# Projective Space

## C.1. The projective line

Let us begin with an example. Consider the function $f(x) = \frac{1}{x}$. We know from Calculus that $f$ is continuous (and differentiable) on all of its domain (i.e. $\mathbb{R}$) except at $x = 0$. Would it be possible to extend the real line, so that $f(x)$ is continuous everywhere? The answer is yes, it is possible, and the solution is to *glue* the "end" of the real line at $\infty$ with the other "end", at $-\infty$. Formally, we need the *projective line*, which is a line with points $\mathbb{R} \cup \{\infty\}$, i.e. a real line plus a single point at infinity that ties the line together (into a circle).

The formal definition of the projective line is as follows. It may seem a little confusing at first, but it is fairly easy to work and compute with it. First, we need to define a relation between vectors of real numbers in the plane. Let $a, b, x, y$ be real numbers, such that neither $(x, y)$ nor $(a, b)$ is the zero vector. We say that $(x, y) \sim (a, b)$ if the vector $(x, y)$ is a non-zero multiple of the vector $(a, b)$. In other words, if considered as points, $(a, b) \sim (x, y)$ if they lie in the same line on the plane. Again:

$(x, y) \sim (a, b)$ if and only if there is $\lambda \in \mathbb{R}$ such that $x = \lambda a$, $y = \lambda b$.

For instance $(\sqrt{2},\sqrt{2}) \sim (1,1)$. We denote by $[x,y]$ the set of all vectors $(a,b)$ such that $(x,y) \sim (a,b)$:

$$[x,y] = \{(a,b) : a,b \in \mathbb{R} \text{ such that } (a,b) \neq (0,0) \text{ and } (x,y) \sim (a,b)\}.$$

Finally, we define the real projective line by:

$$\mathbb{P}^1(\mathbb{R}) = \{[x,y] : x,y \in \mathbb{R} \text{ with } (x,y) \neq (0,0)\}.$$

If you think about it, $\mathbb{P}^1(\mathbb{R})$ is the set of all lines through the origin (each class $[x,y]$ consists of all points -except the origin- on the line that goes through $(x,y)$ and $(0,0)$). The important thing to notice is that if $[x,y] \in \mathbb{P}^1(\mathbb{R})$ and $y \neq 0$, then $(x,y) \sim (\frac{x}{y},1)$, so the class of $[x,y]$ contains a unique representative of the form $(a,1)$, for some $a = \frac{x}{y} \in \mathbb{R}$. This allows the following decomposition of $\mathbb{P}^1(\mathbb{R})$:

$$\mathbb{P}^1(\mathbb{R}) = \{[x,1] : x \in \mathbb{R}\} \cup \{[1,0]\}.$$

The points $\{[x,1]\}$ form a real line and the point $[1,0]$ is called the point at infinity (see Figure 1.)

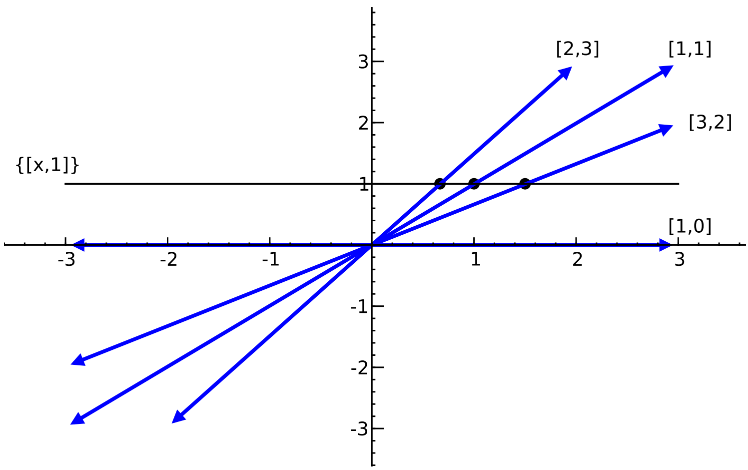


**Figure 1.** Some points in the projective line, e.g. $[2,3] \in \mathbb{P}^1(\mathbb{R})$, and their representatives of the form $[x,1]$, e.g. $[\frac{2}{3},1]$, except for $[1,0]$.

