Chapter 1

Real functions of a real variable

Contents

1.1 Recalls and definitions

1.1.1 Intervals of $\mathbb{R}$

Let $a, b \in \mathbb{R}$ such that $a < b$, we call

❶ The open interval of ends a and b the set

$$]a,b[=\left\{x \in \mathbb{R},\ \ a < x < b\right\}$$

❷ The closed interval of ends a and b the set

$$[a,b]=\left\{x \in \mathbb{R},\ \ a \leq x \leq b\right\}$$

❸ The semi-open interval, the sets

$$[a,b[=\left\{x \in \mathbb{R},\ \ a \leq x < b\right\}$$

$$]a,b]=\left\{x \in \mathbb{R},\ \ a < x \leq b\right\}$$

❹ The open interval of center a the set

$$]a-\varepsilon, a+\varepsilon[\ \ with\ \ \varepsilon > 0$$

$$[a,+\infty[=\left\{x \in \mathbb{R},\ \ x \geq a\right\}$$

$$]a,+\infty[=\left\{x \in \mathbb{R},\ \ x > a\right\}$$

$$]-\infty,a]=\left\{x \in \mathbb{R},\ \ x \leq a\right\}$$

$$]-\infty,a[=\left\{x \in \mathbb{R},\ \ x < a\right\}$$

1.1.2 Real functions of a real variable

Definition 1.1.1.

A real (or numerical) function of a real variable is a map f of a part D of $\mathbb{R}$ to values in $\mathbb{R}$, we note

$$\begin{array}{rcl} f: D & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x) \end{array}$$

Example 1.1.1.

❶ *The square function*

$$\begin{array}{rcl} f: \mathbb{R} & \longrightarrow & \mathbb{R}^+ \\ x & \longmapsto & x^2 \end{array}$$

❷ *The exponential function*

$$\begin{array}{rcl} f: \mathbb{R} & \longrightarrow & \mathbb{R}^+ \\ x & \longmapsto & e^x \end{array}$$

❸ *The cosine function*

$$\begin{array}{rcl} f: \mathbb{R} & \longrightarrow & [-1,1] \\ x & \longmapsto & \cos(x) \end{array}$$

1.1.3 Domain of definition

Definition 1.1.2.

The domain of definition of a function f denoted D_f is the set of elements x for which the function f is defined

$$D_f = \{x \in \mathbb{R},\ \ f(x) \text{ is defined}\}$$

1.1.4 Practical determination of the domain of definition

Let f and g be two functions

❶ 1^{st} case: function of type $\dfrac{f}{g}$ is defined for all $g \neq 0$

❷ 2^{nd} case: function of type $\sqrt{f}$ is defined for all $f \geq 0$

❸ 3^{rd} case: function of type $\dfrac{f}{\sqrt{g}}$ is defined for all $g > 0$

Example 1.1.2.

❶ $f(x) = \dfrac{x^2 + 2x + 5}{x^2 - 1}$

$$\begin{aligned} D_f &= \left\{x \in \mathbb{R},\ \ x^2 - 1 \neq 0\right\} \\ &= \left\{x \in \mathbb{R},\ \ (x-1)(x+1) \neq 0\right\} \\ &= \mathbb{R} - \left\{-1, 1\right\} \end{aligned}$$

❷ $g(x) = \dfrac{3x + 5}{\sqrt{2x - 1}}$

$$\begin{aligned} D_f &= \left\{x \in \mathbb{R},\ \ 2x - 1 > 0\right\} \\ &= \left\{x \in \mathbb{R},\ \ x > 1/2\right\} \\ &=]1/2, +\infty[\end{aligned}$$

General information on functions We suppose that $\forall x \in D, -x \in D$. The function $f : D \longrightarrow \mathbb{R}$ is said

❶ **Even function** if $\forall x \in D, \ f(-x) = f(x)$. The curve C_f in an orthonormal coordinate system is symmetrical with respect to the axis (oy) $(f(x) = x^2)$

❷ **Odd function**if $\forall x \in D, \ f(-x) = -f(x)$. The curve C_f in an orthonormal coordinate system is symmetrical with respect to the origin $(f(x) = \sin(x))$

