Chapter 1 – Numerical sequences..

I- Generalities..

1) Vocabulary.

Here is a list of numbers :	1	3	6	10	15	21	…	(terms)
We can number them:	n°0	n°1	n°2	n°3	n°4	n°5	…	(rows)

So the term of rank 4, in this example, is 15.

If we name this sequence (u_n) (Note: the name of the sequence is noted in parentheses), we will note:

$u_0=1$ (the term of rank 0 is equal to 1) $u_1=3$, $u_2=6$, $u_3=10$ etc...

For a natural number, a , the term of rank n of the sequence, is called a general term. (He is noted without parentheses, unlike the name of the suite.)

Note: you do not have to start the numbering at 0, you can start it at 1 or another rank.

2) Sequences defined explicitly as a function of n..

A sequence (u_n) is explicitly defined as a function of n when it is given, for all n, by a formula of type $u_n=f(n)$.

Exemple : Let u_n the sequence defined for all $n \in \mathbb{N}^*$ by $u_n=\frac{3}{n}+n$.

We can then calculate any term in the sequence..

Exemple u_{100} : $u_{100}=\frac{3}{100}+100$, $u_{100}=100{,}03$.

We can also calculate u_3 : $u_3=\frac{3}{3}+3$, $u_3=4$.

3) Sequences defined by recurrence..

A sequence is defined by induction when it provides:
- Its initial term
- and a relation allowing each term to be calculated from the term which precedes it. (Called recurrence relation)

Exemple : Let (v_n) sequence defined by :

$$\begin{cases} v_0=10 & \leftarrow \text{initial term} \\ \forall n \in \mathbb{N}, v_{n+1}=\frac{1}{2}v_n+3 & \leftarrow \text{recurrence relation} \end{cases}$$

Here, we cannot directly calculate any term. We must calculate them step by step from V_0:

$$v_1=\frac{1}{2}\times v_0+3=\frac{1}{2}\times 10+3=5+3=8$$

$$v_2=\frac{1}{2}\times v_1+3=\frac{1}{2}\times 8+3=4+3=7$$

$$v_3=\frac{1}{2}\times v_2+3=\frac{1}{2}\times 7+3=6{,}5$$

etc...

4) Direction of variation of a sequence..

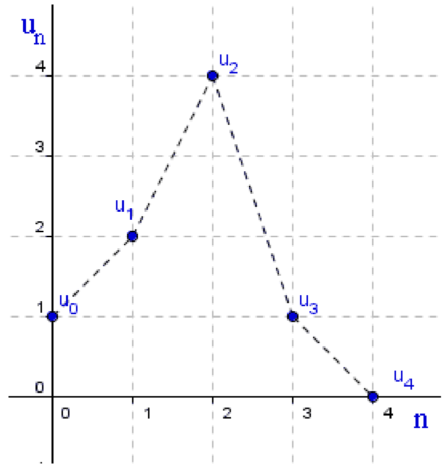
Definition 1: We will say that a sequence (u_n) is:

- Increasing when for all n, $u_{n+1} \geqslant u_n$
- Decreasing when for all n, $u_{n+1} \leqslant u_n$
- Constant when for all n, $u_{n+1} = u_n$

When a sequence is increasing, decreasing or constant, we say that it is monotonic. (This means that its direction of variation is constant).

Example of a non-monotonic sequence: (u_n) telle que $u_0=1$, $u_1=2$, $u_2=4$, $u_3=1$ et $u_4=0$.

(This sequence is increasing for n varying from 0 to 2, then decreasing for n varying from 2 to 4)



Note: if the inequality is strict (> or <), we speak of continuation strictly increasing or strictly decreasing.

In practice: to study the direction of variations of a sequence (u_n), study the sign of $u_{n+1} - u_n$.

Exemple : Let (w_n) the function defined for all n ∈ ℕ by $w_n = \dfrac{1}{3n+1}$.

To study its direction of variation, we can calculate $w_{n+1} - w_n$ for all n ∈ IN

∀ n ∈ IN, $w_{n+1} = \dfrac{1}{3\ n+1\quad 1} = \dfrac{1}{3n+3+1} = \dfrac{1}{3n+4}$.

So ∀ n ∈ ℕ, $w_{n+1} - w_n = \dfrac{1}{3n+4} - \dfrac{1}{3n+1} = \dfrac{3n+1}{(3n+4)(3n+1)} - \dfrac{3n+4}{(3n+1)(3n+4)} = \dfrac{-3}{(3n+1)(3n+4)}$.

