3. Real functions of one real variable

3.1 Notions of function

3.1.1 General definitions

Definition 3.1.1 We call digital function on a set D_f any process which, at all element $x\ of\ D_f$, allows to associate at most one element of the set R, then called image of x and denoted $f(x)$. The elements of D_f which have an image by f form the definition set of f, noted D_f.

Example 3.1.1 The function $f: x \to \sqrt{x-1}$ is defined for all x ∈ R such that x − 1 ≥ 0. So $Df = [1, +\infty[$

Definition 3.1.2. We call a graph, or representative curve, of a function f defined on an interval $D_f \subset R$, the set $\Gamma f = \{(x, f(x)) : x \in D_f\}$ formed from the points $(x, f(x)) \in R^2$ of the plan provided with an orthonormal reference $(o, \vec{\imath}, \vec{\jmath})$

3.1.2 Bounded functions, monotonic function

Definition 3.1.3 Let $f : D_f \to R$ be a function. We say that:

a. f is **bounded from above** if there is a number M such that for all x from D_f: $f(x) \leq K$.
we write: $\exists M \in R, \forall x \in D_f: f(x) \leq M.$

b. f is **bounded from below** if there is a number m such that for all x from D_f: $f(x)$.
we write: $\exists m \in R, \forall x \in D_f: m \leq f(x)$

c. f is **bounded** if it is bounded both from above and below.
It is to say: $\exists M > 0, \forall x \in D_f: |f(x)| \leq M.$

d. Function that is not bounded is called unbounded function

Definition 3.1.4. Let $f : D_f \to R$ be a function. We say that:

a. f is <u>increasing</u> on D_f if: $\forall x, y \in D_f, x < y \Rightarrow f(x) \leq f(y).$
b. f is <u>strictly increasing</u> on D_f if: $\forall x, y \in D, x < y \Rightarrow f(x) < f(y).$
c. f is <u>decreasing</u> on D_f if: $\forall x, y \in D, x < y \Rightarrow f(x) \geq f(y).$
d. f is <u>strictly decreasing</u> on D_f if: $\forall x, y \in D, x < y \Rightarrow f(x) > f(y).$
e. f is <u>monotonic</u> (strictly monotonic, respectively) on D_f if f is <u>increasing **or** decreasing</u> (strictly increasing **or** strictly decreasing, resp) on D_f.

Example 3.1.2.

a. Exponential functions exp: R → R is strictly increasing.

b. The absolute value function $x \to |x|$ defined on R is not monotonic

3.1.3 Even, odd, periodic function

Definition 3.1.5. Let I be an interval of R <u>symmetric with respect to 0</u>.
Let $f : I \to R$ be a function. We say that:

a. f is even (زوجية) if $\forall x \in I: f(-x) = f(x)$.
b. f is odd (فردية) if $\forall x \in I: f(-x) = -f(x)$.

Example 3.1.3.

a. The function defined on R by $x \to x^{2n}$ $(n \in N)$ is even.
b. The function defined on R by $x \to x^{2n+1}$ $(n \in N)$ is odd.

Definition 3.1.6. Let $f: R \to R$ be a function and T a real number, T > 0. The function f is called periodic of period T if:

$$\forall x \in R: f(x + T) = f(x).$$

Example 3.1.4. The functions sin and cos are 2π $periodic$. The tangent function is π periodic.

3.1.4 Algebraic operations on functions

The set of functions from $D \subset R$ to R, is denoted F (D, R).

Definition 3.1.7 Let f and $g \in F(D, R)$ and λ ∈ R. We define:

- Sum of two functions $f + g: x \to (f + g)(x) = f(x) + g(x)$.
- For λ ∈ R, we define $\lambda f: x \to (\lambda f)(x) = \lambda f(x)$.
- Product of two functions $fg: x \to (fg)(x) = f(x)\, g(x)$.

The functions $f + g, \lambda f$ and fg are $functions$ $belonging$ to $F(D, R)$.

Definition 3.1.8 Let f and $g \in F(D, R)$ and $\lambda \in R$. We say that

- $f \leq g$ if: $\forall x \in D, f(x) \leq g(x)$.
- $f < g$ if: $\forall x \in D, f(x) < g(x)$.

Example 3.1.5 *Let f and g* be two functions defined on $]0, 1[$ by:
$f(x) = x, g(x) = x^2$. We have $g < f$, because $\forall x \in]0, 1[, x^2 < x$.

3.2 Limit of a function

3.2.1 Finite limit of a function at a point x_0 نهايـــة منتهيـــة عنـــد نقطـــة

Let $f: D_f \rightarrow R$ be a function. Let $x_0 \in R$ be a point of D_f or an extremity of D_f.