❸ **Periodic function** if there exists $T > 0$ such that $\forall x \in D, x + T \in D$ and $f(x + T) = f(x)$. The smallest value of T is called the period of f as the functions cos and sin are periodic functions of period 2π

❹ **Major function**$\Longleftrightarrow \exists M \in \mathbb{R}, \forall x \in D, f(x) \leq M$

❺ **Minor function** $\Longleftrightarrow \exists m \in \mathbb{R}, \forall x \in D, f(x) \geq m$

❻ **Bounded function** $\Longleftrightarrow \exists m, M \in \mathbb{R}, \forall x \in D, m \leq f(x) \leq M$

❼ **Increasing function** (resp strictly increasing) if $\forall x, y \in D \ x < y \Longrightarrow f(x) \leq f(y)$ (resp $f(x) < f(y)$)

❽ **Decreasing function**(resp strictly decreasing) if $\forall x, y \in D \ x < y \Longrightarrow f(x) \geq f(y)$ (resp $f(x) > f(y)$)

❾ **Monotonous function** if it is increasing or decreasing on D.

1.2 Limit of a function

1.2.1 Neighborhood of a point x_0

Definition 1.2.1.

We call neighborhood of a point x_0 any open interval of the form $]x_0 - \varepsilon, x_0 + \varepsilon[$, $\varepsilon > 0$

$$\begin{aligned} V_\varepsilon(x) &= \left\{x \in \mathbb{R}, \ \ x_0 - \varepsilon < x < x_0 + \varepsilon\right\} \\ &= \left\{x \in \mathbb{R}, \ \ |x - x_0| < \varepsilon\right\} \end{aligned}$$

1.2.2 Function defined in the vicinity of a point x_0

Definition 1.2.2.

We say that a function $f : D \longmapsto \mathbb{R}$ *is defined in the neighborhood of a point* $x_0 \in \mathbb{R}$ *if* $\exists \varepsilon > 0,]x_0 - \varepsilon, x_0 + \varepsilon[\subseteq D$

1.2.3 Limit of a function

Definition 1.2.3.

❶ *We say that a function f defined in the vicinity of a point x_0 admits a limit $l \in \mathbb{R}$ as x tends towards x_0 if*

$$\forall \varepsilon > 0 \;\; \exists \sigma > 0 \; \forall x \neq x_0 \;\; |x - x_0| < \sigma \Longrightarrow |f(x) - l| < \varepsilon$$

and we write $\lim_{x \to x_0} f(x) = l$.

❷ ***limit on the right*** *We say that a function f has a right limit at the point x_0 if*

$$\forall \varepsilon > 0 \;\; \exists \sigma > 0 \; \forall x \;\; x_0 < x < x_0 + \sigma \Longrightarrow |f(x) - l| < \varepsilon$$

and we write $\lim_{x \to x_0^+} f(x) = \lim_{x \overset{>}{\to} x_0} f(x) = l$.

❸ ***Limit on the left*** *We say that a function f has a left limit at the point x_0 if*

$$\forall \varepsilon > 0 \;\; \exists \sigma > 0 \;\; \forall x \;\; x_0 - \sigma < x < x_0 \Longrightarrow |f(x) - l| < \varepsilon$$

and we write $\lim_{x \to x_0^-} f(x) = \lim_{x \overset{<}{\to} x_0} f(x) = l$.

Remark 1.2.1.

❶ *If f admits a limit at the point x_0 then this limit is unique and we have*

$$\lim_{x \to x_0} f(x) = \lim_{x \overset{<}{\to} x_0} f(x) = \lim_{x \overset{>}{\to} x_0} f(x) = l$$

❷ *if* $\lim_{x \overset{<}{\to} x_0} f(x) \neq \lim_{x \overset{>}{\to} x_0} f(x)$ *then* f *does not admit a limit at the point* x_0

1.2.4 Operations on limits

Let f and $g : D \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be two functions satisfying $\lim_{x \to x_0} f(x) = l_1$ and $\lim_{x \to x_0} g(x) = l_2$

❶ $\lim_{x \to x_0} \Big(f(x) + g(x)\Big) = l_1 + l_2$

❷ $\lim_{x \to x_0} \Big(f(x).g(x)\Big) = l_1.l_2$

❸ $\lim_{x \to x_0} af(x) = al_1 \quad \forall a \in \mathbb{R}$

❹ $\lim_{x \to x_0} |f(x)| = |l_1|$

❺ $\lim_{x \to x_0} \dfrac{f(x)}{g(x)} = \dfrac{l_1}{l_2}, \;\; l_2 \neq 0$

❻ Si $f(x) < g(x) \Longrightarrow l_1 < l_2$

1.2.5 Indeterminate forms

There are four algebraic indeterminate forms (IF)

$$\Big(\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, +\infty - \infty\Big)$$

and three exponential forms

$$\Big(0^0, \infty^0, 1^\infty\Big)$$

Example 1.2.1.

