_n = u_0 + u_1 + \cdots + u_n = \sum_{k=0}^{n} u_k \end{cases}$$

S_n is called the n^{th} partial sum of the series $\sum\limits_{n\geq 0} u_n$ and $(S_n)_n$ is called the sequence of partial sums of this series.

Example 1.1.1.

- $\sum\limits_{n\geq 0} aq_n, \quad a \neq 0 \longrightarrow$ *Geometric series of reason q.*
- $\sum\limits_{n\geq 1} \frac{1}{n} \longrightarrow$ *Harmonic Series.*

1.1.1 The convergence of a numerical series

Definition 1.1.2. *The series with real terms* $\sum\limits_{n\geq 0} u_n$ *is said convergent if the sequence of partial sums* $(S_n)_n$ *converges to a limit S called the sum of the series.*

$$S = \lim_{n\to+\infty} S_n = \sum_{n\geq 0} u_n.$$

Example 1.1.2.

Let $\sum_{n\geq 1} u_n$ *be the series with the general term* $u_n = \dfrac{1}{n(n+1)}, \quad n \geq 1$

• *The term* u_n *can be rewritten in the form*

$$u_n = \frac{1}{n} - \frac{1}{n+1}, \quad n \geq 1.$$

• *The partial sum of the series*

$$S_n = \sum_{k=1}^{n} u_k = u_1+u_2+\cdots+u_n = \Big(1-\frac{1}{2}\Big)+\Big(\frac{1}{2}-\frac{1}{3}\Big)+\cdots+\Big(\frac{1}{n-1}-\frac{1}{n}\Big)+\Big(\frac{1}{n}-\frac{1}{n+1}\Big) = 1-\frac{1}{n+1}$$

• *We have* $\lim\limits_{n\to+\infty} S_n = \lim\limits_{n\to+\infty} 1-\frac{1}{n+1} = 1$ *therefore the series* $\sum_{n\geq 1} u_n$ *is convergent with sum* $S = 1$.

Proposition 1.1.1. *If the numerical series* $\sum_{n\geq 0} u_n$ *is convergent then its general term* u_n *tends towards zero i.e.*

$$\sum_{n\geq 0} u_n \ \textit{ is convergent } \implies \lim_{n\to+\infty} u_n = 0.$$

Corollary 1.1.1. *A sufficient condition for a series to be divergent is that its general term does not tend towards zero.*

1.1.2 Operations on numerical series

Let $\sum_{n\geq 0} u_n$ and $\sum_{n\geq 0} v_n$ be two numerical series, then we have the following properties

• If $\sum_{n\geq 0} u_n$ is convergent of sum S_1 and $\sum_{n\geq 0} v_n$ is convergent of sum S_2, then $\sum_{n\geq 0} u_n+v_n$ is convergent of sum $S_1 + S_2$.

• If $\sum_{n\geq 0} u_n$ is convergent of sum S_1 and $a \in \mathbb{R}$ then $\sum_{n\geq 0} au_n$ is convergent of sum aS_1.

• If $\sum_{n\geq 0} u_n$ is convergent and $\sum_{n\geq 0} v_n$ is divergent then $\sum_{n\geq 0} u_n + v_n$ is divergent.

• If the two series $\sum_{n\geq 0} u_n$ and $\sum_{n\geq 0} v_n$ are divergent, then we cannot conclude anything about the nature of the series $\sum_{n\geq 0} u_n + v_n$.