Definition 3.2.1 Let $\ell \in R$. We say that the function f has a limit ℓ at x_0 if we have:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in D_f, |x - x_0| < \delta \Rightarrow |f(x) - \ell| < \varepsilon$$

We write in this case: $\lim_{x \to x_0} f(x) = \ell$.

<u>Explanation of the definition:</u> We say that f has a (finite) limit ℓ in x_0 if, when $x \to x_0$, $f(x) \rightarrow \ell$

Example 3.2.1 Consider the function $f(x) = 2x - 1$ which is defined on R. At the point $x = 1$,

Indeed, for all $\varepsilon > 0$, we have $|f(x) - 1| = 2|x - 1| < \varepsilon$, if we have, a fortiori, |x

$$|x - 1| < \frac{\varepsilon}{2}$$

The right choice will then be to take $\delta = \frac{\varepsilon}{2}$

- **Limit to the right \ to the left x_0** النهايـــة عـــن يميـــن وعن شمـــال

Definition 3.2.2

a. We say that the function f admits ℓ as a limit to the right of x_0, or when x tends towards x_0^+, if $\forall \varepsilon > 0, \exists \delta > 0$, such that: $x_0 < x < x_0 + \delta, \rightarrow |f(x) - \ell| \leq \varepsilon$

We will write, in this case:

$$\lim_{x \to x_0^+} f(x) = \ell$$

b. We say that the function f admits ℓ as a limit to the left of x_0, or when x tends towards x_0^-, if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that: $x_0 - \delta < x < x_0$, results $|f(x) - \ell| \leq \varepsilon$. We will write, in this case:

$$\lim_{x \to x_0^-} f(x) = \ell$$

Example 3.2.2. The function $x \in R^+ \rightarrow \sqrt{x}$ tends to 0 when $x \rightarrow 0^+$

Noticed. If the function f admits a limit ℓ to the left of the point x_0 and a limit ℓ`

to the right of x_0, for f to admit a limit at the point x_0 it is necessary and sufficient that.

$$\boldsymbol{\ell} = \boldsymbol{\ell}`$$

$$\lim_{x \to x_0} f(x) = \ell$$

Example 3.2.3. Consider the function defined by

$$f(x) = \begin{cases} 1, si\ x \geq 0 \\ -1, si\ x < 0 \end{cases}$$

It admits 1 as the limit to the right of 0 and -1 as the limit to the left of 0. But it admits **no limit at the point 0.**

Uniqueness of the limit

Proposal If f admits a limit at the point x_0, this limit is unique.

3.2.2 Infinite limit of a function at a point x_0 نهايــة غير منتهيــة عنــد نقطــة

Let $x_0 \in R$. We will put by definition:

a. $\lim_{x \to x_0} f(x) = +\infty$ if:
$\forall A > 0, \exists \delta > 0, such\ as\ |x - x_0| < \delta \Rightarrow f(x) > A.$

b. $\lim_{x \to x_0} f(x) = -\infty$ if:
$$\forall A > 0, \exists \delta > 0, such\ as\ |x - x_0| < \delta \Rightarrow f(x) < -A.$$

3.2.3 Finite limit of a function at infinity ($x \to \pm\infty$) نهايــة منتهيــة عنـــد المـــالانهايـــة

a. $\lim_{x \to +\infty} f(x) = \ell, if$
$$\forall \varepsilon > 0, \exists A > 0, tel\ que\ x > A \Rightarrow |f(x) - \ell| < \varepsilon.$$

We say that f has a (finite) limit ℓ at $+\infty$ if, when x becomes very large, $f(x)$ becomes very close to ℓ

b. $\lim_{x \to -\infty} f(x) = \ell, if$
$$\forall \varepsilon > 0, \exists A > 0, tel\ que\ x < -A \Rightarrow |f(x) - \ell| < \varepsilon.$$

we say that f has a (finite) limit at $-\infty$ if, when x becomes very large in negative value, $f(x)$ becomes very close to ℓ

3.2.4 Infinite limit at infinity نهايـــة غيـــر منتهيـــة عنـــد المـــالانهايـــة

Let $x_0 \in R$. We will put by definition:

a. $\lim_{x\to+\infty} f(x) = +\infty$ if:

$$\forall A > 0, \exists\, B > 0, such\ as\ x > B \Rightarrow f(x) > A.$$

b. $\lim_{x\to+\infty} f(x) = -\infty$ if:

$$\forall A > 0, \exists\, B > 0, such\ as\ x > B \Rightarrow f(x) < -A.$$

c. $\lim_{x\to-\infty} f(x) = +\infty$ if:

$$\forall A > 0, \exists\, B > 0, such\ as\ x < -B \Rightarrow f(x) > A.$$

d. $\lim_{x\to-\infty} f(x) = -\infty$ if:

$$\forall A > 0, \exists\, B > 0, such\ as\ x < -B \Rightarrow f(x) < -A.$$

Examples:

Prove the following limits using the definition

$$\lim_{x\to1} 2x - 3 = -1$$

$$\lim_{x\to4} \sqrt{x} = 2$$

$$\lim_{x\to1} \frac{1}{(1-x)^2} = +\infty$$

$$\lim_{x\to1} 2x^2 + 3x - 1 = +\infty$$

Indeterminate form

Some forms of limits are called indeterminate. An **indeterminate form** is an expression whose limit cannot be determined solely from the limits of the individual functions.

Example of indeterminate forms: $0/0$, ∞/∞, $+\infty - \infty$, $0\times\infty$, ∞^0, 0^0

3.2.5 Properties of function limits

Let f, g: [a, b] → R and $x_0 \in]a, b[$, we have:

a. If $\lim_{x\to x_0} f(x) = \pm\infty \Longrightarrow \lim_{x\to x_0} \frac{1}{f(x)} = 0$
b. If $f \le g$ and $\lim_{x\to x_0} f(x) = \ell$ $\lim_{x\to x_0} g(x) = \ell' \Longrightarrow \ell \le \ell'$

c. If $f \leq g$ *and* $\lim_{x \to x_0} f(x) = +\infty$ *so*: $\lim_{x \to x_0} g(x) = +\infty$

d. If $f \leq g$ *and* $\lim_{x \to x_0} g(x) = -\infty$ *so*: $\lim_{x \to x_0} f(x) = -\infty$

Theorem Let f, g, h: [a, b] → R and $x_0 \in]a, b[$, if we have:

i. $f(x) \leq g(x) \leq h(x), pour\ tout\ x \in]a, b[$,

ii. $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = \ell \in R.$

$$\Longrightarrow \text{So } \lim_{x \to x_0} g(x) = \ell$$

3.2.6 Operations on the limits

Theorem Let $f, g: [a, b] \to R$ and $x_0 \in]a, b[$, such that: $\begin{cases} \lim_{x \to x_0} f(x) = \ell \\ and \\ \lim_{x \to x_0} g(x) = \ell' \end{cases}$, so:

a. $\lim_{x \to x_0} (f(x) + g(x)) = \ell + \ell'$

b. $\lim_{x \to x_0} (\lambda f(x)) = \lambda \ell$

c. $\lim_{x \to x_0} f(x) g(x) = \ell . \ell'$

d. $\lim_{x \to x_0} |f(x)| = |\ell|$

e. $\lim_{x \to x_0} |f(x) - \ell| = 0$

f. $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{\ell}{\ell'}\ if\ \ell' \neq 0$

Let $f : [a, b] \to [c, d], g : [c, d] \to R\ and\ x_0 \in]a, b[, y_0 \in [c, d]$, such that: $\lim_{x \to x_0} f(x) = y_0, and \lim_{y \to y_0} g(y) = \ell, so$:

$$Alors \lim_{x \to x_0} (g \circ f)(x) = \ell$$

Proposition Let $f, g: [a, b] \to R\ and\ x_0 \in]a, b[$, we have:

a. $if \lim_{x \to x_0} f(x) = +\infty, so \lim_{x \to x_0} \frac{1}{f(x)} = 0$

b. $if \lim_{x \to x_0} f(x) = -\infty, so \lim_{x \to x_0} \frac{1}{f(x)} = 0$

c. $if\ f \leq g \lim_{x \to x_0} f(x) = \ell, and \lim_{x \to x_0} g(x) = \ell', so: \ell \leq \ell'$

d. $if\ f \leq g \lim_{x \to x_0} f(x) = +\infty, so \lim_{x \to x_0} g(x) = +\infty,$

3.3 Continuity of a function

3.3.1 General definitions

Definition 3.3.1. Consider a function $f: D_f \to R$, D_f being an interval of R. We say that f is continuous at the point $x_0 \in D_f$ if:

$$\lim_{x \to x_0} f(x) = f(x_0)$$
$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in I, |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

To remember:

f continuous $\leftrightarrow$

1. $f(x_0)$ exists: f is defined at x_0

   الدالة معرفة عند النقطة x_0

2. $\lim_{x \to x_0^+} f(x) = \ell$ and $\lim_{x \to x_0^-} f(x) = \ell'$: **$\ell$ and ℓ' exist**

   الدالة تقبل نهاية عن يمين وعن شمال النقطة x_0

3. $\boldsymbol{\ell = \ell'}$

   النهايتان عن يمين وعن شمال النقطة x_0 متساويتان

Examples

1. Is the function $f(x) = \frac{x^2-4}{x-2}$ continuous

f is discontinuous at 2 because $f(2)$ is undefined.

