1 Numerical function of a real variable numerical function of a real variable

1. Set of Real Numbers

□ Algebraic order and operations

The set R with the relation “less than or equal” is a totally ordered set. Furthermore, we have the following property: If x, y and z are three real numbers, then

$$x \leq y \Longleftrightarrow x + z \leq y + z$$

if $z > 0$ $\quad x \leq y \Longleftrightarrow xz \leq yz$

if $z < 0$ $\quad x \leq y \Longleftrightarrow xz \geq yz$

□ Set $\overline{\mathbb{R}}$

We call $\overline{\mathbb{R}}$ the set R to which we add the two symbols $+\infty$ and $-\infty$. Let :

$$\overline{\mathrm{R}} = \mathrm{R} \cup \{+\infty\} \cup \{-\infty\}.$$

We extend to $\overline{\mathrm{R}}$ addition, multiplication and order relation of R as follows:

— For $l \in \mathrm{R}$ we pose:

$$l + (+\infty) = +\infty, \qquad -(+\infty) = -\infty, \qquad (+\infty) + (+\infty) = +\infty$$

$$l + (-\infty) = -\infty, \qquad -(-\infty) = +\infty, \qquad (-\infty) + (-\infty) = -\infty, \qquad -\infty < l < +\infty.$$

— For $l \in \mathrm{R}^*$ we pose :

$$\ell \times (+\infty) = \begin{cases} +\infty & \text{si } \ell > 0 \\ -\infty & \text{si } \ell < 0. \end{cases} \qquad \ell \times (-\infty) = \begin{cases} -\infty & \text{si } \ell > 0 \\ +\infty & \text{si } \ell < 0. \end{cases} \qquad \begin{matrix} (+\infty) \times (+\infty) = +\infty \\ (+\infty) \times (-\infty) = -\infty \end{matrix}$$

Despite everything, certain expressions are not defined:

$$0 \times (+\infty), \qquad 0 \times (-\infty), \qquad (+\infty) + (-\infty).$$

These expressions are called indeterminate forms.

□ Set Intervals R

Let a and b be two elements of R such as $a < b$. We call open interval of ends a and b the subset of R denoted$]a, b[$ defined by:

$$]a, b[= \{ x \in \mathrm{R} \mid a < x < b \}\cdot$$

Let a and b be two real numbers such that $a \leq b$. We call closed interval of endpoints a and b the subset of R denoted $[a, b]$ defined by:

$$[a, b] = \{ x \in \mathbb{R} \mid a \leq x \leq b \}.$$

If a and b two real numbers such that $a \leq b$, we similarly define the half-open interval on the right (resp. on the left) of ends a and b by:

$$[a, b[= \{ x \in \mathbb{R} \mid a \leq x < b \} \qquad (\text{resp. }]a, b] = \{ x \in \mathbb{R} \mid a < x \leq b \}).$$

Let a be a real number. Any interval of type is called an open interval with center

$$]a - \varepsilon, a + \varepsilon[$$

or ε denotes a strictly positive real number. Finally, we pose:

$$[a, +\infty[= \{ x \in \mathbb{R} \mid x \geq a \}, \qquad]-\infty, a] = \{ x \in \mathbb{R} \mid x \leq a \}$$

$$]a, +\infty[= \{ x \in \mathbb{R} \mid x > a \}, \qquad]-\infty, a[= \{ x \in \mathbb{R} \mid x < a \}.$$

□ Absolute value

Definition 1.1 *Let x be a real number. The absolute value of x is the positive number, denoted $|x|$, defined by:*

$$|x| = \sup\{x, -x\}$$

It immediately follows from the definition that:

$$\forall x \in \mathbb{R} : \qquad |x| \geq 0, \quad |x| = |-x|, \quad |x| \geq x.$$

Proposition 1.1 *Let x and y be two real numbers, we have:* $|xy| = |x||y|$

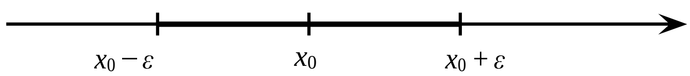
- $|x + y| \leq |x| + |y|$ *(triangular inequality)*

□ Neighborhoods

Definition 1.2 *Let x_0 be a real number. We call fundamental neighborhood of x_0 any open interval not empty with center x_0.*

We notice $V_\varepsilon(x_0)$ the fundamental neighborhood of x of radius ε $(\varepsilon > 0)$:

$$V_\varepsilon(x_0) = \{x \in \mathbb{R} : x_0 - \varepsilon < x < x_0 + \varepsilon\} = \{x \in \mathbb{R} : |x_0 - x| < \varepsilon\}$$



Definition 1.3 *We call neighborhood of a real number x0 any part of R which contains a fundamental neighborhood of x.*

Definition 1.4 *We call the neighborhood of $+\infty$ resp. $-\infty$ any part of R containing an interval of the form* $]a, +\infty[$ *resp.* $]-\infty, a[$ *or* $a \in \mathbb{R}$.