Calculate the following limits

❶ $\lim_{x \to 1} \dfrac{x^2 - 1}{x - 1} = \dfrac{0}{0}(FI)$
$\lim_{x \to 1} \dfrac{x^2 - 1}{x - 1} = \lim_{x \to 1} \dfrac{(x-1)(x+1)}{x-1} = \lim_{x \to 1}(x+1) = 2$

❷ $\lim\limits_{x\to 0^+} \frac{1}{x} - \frac{1}{x^2} = +\infty - \infty (FI)$

$\lim\limits_{x\to 0^+} \frac{1}{x} - \frac{1}{x^2} = \lim\limits_{x\to 0^+} \frac{1}{x}\Big(1 - \frac{1}{x}\Big) = -\infty.$

1.3 Continuous functions

Definition 1.3.1.

❶ *Continuous functions at a point: we say that a function f is continuous at the point $x_0 \in \mathbb{R}$ if it is defined in a neighborhood of x_0 and $\lim\limits_{x\to x_0} f(x) = f(x_0)$.*

❷ *Continuous functions on a set: we say that a function f is continuous on a set $D \subset \mathbb{R}$ if it is continuous at every point of D.*

1.3.1 Operations on continuous functions

Let f and g be two continuous functions in x_0 and let $\lambda \in \mathbb{R}$, then $\big(f \pm g\big), \big(f \times g\big), \big(\lambda f\big), \big(|f|\big)$ and $\Big(\frac{f}{g}\Big)$ $(g(x_0) \neq 0)$ are continuous functions in x_0.

1.3.2 Extension by continuity

Let f be a function defined and continuous on $I - \{x_0\}$, if $\lim\limits_{x\to x_0} f(x) = l$ (l exists and finite) then $f(x)$ can be extended by continuity at the point x_0 to the function g defined by

$$g(x) = \begin{cases} f(x) & \text{if } x \neq x_0 \\ \lim\limits_{x\to x_0} f(x) & \text{if } x = x_0 \end{cases}$$

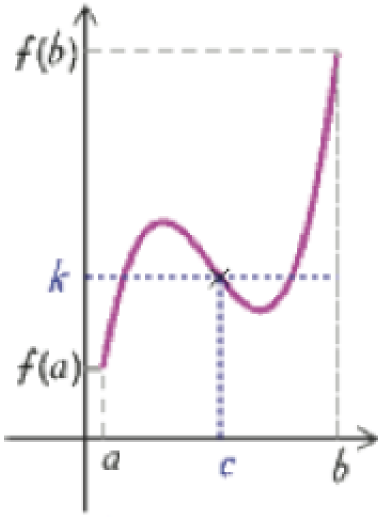
Example 1.3.1.

The function $f(x) = e^{-\frac{1}{x^2}}$ is continuous on $\mathbb{R} - \{0\}$ and we have $\lim\limits_{x\to 0} f(x) = 0$ therefore

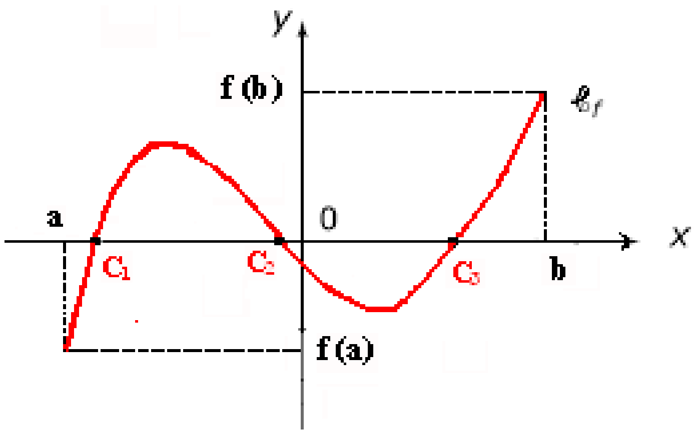
$f(x)$ can be extended by continuity at the point 0 to the function

$$g(x) = \begin{cases} e^{-\frac{1}{x^2}} & if \;\; x \neq 0 \\ 0 & if \;\; x = 0 \end{cases}$$