2. Determine whether the function $f(x) = \begin{cases} -x^2 - 4 & if\ x \leq 3 \\ 4x - 8 & if\ x > 3 \end{cases}$ is continuous at x=3

- ✓ We calculate $f(3)$: $f(3) = -3^2 - 4 = -5$, Thus, <u>$f(3)$ is defined.</u>
- ✓ We calculate : $\lim_{x \to >3} f(x)$ and $\lim_{x \to <3} f(x)$

$$\lim_{x \to <3} f(x) = -3^2 - 4 = -5$$

$$\lim_{x \to >3} f(x) = \lim_{x \to >3} 4(3) - 8 = 4$$

- ✓ $-5 \neq 4 \Leftrightarrow not\ continuous\ at\ 3$

Definition 3.3.2. A function defined on an interval I is continuous on I if it is continuous at every point of I. The set of continuous functions on I is denoted by C (I).

Continuity on the left \ on the right

Definition 3.3.3 Consider a function $f: D_f \to R$, I being an interval of R.

1. The function is continuous to the right at x_0 if

$\lim_{x \to > x_0} f(x) = f(x_0) \Leftrightarrow$

$$\forall \varepsilon > 0, \exists \delta > 0, such\ that: \boldsymbol{x_0} < \boldsymbol{x} < \boldsymbol{x_0} + \boldsymbol{\delta}, \to |f(x) - f(x_0))| \leq \varepsilon.$$

2. The function f is said to be left continuous at x_0 if

$\lim_{x \to < x_0} f(x) = f(x_0) \Leftrightarrow$

$$\forall \varepsilon > 0, \exists \delta > 0, such\ that: \boldsymbol{x_0} - \boldsymbol{\delta} < x < x_0, \to |f(x) - f(x_0))| \leq \varepsilon.$$

3.3.2 Operations on continuous functions

Theorem 3.3.1. Let D_f be an interval, and $f\ and\ g$ be functions defined on D_f and continuous at $x_0 \in D_f$ Then

1. $\lambda f\ is\ continuous\ at\ x_0, (\lambda \in R)$.
2. $f + g\ is\ continuous\ at\ x_0$.
3. $f . g\ is\ continuous\ at\ x_0$.
4. $\frac{f}{g}\ (if\ g\ (x0)\ 6 =\ 0)\ is\ continuous\ at\ x_0$.

3.4 Differentiability of a function

Definition and properties

Definition 3.4.1. Let Df be an interval of R, x_0 a point of $Df, and\ f$ a function $(f: Df \to R)$

We say that f is **differentiable (derivable)** at the point x_0 if the limit $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ **exists** **finitely**

تعـريف نقـول ان الدالــة f **قابلة للاشتقــاق** في النقطة x_0 ادا كانــت النهـــاية $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ **مـوجودة ومنتهيــة**

This limit is called the derivative of $f\ at\ x_0$ and is noted $f'(x_0)$

تسمـــى هــذه النهـــاية مشتـــق الدالـــة f عند النقطـــة x_0 ونرمـــز لهـــا بـــ $f'(x_0)$

If we put $x - x_0 = h$ we obtain:

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}= f'(x_0)$$

Graphically (بيـــانيــا)

The derivative of $f\ at\ x_0$ is the **slope of the tangent line** at this point ($x_0, f(x_0)$), which is the limit as h → 0 of the slopes of the lines through

Example Let f be the real function defined on R by $f\ (x)\ =\ x^2$. The derivative of f at a point $x^0 \in\ R$ is:

$$\lim_{h\to 0}\frac{f(h+x_0)-f(x_0)}{h}=\lim_{h\to 0}\frac{(h+x_0)^2-{x_0}^2}{h}=\lim_{h\to 0}\frac{{x_0}^2+h^2+2x_0.h-{x_0}^2}{h}$$
$$=\lim_{h\to 0}h+2x_0=2x_0$$

For $x_0=1, f'(x_0)=2$

The function $y\ =\ f\ (x)$ is said to be differentiable in an open interval I if it is differentiable at every point of I

Proposition 3.4.1. Every differentiable function is continuous, but the converse is not true. (differentiable $\Longrightarrow$ continuous)