According to the rule of signs, $w_{n+1} - w_n$ is strictly negative for everything n (car -3 is negative, $3n+1$ is positive since $n \geqslant 0$ and $3n+4$ Also). So the rest (w_n) is strictly decreasing.

II- Arithmetic sequences.

1) Definition.

Definition 1: A sequence (u_n) is arithmetic when there exists a constant real r such that, for all n, $u_{n+1} = u_n + r$. the real r called the reason of the arithmetic sequence.

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Exemple : $u_0=3$ $u_1=5$ $u_2=7$ $u_3=9$ $u_4=11$ etc...

For all n ∈ N, $u_{n+1} = u_n + 2$.

(u_n) is an arithmetic sequence of reason 2.

2) Calculation of the general term of an arithmetic sequence.

a) When the initial term is the term of rank 0.

Exemple : Let (u_n) the arithmetic sequence of initial term $u_0=5$ and reason $r=7$. $u_0=5$ (For n=0) $u_1=5+7$ (For n=1) $u_2=u_1+7=5+7+7=5+2\times7$ (For n=2) $u_3=5+2\times7+7=5+3\times7$ (For n=3) $u_4=5+4\times7$ etc... (For n=4) For all n ∈ N, $u_n=5+n\times7$ Or $u_n=5+7n$.	General formula: Soit (u_n) an arithmetic sequence initial term u_0 and reason r. $u_0=u_0$ (For n=0) $u_1=u_0+r$ (For n=1) $u_2=u_1+r=u_0+2r$ (For n=2) $u_3=u_0+3r$ (For n=3) $u_4=u_0+4r$ etc (For n=4) For all n ∈ N, $u_n=u_0+nr$.

Theorem 1: If (u_n) is an arithmetic sequence with initial term u_0 and reason r, then, for all n ∈ ℕ, $u_n=u_0+nr$.

b) When the initial term is the term of rank 1.

Exemple : Let (u_n) the arithmetic sequence of initial term $u_1=7$ and reason $r=10$. $u_1=7$ (For n=1) $u_2=7+10$ (For n=2) $u_3=u_2+10=7+10+10=7+2\times10$ (For n=3) $u_4=7+2\times10+10=7+3\times10$ (For n=4) $u_5=7+4\times10$ etc... (For n=5) For all n ∈ N*, $u_n=7+(n-1)\times10$.	General formula: Let (u_n) an arithmetic sequence initial term u_1 and reason r. $u_1=u_1$ (For n=1) $u_2=u_1+r$ (For n=2) $u_3=u_2+r=u_1+2r$ (For n=3) $u_4=u_1+3r$ (For n=4) $u_5=u_1+4r$ etc (For n=5) For all n ∈ N, $u_n=u_1+(n-1)r$.

Theorem 2: If (u_n) is an arithmetic sequence with initial term u_1 and reason r, then, for all n ∈ ℕ, $u_n=u_0+(n-1)r$.

Note: if the initial term is the term of rank p, then for all $n\geqslant p$, $\boxed{u_n=u_p+(n-p)r}$.

3) How to prove that a sequence is arithmetic?

a) By proving that its absolute variation is constant.

Definition 2: We call absolute variance of the sequence (u_n) the difference $u_{n+1}-u_n$.

Theorem 3: A sequence (u_n) is arithmetic if its absolute variation $u_{n+1}-u_n$ is constant.

Proof :

- If $u_{n+1}-u_n$ is a constant equal to a, then for all n, $u_{n+1}-u_n=a \Leftrightarrow u_{n+1}=u_n+a$.

So by definition, (u_n) is an arithmetic sequence of reason a.

Conversely, if (u_n) is an arithmetic sequence of reason r, then for all n, $u_{n+1}=u_n+r$ Let $u_{n+1}-u_n=r$, therefore its absolute variation is constant.

4) Sens of variations of an arithmetic sequence.

> Theorem 4: An arithmetic sequence is:
> - (strictly) increasing when its reason is (strictly) positive.
> - (strictly) decreasing when its reason is (strictly) negative.
> - constant when its reason is zero.

Proof: Immediate because $u_{n+1}-u_n$ is equal to reason, therefore of the sign

ofreason.

III- Geometric sequences.

1) Definition.

Definition 1: A sequence (u_n) is geometric when there exists a constant real q such that, for all n, $u_{n+1}=u_n\times q$. the real q called the reason of the geometric sequence.

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Exemple : $u_0=0{,}003$ $u_1=0{,}03$ $u_2=0{,}3$ $u_3=3$ $u_4=30$ etc...

