Polynomials and rational fractions

.1 Generalities about polynomials

1.1 D efinition of a polynomial with real coefficients

Definition IV.1.1 We will call polynomial with real coefficients any function P: **R** $\longrightarrow$ **R** of the form:

$$x \longrightarrow P(x) = \sum_{i=0}^{n} a_i x^i = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x^1 + a_0$$

where n is a natural number, and $a_n, a_{n-1}, a_{n-2}, \ldots, a_2, a_1, a_0$ are given real numbers called coefficients of the polynomial. The number a_0 is called the constant term.

The set of polynomials with real coefficients is denoted $\mathbb{R}[X]$.

Exemples :

1. *The coefficients of the polynomial* $P(x) = x^4 + x$ sont 1, 0, 0, 1, 0 ;
2. *those of the polynomial P (x) = 5x are 5, 0;*
3. and those of the polynomial $P(x) = 2(3x + 1)(x^2 - 4) = 6x^3 + 2x^2 - 24x - 8$ are 6, 2, −24, −8.

Theorem IV.1.2 $P(x) = \sum_{i=0}^{n} a_i x^i$ is zero if and only if all its coefficients are zero.

1.2 *Degree of a polynomial*

Definition IV.1.3 We call the degree of a non-zero polynomial P, denoted deg(P), the highest power of the variable x actually present in the polynomial.

In other words, let $P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x^1 + a_0$, if $a_n \neq 0$ *so deg(P) = n.*
si a_n

- *If* $a_n = 1$*, we say that the polynomial is unitary.*
- *If* $P(x) = a_n x^n$ *, we say that* $P(x)$ is a monome of degree n.
- *The set of polynomials whose degree is equal to 0 is made up of constant polynomials .*

Note : The zero polynomial P (x) = 0 has no degree..

Exemples :

1. *Let the polynomial* $P(x) = x^4 + x$, then $deg(P) = 4$;
2. *Let the polynomial* $P(x) = 5x$, then $deg(P) = 1$;
3. *Let the polynomial* $P(x) = 2x^2 + 6x^3 - 24x - 8$, then $deg(P) = 3$;
4. *Let the polynomial* $P(x) = 3$, then $deg(P) = 0$.

1.3 Valuation of a polynomial

Definition IV.1.4 *We call the valuation of a polynomial P, denoted val(P), the smallest power of the variable x actually present in the polynomial.*
In other words, let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_{n_0+1}x^{n_0+1} + a_{n_0}x^{n_0}$, *if* $a_{n_0} \neq 0$ *then* $val(P) = n_0$.

Exemples :

1. *Let the polynomial* $P(x) = x^4 + x^2$, then $val(P) = 2$;
2. *Let the polynomial* $P(x) = 9x^6$, then $val(P) = deg(P) = 6$;
3. *Let the polynomial* $P(x) = 6x^3 + 2x^2 - 24x - 8$, then $val(P) = 0$;
4. *Let the polynomial* $P(x) = 3$, then $val(P) = deg(P) = 0$.

1.4 Eperation on polynomials

Two non-zero polynomials are equal if and only if they have the same degree and if the coefficients of their terms of the same power are equal.

1.5 Operation on polynomials

a/ Sum of polynomials

Definition IV.1.5 *let* $P(x) = \sum_{i=0}^{n} a_i x^i$ *and* $Q(x) = \sum_{i=0}^{m} b_i x^i$ *two polynomials with real coefficients.*
The sum P + Q is a polynomial defined by:

$$P(x) + Q(x) = \sum_{i=0}^{s} (a_i + b_i)x^i \quad \text{and } s \leq \max\{n, m\}.$$

In general, deg(P + Q) ≤ max{deg(P), deg(Q)} and we have equality if the terms of higher degree do not eliminate each other..

Exemples :

1. Let $P(x) = 4x^3 + 3x^2 + 4$ et $Q(x) = -x^2 + 2x + 5$.
   then, $P(x) + Q(x) = 4x^3 + 2x^2 + 2x + 9$ and $deg(P + Q) = \max\{deg(P), deg(Q)\}$;
2. Let $P(x) = x^2 + 4$ and $Q(x) = -x^2 + x + 6$.
   then, $P(x) + Q(x) = x + 10$ and $deg(P + Q) < \max\{deg(P), deg(Q)\}$.

b/ Difference of polynomials

Definition IV.1.6 *Let* $P(x) = \sum_{i=0}^{n} a_i x^i$ *and* $Q(x) = \sum_{i=0}^{m} b_i x^i$ be two polynomials with real coefficients.