Theorem 1.3.1. *(Intermediate Value Theorem) Let f be a defined and continuous function on any interval $I \subset \mathbb{R}$ and let $a, b \in I$ $(a < b)$. Then for every real k strictly included between $f(a)$ and $f(b)$ there exists $c \in]a, b[$ such that $f(c) = k$*



Theorem 1.3.2. *Let f be a defined and continuous function on $[a, b]$ and $f(a)f(b) < 0$, then there exists at least one point $c \in]a, b[$ such that $f(c) = 0$.*



1.4 Differentiable functions

Definition 1.4.1.

Let $f : I \longrightarrow \mathbb{R}$ *and let* $x_0 \in I$

❶ *We say that the function* f *is differentiable at the point* x_0 *if the limit*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists and finite. This limit is called derivative of f *at the point* x_0 *and we denote it* $f'(x_0)$.

if $h = x - x_0$, *we obtain*

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

❷ *We say that the function* f *is differentiable on the right at* x_0 *if the limit*

$$\lim_{x \overset{>}{\to} x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists and finite and we denote it by $f'_d(x_0)$ *or* $f'_+(x_0)$.

❸ *We say that the function* f *is differentiable on the left at* x_0 *if the limit*

$$\lim_{x \overset{<}{\to} x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists and finite and we denote it by $f'_g(x_0)$ *or* $f'_-(x_0)$.

Remark 1.4.1.

$$f \text{ is differentiable at the point } x_0 \iff \begin{cases} f'_d(x_0) \text{ exists and finite} \\ f'_g(x_0) \text{ exists and finite} \\ f'_d(x_0) = f'_g(x_0) \end{cases}$$

Example 1.4.1.

$f(x) = |x| + x^2$, $x_0 = 0$

$$f \text{ is differentiable at the point } x_0 = 0 \iff \begin{cases} f'_d(0) \text{ exists and finite} \\ f'_g(0) \text{ exists and finite} \\ f'_d(0) = f'_g(0) \end{cases}$$

We have

$$\begin{aligned} f'_d(0) &= \lim_{x \overset{>}{\to} 0} \frac{f(x) - f(0)}{x} \\ &= \lim_{x \overset{>}{\to} 0} \frac{|x| + x^2}{x} \\ &= \lim_{x \overset{>}{\to} 0} \frac{x + x^2}{x} \\ &= \lim_{x \overset{>}{\to} 0} 1 + x \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} f'_g(0) &= \lim_{x \overset{<}{\to} 0} \frac{f(x) - f(0)}{x} \\ &= \lim_{x \overset{<}{\to} 0} \frac{|x| + x^2}{x} \\ &= \lim_{x \overset{<}{\to} 0} \frac{-x + x^2}{x} \\ &= \lim_{x \overset{<}{\to} 0} -1 + x \\ &= -1 \end{aligned}$$

we have $f'_d(0)$ and $f'_g(0)$ exist but $f'_d(0) \neq f'_g(0)$, then the function f is not differentiable at the point $x_0 = 0$

1.4.1 Operations on differentiable functions

Let $f, g : I \longrightarrow \mathbb{R}$ be two differentiable functions in x_0 and let $\lambda \in \mathbb{R}$, then

❶ $\Big(\lambda f\Big)'(x_0) = \lambda f'(x_0)$

❷ $\Big(f \pm g\Big)'(x_0) = f'(x_0) \pm g'(x_0)$

❸ $\Big(f.g\Big)'(x_0) = f'(x_0).g(x_0) + f(x_0).g'(x_0)$

❹ If $g(x_0) \neq 0$ then $\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0).g(x_0) - f(x_0).g'(x_0)}{g^2(x_0)}$