كـل دالـــة قابلـــة للاشتقـــاق عند نقطة مـــا فهي مستمرة عند تلـك النقطـــة. العكـــس غيـــر صحيـــح

3.4.1 Derivation operations

Theorem 3.4.1 Let f and g be two functions defined on the interval $I\ \subset\ R$ with and x0 ∈ I. If the functions $f\ and\ g$ are differentiable at x_0, then

1. ∀α ∈ R, αf is differentiable at x0 and we have:
$$(\alpha f)'(x_0)=\ \alpha f'(x_0)$$
2. f + g is differentiable at x0 and we have
$$(f+g)'(x_0)=f'(x_0)+g'(x_0)$$
3. fg is differentiable at x0 and we have
$$(f.g)'(x_0)=f'(x_0).g(x_0)+f(x_0).g'(x_0)$$
4. If $g'(x_0)\neq 0$ the function $\frac{f}{g}$ is differentiable
$$\left(\frac{f}{g}\right)'(x_0)=\frac{f'(x_0)g(x_0)-f(x_0)g'(x_0)}{g^2(x_0)}$$
$$(1\backslash g)'(x_0)=\frac{-g'(x_0)}{g^2(x_0)}$$

Theorem Let J be an interval of R, $f: I \rightarrow J$ and $g: J \rightarrow R$. If f is differentiable in $x_0 \in I$ and g differentiable at $f(x_0) \in J$, the composite function $g \circ f : I \rightarrow R$ is differentiable at x_0 and

$$(g \circ f)'(x) = f'(x).g'(f(x))$$

3.4.2 Derivatives of standard functions

Fonction f	Dérivée f'	Fonction f	Dérivée f'
x^n	nx^{n-1}	u^n	$nu'u^{n-1}$, $n \in \mathbb{N}^*$
$\frac{1}{x}$	$-\frac{1}{x^2}$	$\frac{1}{u}$	$-\frac{u'}{u^2}$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$	$\sqrt{u}$	$\frac{u'}{2\sqrt{u}}$
$\ln x$	$\frac{1}{x}$	$\ln u$	$\frac{u'}{u}$
e^x	e^x	e^u	$u'e^u$
$\sin(x)$	$\cos(x)$	$\sin(u)$	$u'\cos(u)$
$\cos(x)$	$-\sin(x)$	$\cos(u)$	$-u'\sin(u)$

Proposition Let $f : I \rightarrow R$ be a differentiable function on an Intervale I we have:

(1) $\forall x \in I, f'(x) = 0$ if and only if f is constant on I

(2) $If\ \forall x \in I, f'(x) \geq 0\ (resp\ f'(x) > 0)$, then f is increasing (resp strictly increasing).

(3) if $\forall x \in I, f'(x) \leq 0\ (resp\ f'(x) < 0)$, then f is decreasing (resp strictly decreasing)

3.4.3 L'hopital's rule

Theorem 3.4.7 Let f and g be two functions differentiable on I and both tending towards 0 (or both tend to ∞) $for\ x \rightarrow a\ (or\ \infty)$. We assume that $g(x) \neq 0$ does not vanish in a neighborhood of a and that: $\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = l$ ($\lim_{x \to x_0} \frac{f'(x)}{g'(x)}$ exists)

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = l$$

In other words, if the limit: $lim \frac{f(x)}{g(x)}$ is of indeterminate type $\frac{0}{0}\ or\ \frac{\infty}{\infty}$ for : $x \rightarrow a\ or\ \infty$ then we have: $lim \frac{f(x)}{g(x)} = lim \frac{f'(x)}{g'(x)} = l$

Example Find $\lim_{x\to 0}\frac{x^2}{sinx}$ and $\lim_{x\to 0}\frac{3x^2+x+4}{5x^2+8x}$

Solution As observed above, this limit is of indeterminate type $\frac{0}{0}$, so l'Hôpital's rule applies. We have

$$\lim_{x\to 0}\frac{x^2}{\sin x}\ \left(\frac{0}{0}\right) \overset{\text{l'H}}{=} \lim_{x\to 0}\frac{2x}{\cos x} = \frac{0}{1} = 0,$$

where we have first used l'Hôpital's rule and then the substitution rule. □

$$\lim_{x\to\infty}\frac{3x^2+x+4}{5x^2+8x}\ \left(\frac{\infty}{\infty}\right) \overset{\text{l'H}}{=} \lim_{x\to\infty}\frac{6x+1}{10x+8}\ \left(\frac{\infty}{\infty}\right) \overset{\text{l'H}}{=} \lim_{x\to\infty}\frac{6}{10} = \frac{6}{10} = \frac{3}{5}.$$