The difference P − Q is a polynomial defined by:

$$P(x) - Q(x) = \sum_{i=0}^{s} (a_i - b_i)x^i \qquad \text{where } s \leq \max\{n, m\}.$$

In geral, $deg(P - Q) \leq \max\{deg(P), deg(Q)\}$ *and we have equality if the higher degree term is not eliminated.*

Exemples :

1. Let $P(x) = 4x^3 + 3x^2 + 4$ et $Q(x) = -x^2 + 2x + 5$.
   then, $P(x) - Q(x) = 4x^3 + 4x^2 - 2x - 1$ e $deg(P + Q) = \max\{deg(P), deg(Q)\}$;

2. Let $P(x) = x^2 + 6$ et $Q(x) = x^2 + x + 2$.

then, $P(x) - Q(x) = -x + 4$ and $deg(P + Q) < \max\{deg(P), deg(Q)\}$.

c/ Product of a polynomial by a scalar

Definition IV.1.7 *Let the polynomial be $P(x) = \sum_{i=1}^{n} a_i x^i$ and $\lambda \in \mathbb{R}^\star$. We have (λP) is also a polynomial definedi by :*

$$(\lambda P)(x) = \sum_{i=0}^{n} (\lambda a_i) x^i \quad \text{and } deg(\lambda P) = deg(P)$$

Exemple : Let $P(x) = x^5 - 7x^3 + 14$, then $(2P)(x) = 2x^5 - 14x^3 + 28$.

d/ Product of polynomials

Definition IV.1.8 *Let $P(x) = \sum_{i=0}^{n} a_i x^i$ and $Q(x) = \sum_{i=0}^{m} b_i x^i$ be two non-zero polynomials with coefficients real and of degrees n and m respectively. The product P.Q is a non-zero polynomial of degree s = n + m defined by :*

$$P(x).Q(x) = \sum_{k=0}^{s} \left(\sum_{i=0}^{k} a_i b_{k-i} \right) x^k$$

Exemple :

The product of $P(x) = x^3 + 3x^2 + 4$ et de $Q(x) = -x^2 + 2x + 5$ is the polynomial of degree 5 defined by:

$$\begin{aligned} (x^3 + 3x^2 + 4) \times (-x^2 + 2x + 5) &= -x^5 + 2x^4 + 5x^3 - 3x^4 + 6x^3 + 15x^2 - 4x^2 + 8x + 20 \\ &= -x^5 - 5x^4 + 11x^3 + 11x^2 + 8x + 20. \end{aligned}$$

1.6 Divisibility in R[X]

a/ Division according to the decreasing powers of the variable (Euclidean division)

Theorem IV.1.9 *Given a polynomial A(x) and a non-zero polynomial B(x), there exists a unique pair (Q(x), RJx)) of polynomials such that::*

$$A(x) = B(x)Q(x) + R(x)$$

and degree $(R) <$ degree (B)

$$\begin{array}{c|c} \underbrace{A(x)}_{Dividend} & \underbrace{B(x)}_{Divider} \\ \vdots & \hline & \underbrace{Q(x)}_{Quotient} \\ \underbrace{R(x)}_{Rest} & \end{array}$$

Exemple : Let us divide the polynomial A(x) by the polynomial B(x) with:

$$A(x) = 2x^4 - 3x^3 + 5x^2 + 7x - 2 \quad \text{et} \quad B(x) = x^2 + x - 2.$$

Beforehand, we will have ordered the two polynomials according to the decreasing powers of x. We then arrange the polynomials in the following way:

$$\begin{array}{rrrrrrrrr|l} 2x^4 & - & 3x^3 & + & 5x^2 & + & 7x & - & 2 & x^2 + x - 2 \\ \cline{10-10} \end{array}$$

Definition of a root of a polynomial

Definition IV.1.11 *We say that the number a is the root (or zero) of the polynomial P (x) if and only if P (a) = 0.*

Exemples :

- 1 and 2 are two roots of the polynomial $P(x) = x^3 - 3x + 2$ since $P(1) = 0$ and $P(2) = 0$;
- 1 is root of the polynomial $P(x) = x^3 + x - 2$ since $P(1) = 0$.

ii/ Factorization of a polynomial by (x − a)

Definition IV.1.12 *We say that the non-zero polynomial P (x) can be factorized by (x − a) or even P (x)is divisible by (x − a) if there exists a polynomial Q(x) such that P (x) = (x − a)Q(x).*

Theorem IV.1.13 *The polynomial P (x) admits the number* ***a*** *as root if and only if P (x) is divisible by (x − a).*

Proof : Using the Euclidean division of P (x) by (x − a) we write P (x) = (x − a)Q(x)+ R(x) with deg(R) = 0, Therefore R (x) is a real number and it is worth P(a).