❺ Derivative of a composite functions: let $f : I \longrightarrow \mathbb{R}$ and $g : J \longrightarrow \mathbb{R}$ or $f(I) \subset J$ if f is differentiable at x_0 and g is differentiable at the point $f'(x_0)$ then the composition $g \circ f$ is differentiable at x_0 and we have

$$\Big(g \circ f\Big)'(x_0) = f'(x_0) g'\Big(f(x_0)\Big)$$

1.4.2 Derivatives of usual functions

Fonction	Dérivée	Intervalle de dérivabilité
$f(x) = k$ avec k constante	$f'(x) = 0$	$\mathbb{R}$
$f(x) = x$	$f'(x) = 1$	$\mathbb{R}$
$f(x) = ax + b$	$f'(x) = a$	$\mathbb{R}$
$f(x) = x^2$	$f'(x) = 2x$	$\mathbb{R}$
$f(x) = x^n$ avec $n \in \mathbb{N}^*$	$f'(x) = nx^{n-1}$	$\mathbb{R}$
$f(x) = \frac{1}{x}$	$f'(x) = -\frac{1}{x^2}$	$\mathbb{R}^*$
$f(x) = \frac{1}{x^n} = x^{-n}$ avec $n \in \mathbb{N}$	$f'(x) = -\frac{n}{x^{n+1}} = -nx^{-n-1}$	$\mathbb{R}^*$
$f(x) = \sqrt{x}$	$f'(x) = \frac{1}{2\sqrt{x}}$	$]0, +\infty[$
$f(x) = x^\alpha$ avec $\alpha \in \mathbb{R}$	$f'(x) = \alpha x^{\alpha-1}$	$\mathbb{R}$ si $\alpha \geqslant 0$ et $\mathbb{R}^*$ si $\alpha < 0$
$f(x) = \cos(x)$	$f'(x) = -\sin(x)$	$\mathbb{R}$
$f(x) = \sin(x)$	$f'(x) = \cos(x)$	$\mathbb{R}$
$f(x) = \tan(x)$	$f'(x) = \frac{1}{\cos^2(x)} = 1 + \tan^2(x)$	$\left]\frac{\pi}{2} + k\pi; \frac{\pi}{2} + (k+1)\pi\right[$ avec $k \in \mathbb{Z}$
$f(x) = e^x$	$f'(x) = e^x$	$\mathbb{R}$
$fx) = \ln(x)$	$f'(x) = \frac{1}{x}$	$]0, +\infty[$

1.4.3 Fundamental theorems of differentiable functions

Theorem 1.4.1. *(Rolle's theorem)*

Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function which verifies

❶ *f is continuous on $[a, b]$*

❷ *f is differentiable on $]a, b[$*

❸ *$f(a) = f(b)$*

then there exists $c \in]a, b[$ such that $f'(c) = 0$

Theorem 1.4.2. *(Mean value theorem)*

Let $f : [a, b] \longrightarrow \mathbb{R}$ *be a function which verifies*

❶ f *is continuous on* $[a, b]$

❷ f *is differentiable on* $]a, b[$

then there exists at least one point $c \in]a, b[$ *such that* $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

1.4.4 Application of the derivative to the study of monotonicity

Theorem 1.4.3. *Let* f *be a differentiable function on* $I \subset \mathbb{R}$*, then*

❶ f *is constant on* $I \Longleftrightarrow f'(x) = 0, \forall x \in I$

❷ f *is increasing on* $I \Longleftrightarrow f'(x) \geq 0, \forall x \in I$

❸ f *is strictly increasing on* $I \Longrightarrow f'(x) > 0, \forall x \in I$

❹ f *is decreasing on* $I \Longleftrightarrow f'(x) \leq 0, \forall x \in I$

❺ f *is strictly decreasing on* $I \Longrightarrow f'(x) < 0, \forall x \in I$

1.4.5 Application of the derivative to the calculate the limits

Theorem 1.4.4. *(Hopital's rule)*

Let f, g be two defined and differentiable functions in a neighborhood v of x_0 such that

❶ $g(x) \neq 0$, $g'(x) \neq 0$ *on* v.

❷ $f(x_0) = g(x_0) = 0$.

If $\lim_{x \to x_0} \frac{f'(x)}{g'(x)}$ *exists finite or infinite, then*

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

This result applies to indeterminate forms $\left(\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right)$

Example 1.4.2.