Let us see that, with this definition, the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 1/x$ is continuous everywhere when extended to $\mathbb{P}^1(\mathbb{R})$. We define instead an extended function $F : \mathbb{P}^1(\mathbb{R}) \to \mathbb{P}^1(\mathbb{R})$ by

$$F([x, y]) = [y, x].$$

Notice that a point on the real line of $\mathbb{P}^1$, i.e. a point of the form $[x, 1]$, is sent to the point $[1, x]$ of $\mathbb{P}^1$, and $(1, x) \sim (\frac{1}{x}, 1)$ as long as $x \neq 0$. So $[x, 1]$ with $x \neq 0$ is sent to $[\frac{1}{x}, 1]$ via $F$ (i.e. the real point $x$ is sent to $\frac{1}{x}$). Hence, $F$ coincides with $f$ on $\mathbb{R} - \{0\}$. But $F$ is perfectly well defined on $x = 0$, i.e. on the point $[0, 1]$, and $F([0, 1]) = [1, 0]$ so that $[0, 1]$ is sent to the point at infinity. Moreover, both sided limits coincide:

$$\lim_{x\to 0^+} F([x, 1]) = \lim_{x\to 0^-} F([x, 1]) = F([0, 1]) = [1, 0].$$

## C.2. The projective plane

We may generalize the construction above of the projective line, in order to construct a projective plane which will consist of a real plane plus a number of points at infinity, one for each direction in the plane, i.e. the projective plane will be a real plane plus a projective line of points at infinity.

Let $a, b, c, x, y, z \in \mathbb{R}$ such that neither $(a, b, c)$ nor $(x, y, z)$ are the zero vector:

$(x, y, z) \sim (a, b, c)$ if and only if there is $\lambda \in \mathbb{R}$ such that $x = \lambda a$, $y = \lambda b$, $z = \lambda c$.

We also define classes of similar vectors by:

$[x, y, z] = \{(a, b, c) : a, b, c \in \mathbb{R}$ such that $(a, b, c) \neq \vec{0}$ and $(x, y, z) \sim (a, b, c)\}$.

Notice that, as before, the class $[x, y, z]$ contains all the points in the line that goes through $(x, y, z)$ and $(0, 0, 0)$, except the origin. We define the projective plane to be the collection of all such lines:

$$\mathbb{P}^2(\mathbb{R}) = \{[x, y, z] : x, y, z \in \mathbb{R} \text{ such that } (x, y, z) \neq (0, 0, 0)\}.$$

If $z \neq 0$ then $(x, y, z) \sim (\frac{x}{z}, \frac{y}{z}, 1)$. Thus:

$$\mathbb{P}^2(\mathbb{R}) = \{[x, y, 1] : x, y \in \mathbb{R}\} \cup \{[a, b, 0] : a, b \in \mathbb{R}\}.$$

The points of the set $\{[x, y, 1] : x, y \in \mathbb{R}\}$ are in 1-to-1 correspondence with the real plane, and the points in $\{[a, b, 0] : a, b \in \mathbb{R}\}$ are called the points at infinity, and form a $\mathbb{P}^1(\mathbb{R})$, a projective line.

One interesting consequence of the definitions is that any two parallel lines in the real plane intersect at a point at infinity. Indeed, let $L : y = mx + b$ and $L' : y = mx + b'$ be distinct parallel lines in the real plane. If points in the real plane correspond to lines in $\mathbb{P}^2(\mathbb{R})$, lines in the real plane correspond to *planes* in $\mathbb{P}^2(\mathbb{R})$:

$$L = \{[x, y, z] : mx - y + bz = 0\}, \quad L' = \{[x, y, z] : mx - y + b'z = 0\}.$$

What is $L \cap L'$? The intersection points are those $[x, y, z]$ such that $mx - y + bz = mx - y + b'z = 0$, which implies that $(b - b')z = 0$. Since $L \neq L'$, we have $b \neq b'$ and, therefore, we must have $z = 0$. Hence:

$$L \cap L' = \{[x, mx, 0] : x \in \mathbb{R}\} = \{[1, m, 0]\}$$

so the intersection consists of a single point at infinity: $[1, m, 0]$.

## C.3. Over an arbitrary field

The projective line and plane can be defined over any field. Let $K$ be a field (e.g. $K = \mathbb{Q}, \mathbb{R}, \mathbb{C}$ or $\mathbb{F}_p$). The usual *affine plane* (or Euclidean plane) is defined by:

$$\mathbb{A}^2(K) = \{(x, y) : x, y \in K\}.$$

The projective plane over $K$ is defined by:

$$\mathbb{P}^2(K) = \{[x, y, z] : x, y, z \in K \text{ such that } (x, y, z) \neq (0, 0, 0)\}.$$

As before, $(x, y, z) \sim (a, b, c)$ if and only if there is $\lambda \in K$ such that $(x, y, z) = \lambda \cdot (a, b, c)$.