Corollary IV.1.14 *if the polynomial P (x) has k distinct roots $a_1, a_2, \cdots, a_k$ then $P(x)$ is divisible by $(x - a_1)(x - a_2)\cdots(x - a_k)$*

Proof : if a_1 is the root of $P(x)$, we can write $P(x) = (x - a_1)P_1(x)$; since $a_2 \neq a_1$ is the root of $P(x)$ it is also the root of $P_1(x)$ and we have $P_1(x) = (x - a_2)P_2(x)$ Hence,
$P(x) = (x - a_1)P_1(x) = (x - a_1)(x - a_2)P_2(x)$. And we continue the process.

Corollary IV.1.15 *A polynomial in R[X] of degree n has at most n distinct roots.*

.2 Fational fractions in R[X]

2.1 Definition of rational fractions

Rational functions are to polynomials what fractions are to integers.

Definition IV.2.1 *The function f (x) is a rational function if there exist two polynomials P (x) and Q(x) prime among them such that :*

$$f(x) = \frac{P(x)}{Q(x)}$$

And we have : $deg(f) = deg(P) - deg(Q)$.

As with any fraction, the top (the polynomial P (x)) is called the numerator and the bottom (the polynomial Q(x)) the denominator.

Exemples :

– The rational fraction defined for all x by:

$$\frac{P(x)}{Q(x)} = \frac{2x^2 - 5x + 5}{x + 3}$$

has degree $1 = 2 - 1$.

– The rational fraction defined for all x by:

$$\frac{P(x)}{Q(x)} = \frac{2x}{x + 1}$$

Has degree $0 = 1 - 1$.

$$\frac{2x^2 - 4x + 5}{x^2 + 1} + \frac{2x}{x + 3} = \frac{4x^3 + 2x^2 - 5x + 15}{x^3 + 3x^2 + x + 3}$$

Its degree is worth $0 = 3 - 3$.

Theorem IV.2.2 *Any rational fraction is written uniquely as the sum of a polynomial (called integer part) and simple elements (called polar part) whose type is determined by determined by the denominator of the rational fraction that we decompose.*

2.2 Whole part of a rational fraction

Theorem IV.2.3 *Consider two polynomials P (x) of degree m and Q(x) of degree n with m ≥ n. Then, for all x such that Q(x) /= 0, we can write :*

$$\frac{P(x)}{Q(x)} = E(x) + \frac{R(x)}{Q(x)}$$

where E(x) is the quotient of the Euclidean division of P (x) by Q(x) of degree m − n, R(x) is the rest , and $\frac{R(x)}{Q(x)}$ is a rational fraction. *The polynomial* $E(x)$ *est dit partie entière de* $\frac{P(x)}{Q(x)}$.

Exemple :

Consider the rational fraction $\frac{5x^4 + 3x + 2}{x^2 \_ 2x + 1}$. Noting that 1 is not the root of the numerator, we deduces that the polynomials are coprime. For $x \neq 1$, we have :

	$5x^4$	+	$0x^3$	+	$0x^2$	+	$3x$	+	2	x^2	-	$2x$	+	1
-	$5x^4$	+	$10x^3$	-	$5x^2$					$5x^2$	+	$10x$	+	15
			$10x^3$	-	$5x^2$	+	$3x$	+	2					
		-	$10x^3$	+	$20x^2$	-	$10x$							
					$15x^2$	-	$7x$	+	2					
				-	$15x^2$	+	$30x$	-	15					
							$23x$	-	13					

Also,

$$\frac{5x^4 + 3x + 2}{x^2 - 2x + 1} = 5x^2 + 10x + 15 + \frac{23x - 13}{x^2 - 2x + 1}.$$

2.2 Polar part of a rational fraction

Proposition IV.2.4 Let $\frac{P(x)}{Q(x)}$ *be a rational fraction and let it have a root* ^{3} *of Q(x) of multiplicity*

m. Let us write $Q(x) = (x - a)^m Q_1(x)$ *with* $Q_1(a)$ 0*. here exists a unique decomposition in the form:*

$$\frac{P(x)}{Q(x)} = \frac{A(x)}{(x-a)^m} + \frac{B(x)}{Q_1(x)},$$

with $A(x)$ *and* $B(x)$ *two p polynomials,* $A(x)$ *being such that* $deg(A) < m$*. The fraction* $\frac{A(x)}{(x-a)^m}$

The fraction A(x) is called the polar part of the rational fraction.