❶ $\lim_{x \to +\infty} \frac{x^2}{e^x} = \frac{\infty}{\infty}$*(IF), according to the Hopital's rule*

$$\lim_{x \to +\infty} \frac{x^2}{e^x} = \lim_{x \to +\infty} \frac{2x}{e^x} = \lim_{x \to +\infty} \frac{2}{e^x} = 0.$$

❷ $\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \frac{0}{0} =$ *(IF), according to the Hopital's rule*

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{2x}{1} = 2.$$

1.4.6 Maximum, minimum (extremum)

Let $f : I \longrightarrow \mathbb{R}$ and let $x_0 \in I$

❶ We say that f admits a local maximum at x_0 if

$$\exists J \subset I \text{ of center } x_0, \forall x \in J \quad f(x) \leq f(x_0)$$

❷ We say that f admits a local minimum at x_0 if

$$\exists J \subset I \text{ of center } x_0, \forall x \in J \quad f(x) \geq f(x_0)$$

❸ We say that f admits a local extremum at x_0 if f admits at x_0 a local maximum or minimum

❹ If f is differentiable at x_0 and $f'(x_0) = 0$, then x_0 is called the critical point of f.

❺ If $f'(x_0) = 0$ and $f''(x_0) \neq 0$, then f admits an extremum at x_0 (maximum if $f''(x_0) < 0$ and minimum if $f''(x_0) > 0$).

Example 1.4.3.

❶ $f(x) = x^2 - 1$

- The critical points

We have

$$f'(x) = 2x$$

So

$$f'(x) = 0 \Longleftrightarrow 2x = 0 \Longleftrightarrow x = 0 \textit{ is a critical point of } f$$

- The nature of critical points

$f''(x) = 2 \neq 0$ *therefore f admits a minimum at the point* $x_0 = 0$ *because* $f''(0) > 0.$

❷ $f(x) = x^2 e^{-x}$

- The critical points

we have

$$f'(x) = xe^{-x}(2 - x)$$

So

$$f'(x) = 0 \Longleftrightarrow xe^{-x}(2-x) = 0 \Longleftrightarrow x = 0 \text{ or } x = 2$$

- The nature of critical points

We have

$$f''(x) = e^{-x}(1-x)(2-x) - xe^{-x}$$

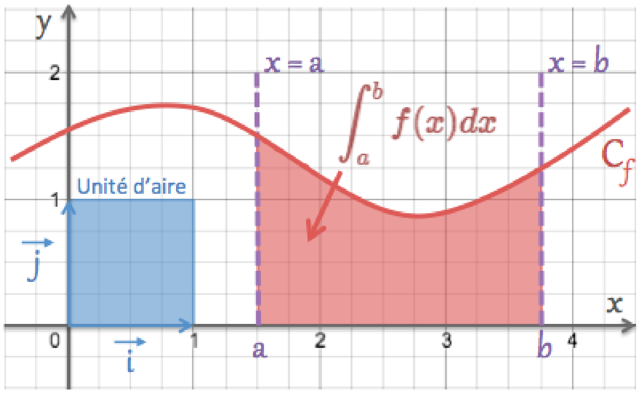
for $x = 0$, $f''(0) = 2 > 0$ *so* $x = 0$ *is a minimum.*

for $x = 2$, $f''(2) = -2e^{-2} < 0$ *so* $x = 2$ *is a maximum.*

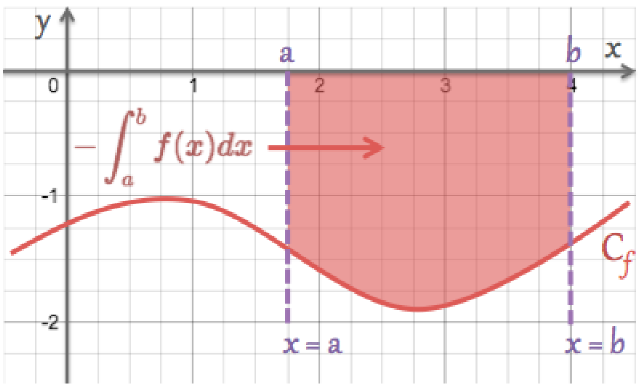
1.5 Integrals and primitives

Integration is linked to the problem of calculating a surface A delimited by the curve of a function f defined on a segment $[a, b]$ and the lines $x = a$, $x = b$ and $y = 0$