## C.4. Curves in the projective plane

Let $K$ be a field and let $C$ be a curve in affine space, given by a polynomial in two variables:

$$C : f(x, y) = 0$$

for some $f(x, y) \in K[x, y]$, e.g. $C : y^2 - x^3 - 1 = 0$. We want to extend $C$ to a curve in the projective plane $\mathbb{P}^2(K)$. In order to do this, we consider the points in the curve $(x, y)$ to be points in the plane $[\frac{x}{z}, \frac{y}{z}, 1]$ of $\mathbb{P}^2(K)$. Thus, we have:

$$C : \left(\frac{y}{z}\right)^2 - \left(\frac{x}{z}\right)^3 - 1 = 0$$

or, equivalently, $zy^2 - x^3 - z^3 = 0$. Notice that the polynomial $F(x, y, z) = zy^2 - x^3 - z^3$ is homogeneous in its variables (each monomial has degree 3) and $F(x, y, 1) = f(x, y)$. The curve in $\mathbb{P}^2(K)$ given by:

$$\widehat{C} : F(x, y, z) = zy^2 - x^3 - z^3 = 0$$

is the curve we were looking for, which extends our original curve $C$ in the affine plane. Notice that if the points $(x, y) \in C$ then $[x, y, 1] \in \widehat{C}$. However, there may be some extra points in $\widehat{C}$ which were not present in $C$, namely those points of $\widehat{C}$ at infinity. Recall that the points at infinity are those with $z = 0$, so $F(x, y, 0) = -x^3 = 0$ implies that $x = 0$ also, and the only point at infinity in $\widehat{C}$ is $[0, 1, 0]$.

In general, if $C \subseteq \mathbb{A}^2(K)$ is given by $f(x, y) = 0$, and $d$ is the highest degree of a monomial in $f$, then $\widehat{C} \in \mathbb{P}^2(K)$ is given by

$$\widehat{C} : F(x, y, z) = 0$$

where $F(x, y, z) = z^d \cdot f\left(\frac{x}{z}, \frac{y}{z}\right)$. Conversely, if $\widehat{C} : F(x, y, z) = 0$ is a curve in the projective plane, then $C : F(x, y, 1) = 0$ is a curve in the affine plane. In this case, $C$ is the projection of $\widehat{C}$ onto the chart $z = 1$; we may also look at other charts, e.g. $x = 1$ which would yield a curve $C' : F(1, y, z) = 0$.

Here is another example. Let $C$ be given by:

$$C : y - x^2 = 0$$

so that $C$ is a parabola. Then $\widehat{C}$ is given by

$$\widehat{C} : F(x, y, z) = z^2 f\left(\frac{x}{z}, \frac{y}{z}\right) = zy - x^2 = 0.$$

The curve $\widehat{C}$ has a unique point at infinity, namely $[0, 1, 0]$. This means that the two "arms" of the parabola meet at a single point at infinity. Thus, a parabola has the shape of an ellipse in $\mathbb{P}^2(K)$. How about hyperbolas? Let

$$C : x^2 - y^2 = 1.$$

Then $\widehat{C} : x^2 - y^2 = z^2$ and there are two points at infinity, namely $[1, 1, 0]$ and $[1, -1, 0]$. Thus, the four arms of the hyperbola in the affine plane meet in two points, and the hyperbola also has the shape of an ellipse in the projective plane, $\mathbb{P}^2(K)$.

## C.5. Singular and smooth curves

We say that a projective curve $C : F(x, y, z) = 0$ is singular at a point $P \in C$ if and only if $\frac{\partial F}{\partial x}(P) = \frac{\partial F}{\partial y}(P) = \frac{\partial F}{\partial z}(P) = 0$. In other words, $C$ is singular at $P$ if the tangent vector at $P$ vanishes. Otherwise, we say that $C$ is non-singular at $P$. If $C$ is non-singular at every point, we say that $C$ is a smooth (or non-singular) curve.