For all n ∈ fl, $u_{n+1}=u_n\times 10$. (u_n) is a geometric sequence of reason 10.

2) Calculation of the general term.

a) When the initial term is the term of rank 0.

Exemple : Let (u_n) the geometric sequence of initial term $u_0=3$ and reason $q=5$.		General formula: Let (u_n) a geometric sequence initial term u_0 and reason q.	
$u_0=3$	(For n=0)	$u_0=u_0$	(For n=0)
$u_1=3\times 5$	(For n=1)	$u_1=u_0\times q$	(For n=1)
$u_2=u_1\times 5=3\times 5\times 5=3\times 5^2$	(For n=2)	$u_2=u_1\times q=u_0\times q\times q=u_0\times q^2$	(For n=2)
$u_3=3\times 5^2\times 5=3\times 5^3$	(For n=3)	$u_3=u_0\times q^2\times q=u_0\times q^3$	(For n=3)
$u_4=3\times 5^4$ etc...	(For n=4)	$u_4=u_0\times q^4$ etc	(For n=4)

For all $n \in \mathbb{N}$, $u_n = 3\times 5^n$.	For all $n \in \mathbb{N}$, $u_n = u_0 \times q^n$.

Theorem 5: If (u_n) is an geometric sequence with initial term u_0 and reason r, then, for all $n \in \mathbb{N}$, $u_n = u_0 \times q^n$.

b) When the initial term is the term of rank 1.

Exemple : Let (u_n) the geometric sequence of initial $u_1=5$ and reason $q=8$. $u_1=5$ (For n=1) $u_2=5\times 8$ (For n=2) $u_3=u_2\times 8=5\times 8\times 8=5\times 8^2$ (For n=3) $u_4=5\times 8^2\times 8=5\times 8^3$ (For n=4) $u_5=5\times 8^4$ etc... (For n=5) For all $n \in \mathbb{N}^*$, $u_n=5\times 8^{n-1}$.	General formula: Let (u_n) a geometric sequence initial term u_1 and reason q. $u_1=u_1$ (For n=1) $u_2=u_1\times q$ (For n=2) $u_3=u_2\times q=u_1\times q\times q=u_1\times q^2$ (For n=3) $u_4=u_1\times q^3$ (For n=4) $u_5=u_1\times q^4$ etc (For n=5) For all $n \in \mathbb{N}$, $u_n=u_1\times q^{n-1}$.

Theorem 2: If (u_n) is a geometric sequence with initial term u_1 and reason q, then, for all $n \in \mathbb{N}$, $u_n = u_1 \times q^{n-1}$.

Note: if the initial term is the term of rank p, then for all $n \geq p$, $u_n = u_p \times q^{n-p}$

3) How to prove that a sequence is geometric?

a) We prove that there exists a constant number q such that for all n, $u_{n+1}=u_n\times q$. This is the definition of a geometric sequence!

Variant: if we have proven (or if we know) beforehand that all the terms of the sequence are non-zero, we can calculate the ratio $\frac{u_{n+1}}{u_n}$ and prove that it is constant (then equal to reason).

Theorem 7: A sequence with all non-zero terms is geometric if and only if relative variation $\frac{u_{n+1}-u_n}{u_n}$ is constant

Note: the constant in question is $q-1$, where q is the ratio of the geometric sequence.

Proof : Let (u_n) a geometric sequence with non-zero terms of reason q (Note: q is necessarily non-zero since the terms of the sequence are).

For all n, $u_{n+1}=u_n\times q$ so $\frac{u_{n+1}-u_n}{u_n}=\frac{u_n\times q-u_n}{u_n}=\frac{u_n\times(q-1)}{u_n}=q-1$.

(We have the right to simplify by u_n because $u_n\neq 0$ for all n)

The relative variation $\frac{u_{n+1}-u_n}{u_n}$ is therefore a constant, equal to $q-1$.

• Conversely, if (u_n) is a sequence with non-zero terms such that, for all n, $\frac{u_{n+1}-u_n}{u_n}$ is equal to one constant C. $\frac{u_{n+1}-u_n}{u_n}=C \Leftrightarrow u_{n+1}-u_n=C\times u_n \Leftrightarrow u_{n+1}=C\times u_n+u_n \Leftrightarrow u_{n+1}=(C+1)u_n$.

(u_n) is therefore a geometric sequence of reason C+1.

4) Direction of variation of a geometric sequence..

We will only deal with the case where the first term and the reason are positive, therefore the cases of geometric sequences with positive terms.