❶ If f is positive then $A = \int_a^b f(x)dx$



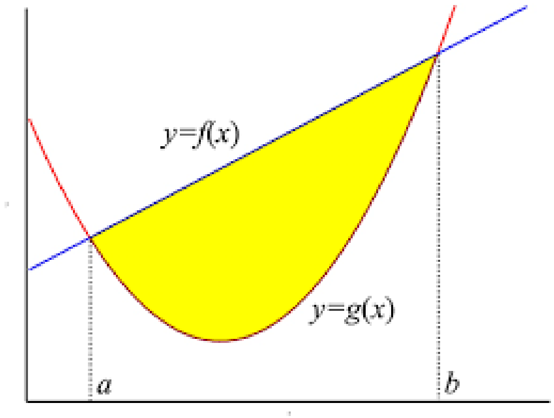
❷ If f is negative then $-f$ is positive and the curves C_f and C_{-f} are symmetrical about the x-axis $A = \int_a^b -f(x)dx$



❸ Surface between two curves

$$A = \int_a^b (f - g)(x)dx \quad si\ f > g$$

$$A = \int_a^b (g - f)(x)dx \quad si\ f < g$$



Definition 1.5.1.

We call the primitive of a function $f : [a, b] \longrightarrow \mathbb{R}$, *any differentiable function* $F : [a, b] \longrightarrow \mathbb{R}$ *such that*

$$F'(x) = f(x), \quad \forall x \in [a, b]$$

Theorem 1.5.1. *If a function* f *admits an primitive* F *on* $[a, b]$ *then the set*

$$\{F + c, \quad c \in \mathbb{R}\}$$

is the set of all primitives of f *on* $[a, b]$

Example 1.5.1.

❶ $f(x) = 2x + 1 \Longrightarrow F(x) = x^2 + x + c$ *where* c *is a constant*

❷ $f(x) = x^2 + \cos(x) \Longrightarrow F(x) = \frac{x^3}{3} + \sin(x) + c$ *where* c *is a constant.*

Theorem 1.5.2. *Every continuous function on* $[a, b]$ *admits a primitive on* $[a, b]$

Remark 1.5.1.

The set of all primitives of the function $f : [a, b] \longrightarrow \mathbb{R}$ is called the indefinite integral of f and denoted by $\int f(x)dx$

$$\int f(x)dx = F + c, \quad c \in \mathbb{R}$$

Example 1.5.2.

$$\int \sin(x)dx = -\cos(x) + c, \quad c \in \mathbb{R}$$

Definition 1.5.2.

Let f be a continuous function on $[a, b]$ and F one of its primitives. We call the integral of f between a and b the quantity

$$\int_a^b f(x)dx = \Big[F(x)\Big]_a^b = F(b) - F(a)$$

Example 1.5.3.

Using the primitives, we obtain

$$\int_0^1 x^2dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

1.5.1 Properties of the integral

Chasles relation

Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function and let $c \in]a, b[$, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$\int_a^a f(x)dx = 0$ and $\int_a^b f(x)dx = -\int_b^a f(x)dx$

Linearity of the integral

Let f, g be two continuous functions on $[a, b]$ and $\alpha, \beta \in \mathbb{R}$, then

$$\int_a^b \alpha f(x) + \beta g(x)dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx$$

Positivity of the integral

Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function, then

$$f \text{ is positive} \Longrightarrow \int_a^b f(x)dx \geq 0$$

Increasing of the Integral

Let f, g be two continuous functions on $[a, b]$, then

$$f(x) \leq g(x) \Longrightarrow \int_a^b f(x)dx \leq \int_a^b g(x)dx$$

Remark 1.5.2.

$$\int_a^b f(x)g(x)dx \neq \int_a^b f(x)dx \int_a^b g(x)dx$$

and

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx$$

1.5.2 Integration methods

Direct integration

Table of usual primitives

Primitives des fonctions usuelles

Dans chaque ligne, F est **une** primitive de f sur l'intervalle I. Ces primitives sont **uniques à une constante près** notée C.