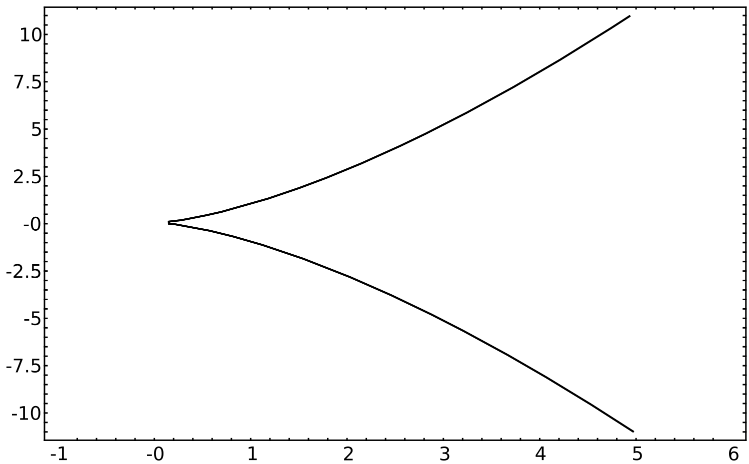


**Figure 2.** The chart $\{[x, y, 1]\}$ of the curve $zy^2 = x^3$.

For example, $C : zy^2 = x^3$ is singular at $P = [0, 0, 1]$ because $F(x, y, z) = zy^2 - x^3$ and:

$$\frac{\partial F}{\partial x} = -x^2, \ \frac{\partial F}{\partial y} = 2yz, \ \frac{\partial F}{\partial z} = y^2$$

Thus, $\frac{\partial F}{\partial x}(P) = \frac{\partial F}{\partial y}(P) = \frac{\partial F}{\partial z}(P) = 0$ for $P = [0, 0, 1]$.

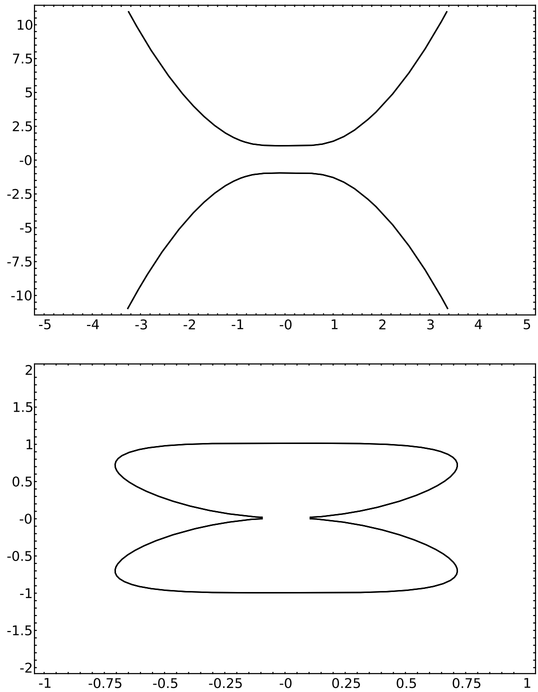


**Figure 3.** The chart $\{[x, y, 1]\}$ of the curve $z^2y^2 = x^4 + z^4$ (above, non-singular) and the chart $\{[x, 1, z]\}$ (below, singular).

Here is another example: the curve $D : z^2y^2 = x^4 + z^4$ has partial derivatives:

$$\frac{\partial F}{\partial x} = -4x^3, \ \frac{\partial F}{\partial y} = 2yz^2, \ \frac{\partial F}{\partial z} = 2y^2z - 4z^3.$$

Thus, if $P = [x, y, z] \in D(\mathbb{Q})$ is singular then

$$-4x^3 = 0, \quad 2yz^2 = 0, \quad \text{and} \ \ 2y^2z - 4z^3 = 0.$$

The first two equalities imply that $x = 0$ and $yz = 0$ (what would happen if we were working over a field of characteristic 2, such as $\mathbb{F}_2$?). If $y = 0$ then $z = 0$ by the third equation, but $[0, 0, 0]$ is not a well-defined point in $\mathbb{P}^2(\mathbb{Q})$ so this is impossible. However, if $x = z = 0$ then $y$ may take any value. Hence, $P = [0, 1, 0]$ is a singular point.

Notice that the affine curve that corresponds to the chart $z = 1$ of $D$, given by $y^2 = x^4 + 1$, is non-singular at all points in the affine plane, but it is singular at a point at infinity, namely $P = [0, 1, 0]$.

An elliptic curve of the form $E : y^2 = x^3 + Ax + B$, or in projective coordinates given by $zy^2 = x^3 + Axz^2 + Bz^3$, is non-singular if and only if $4A^3 + 27B^2 \neq 0$. The quantity $\Delta = -16 \cdot (4A^3 + 27B^2)$ is called the discriminant of $E$.