Theorem 8: A geometric (u_n) sequence of strictly positive initial term and reason $q>0$.

- if $q>1$, then (u_n) is strictly increasing
- if $q=1$, then (u_n) is constant
- if $0<q<1$, then (u_n) is strictly decreasing

Proof: Let (u_n) a geometric sequence with first strictly positive term and reason q>0.

As each term is obtained by multiplying the previous one by q and as the first term is strictly positive, step by step and according to the rule of signs, all the terms in the sequence will be strictly positive.

For all n, $u_{n+1}=u_n\times q$.

If $q>1$, For all n, $u_n\times q>u_n$ (en multiplying both members by u_n which is strictly positive)) Let $u_{n+1}>u_n$. (u_n) is strictly increasing.	If $q=1$, For all n, $u_{n+1}=1\times u_n$ Let $u_{n+1}=u_n$. (u_n) is therefore constant.	If $0<q<1$, then, for all n, $0<q\times u_n<u_n$ (By multiplying the 3 members by u_n which is strictly positive) So $u_{n+1}<u_n$. (u_n) is therefore strictly decreasing.

IV- Some results about the sequence (q^n) , Or $q>1$.

Note: The following (q^n) is a geometric sequence with initial term 1 (for n=0) and reason q.

1. Limit of the sequence (q^n) .

The definition of the concept of limit is outside the program.

Intuitively, we will say:

- That the limit of a sequence is $+\infty$ if the numbers u_n end up exceeding a number as large as one wants when the values of n become large enough. We then note $\lim_{n\to+\infty} u_n=+\infty$ [2].
- That the limit of a sequence is $-\infty$ if the numbers u_n eventually exceed such a ~~small negative~~ number that we want when the values of n become sufficiently large. We then note $\lim_{n\to+\infty} u_n=-\infty$ [3].
- That the limit of a sequence is a real L when the numbers u_n end up accumulating around a fixed number L when n becomes very large, we then note $\lim_{n\to+\infty} u_n=L$ [4]

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- if $q>1$, then $\lim_{n\to+\infty} q^n=+\infty$
- if q=1, then $\lim_{n\to+\infty} q^n=1$
- if $0<q<1$, then $\lim_{n\to+\infty} q^n=0$

Propriétés (admises) :

Si $\lim_{n\to+\infty} u_n =$	$+\infty$	$L \in \mathbb{R}$	$-\infty$
Alors, pour tout réel b, $\lim_{n\to+\infty} u_n+b =$	$+\infty$	L+b	$-\infty$

Si $\lim_{n\to+\infty} u_n =$ Alors $\lim_{n\to+\infty} a\times u_n =$	$+\infty$	$L \in \mathbb{R}$	$-\infty$
Si a>0	$+\infty$	aL	$-\infty$
Si a=0		0	
Si $a<0$	$-\infty$	aL	$+\infty$

Exemple : If $\lim_{n\to+\infty} u_n=+\infty$, then $\lim_{n\to+\infty} -2\times u_n=-\infty$ and $\lim_{n\to+\infty} -2u_n+100=-\infty$.

Application: determination of the limit of a geometric sequence of reason q>0.

Exemple : If (v_n) the geometric sequence of initial term $v_0=8$ and reason 0,1.

As $0<0{,}1<1$, $\lim_{n\to+\infty} 0{,}1^n=0$, so $\lim_{n\to+\infty} 8\times 0{,}1^n=0$ then $\lim_{n\to+\infty} v_n=0$.

Let (w_n) the geometric sequence of initial term $w_0=-10$ and reason 7.

As $7>1$, $\lim_{n\to+\infty} 7^n=+\infty$, so $\lim_{n\to+\infty} -10\times 7^n=-\infty$, let $\lim_{n\to+\infty} w_n=-\infty$.

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2 And we say "The limit of the continuation (u_n) when n tends to $+\infty$ is $+\infty$ » or «The sequence (u_n) diverges towards $+\infty$. »

3 And we say "The limit of the continuation (u_n) when n tends to $+\infty$ is $-\infty$ » or «The sequence (u_n) diverge vers $-\infty$. »

4 And we say "The limit of the continuation (u_n) when n tends to $+\infty$ is L. » or «The sequence (u_n) diverge vers L ».

1) Sum of (n+1) first terms of the sequence (q^n) .

Theorem 10: Let q be a real number different from 0 and 1.

then the sum $S_n = 1 + q + q^2 + q^3 + \ldots + q^n$ is equal to $\frac{1 - q^{n+1}}{1 - q}$.