$f(x)$	I	$F(x)$
λ (constante)	$\mathbb{R}$	$\lambda x + C$
x	$\mathbb{R}$	$\frac{x^2}{2} + C$
x^n $(n \in \mathbb{N}^*)$	$\mathbb{R}$	$\frac{x^{n+1}}{n+1} + C$
$\frac{1}{x}$	$]-\infty, 0[$ ou $]0, +\infty[$	$\ln\|x\| + C$
$\frac{1}{x^n}$ où $n \in \mathbb{N}$, $n \geqslant 2$	$]-\infty, 0[$ ou $]0, +\infty[$	$-\frac{1}{(n-1)x^{n-1}} + C$
$\frac{1}{\sqrt{x}}$	$]0, +\infty[$	$2\sqrt{x} + C$
$\ln x$	$\mathbb{R}_+^*$	$x \ln x - x + C$
e^x	$\mathbb{R}$	$e^x + C$
$\sin x$	$\mathbb{R}$	$-\cos x + C$
$\cos x$	$\mathbb{R}$	$\sin x + C$
$1 + \tan^2 x = \frac{1}{\cos^2 x}$	$\left]-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right[$, $k \in \mathbb{Z}$	$\tan x + C$

Integration by parts

Let f and g be two differentiable functions on $[a, b]$, then we have

$$\int_a^b f'(x)g(x)dx = \Big[f(x)g(x)\Big]_a^b - \int_a^b f(x)g'(x)dx$$

This result is a direct consequence of the derivative of the product of two functions.

Example 1.5.4.

Calculate $\int_0^1 xe^x dx$

We pose

$$\begin{cases} f(x) = x & f'(x) = 1 \\ g'(x) = e^x & g(x) = e^x \end{cases}$$

then

$$\int_0^1 xe^x dx = \Big[xe^x\Big]_0^1 - \int_0^1 e^x dx = e - \Big[e^x\Big]_0^1 = 1$$

Change of variable

Let f be a continuous function on $[a, b]$ and g' exists and continuous. This method consists of putting $x = g(t)$ in the integral and replacing dx by $g'(t)dt$

$$\int_{g(a)}^{g(b)} f(x)dx = \int_a^b f(g(t))\, g'(t)dt$$

Example 1.5.5.
Calculate $\int_1^4 \frac{1}{x+\sqrt{x}}dx$
by the change of variable $t = \sqrt{x} \Longrightarrow x = t^2 \Longrightarrow dx = 2tdt$

$$\begin{cases} Si\ \ x = 1 & t = 1 \\ Si\ \ x = 4 & t = 2 \end{cases}$$

thus

$$\int_1^4 \frac{1}{x+\sqrt{x}}dx = \int_1^2 \frac{2t}{t^2+t}dt = 2\Big[\ln|t+1|\Big]_1^2 = 2\ln\frac{3}{2}$$

Proposition 1.5.1. *Let f be a continuous function on $[a, b]$, then*

❶ *If f is a even function, then*

$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$$

❷ *If f is an odd function, then*

$$\int_{-a}^{a} f(x)dx = 0$$

1.5.3 Opérations et primitives

Opérations et primitives

On suppose que u est une fonction dérivable sur un intervalle I

- Une primitive de $u'u^n$ sur I est $\frac{u^{n+1}}{n+1}$ $(n \in \mathbb{N}^*)$
- Une primitive de $\frac{u'}{u^2}$ sur I est $-\frac{1}{u}$.
- Une primitive de $\frac{u'}{u^n}$ sur I est $-\frac{1}{(n-1)\,u^{n-1}}$.$(n \in \mathbb{N}, n \geqslant 2$.
- Une primitive de $\frac{u'}{\sqrt{u}}$ sur I est $2\sqrt{u}$ (En supposant $u > 0$ sur I.)
- Une primitive de $\frac{u'}{u}$ sur I est $\ln |u|$.
- Une primitive de $u'e^u$ sur I est e^u.

En particulier, si $u > 0$ sur I et si $\alpha \in \mathbb{R} \setminus \{-1\}$, une primitive de $u'u^\alpha$ sur I est :

$$\int u'u^\alpha = \begin{cases} \frac{1}{\alpha+1} u^{\alpha+1} + C & \text{si } \alpha \in \mathbb{R} \setminus \{-1\} \\ \ln u \ + C & \text{si } \alpha = -1 \end{cases}$$

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