1.2 Series with positive term

Definition 1.2.1. *We call a series with positive term any series whose general term $u_n \geq 0$ for all $n \geq 0$.*

Proposition 1.2.1. *Let $\sum_{n\geq 0} u_n$ be a series with positive real terms, then this series converges to S if and only if $S_n \leq S$ for all $n \geq 0$.*

1.2.1 Comparison theorems

Theorem 1.2.1. *Let $\sum_{n\geq 0} u_n$ and $\sum_{n\geq 0} v_n$ be two series with positive terms satisfying*

$$\exists n_0 \in \mathbb{N}, \text{such that for all } \; n \geq n_0, \;\; u_n \leq v_n$$

• *If the series $\sum_{n\geq 0} v_n$ is convergent then the series $\sum_{n\geq 0} u_n$ is convergent.*

• *If the series $\sum_{n\geq 0} u_n$ is divergent then the series $\sum_{n\geq 0} v_n$ is divergent.*

Corollary 1.2.1. *Let $\sum_{n\geq 0} u_n$ and $\sum_{n\geq 0} v_n$ be two series with positive terms, if there exists $a, b > 0$ satisfying*

$$au_n \leq v_n \leq bu_n$$

then $\sum_{n\geq 0} u_n$ and $\sum_{n\geq 0} v_n$ are of the same nature.

Theorem 1.2.2. *Let* $\sum_{n\geq 0} u_n$ *and* $\sum_{n\geq 0} v_n$ *be two series with positive terms, if there exists a real* l *(or* $l = +\infty$*) such that* $\lim_{n\to+\infty} \frac{u_n}{v_n} = l$*, then*

- *If* $l = 0$ *and the series* $\sum_{n\geq 0} v_n$ *is convergent then the series* $\sum_{n\geq 0} u_n$ *is convergent.*
- *If* $l = +\infty$ *and the series* $\sum_{n\geq 0} v_n$ *is divergent then the series* $\sum_{n\geq 0} u_n$ *is divergent.*
- *If* $l \neq 0$ *and* $l \neq +\infty$ *then the two series* $\sum_{n\geq 0} u_n$ *and* $\sum_{n\geq 0} v_n$ *are of the same nature.*

1.2.2 Usual rules of convergence

Definition 1.2.2. *We call a Riemann series any numerical series whose general term* $u_n = \frac{1}{n^\alpha}$.

Proposition 1.2.2. *The Riemann series is convergent for all* $\alpha > 1$.

Proposition 1.2.3. *(Riemann's rule)*

The Riemann rule amounts to comparing a series with positive terms to a Riemann series.

Let $\sum_{n\geq 1} u_n$ *be a series with positive real terms and let* $\alpha \in \mathbb{R}$*, suppose that there exists a positive real* l *(or* $l = +\infty$*) such that* $\lim_{n\to+\infty} n^\alpha u_n = l$

- *If* $l = 0$ *and* $\alpha > 1$ *then the series* $\sum_{n\geq 1} u_n$ *is convergent.*
- *If* $l = +\infty$ *and* $\alpha \leq 1$ *then the series* $\sum_{n\geq 1} u_n$ *is divergent.*
- *If* $l \neq 0$ *and* $l \neq +\infty$ *then the two series* $\sum_{n\geq 1} u_n$ *and* $\sum_{n\geq 1} \frac{1}{n^\alpha}$ *are of the same nature.*

Proposition 1.2.4. *(D'Alembert's rule)*

Let $\sum_{n\geq 1} u_n$ *be a series with strictly positive real terms, suppose that there exists a positive real* l *(or* $l = +\infty$*) such that* $\lim_{n\to+\infty} \frac{u_{n+1}}{u_n} = l$

- *If* $l < 1$ *then the series* $\sum_{n\geq 1} u_n$ *is convergent.*
- *If* $l > 1$ *then the series* $\sum_{n\geq 1} u_n$ *is divergent.*

Proposition 1.2.5. *(Cauchy rule)*

Let $\sum_{n\geq 1} u_n$ *be a series with strictly positive real terms, suppose that there exists a positive real* l *(or* $l = +\infty$*) such that* $\lim_{n\to+\infty} (u_n)^{\frac{1}{n}} = l$

- *If* $l < 1$ *then the series* $\sum_{n\geq 1} u_n$ *is convergent.*
- *If* $l > 1$ *then the series* $\sum_{n\geq 1} u_n$ *is divergent.*