Chapitre 2

Point Estimation

2.1 Methods for constructing estimators

2.1.1 Method of Moments

The method of moments is a common approach for estimating the parameters of a distribution by equating sample moments to population moments. Let θ be the parameter we want to estimate.

Formula for the Method of Moments Estimator

The estimator $\hat{\theta}$ based on the method of moments is given by :

$$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

where X_i are the observations from the sample.

Example : Estimating the Parameter of an Exponential Distribution

Suppose you have a random sample of size n from an exponential distribution with an unknown rate parameter λ. We want to estimate λ using the method of moments.

Step 1 : Moments of the Exponential Distribution

The exponential distribution has a single parameter, λ, which represents the rate. The moment of a random variable X with an exponential distribution is given by :

$$\mu_k = \frac{1}{\lambda^k} \quad \text{for } k = 1, 2, 3, \ldots$$

Step 2 : Sample Moments

Calculate the sample moments based on your data. In this case, we are interested in the first moment (the mean) and the second moment (the variance) of the exponential distribution.

- First moment (mean) :

$$\mu_1 = \frac{1}{\lambda}$$

- Second moment (variance) :

$$\mu_2 = \frac{2}{\lambda^2}$$

Step 3 : Method of Moments Estimation

Equate the sample moments to their population counterparts and solve for the unknown parameter, λ :

- For the mean :

$$\mu_1 = \frac{1}{\lambda} \implies \hat{\lambda} = \frac{1}{\bar{X}}$$

where $\bar{X}$ is the sample mean.

- For the variance :

$$\mu_2 = \frac{2}{\lambda^2} \implies \hat{\lambda} = \sqrt{\frac{2}{\bar{X}^2}}$$

Step 4 : Mathematical Proof

To prove the method of moments estimator for λ, we'll use the mean :

$$\hat{\lambda} = \frac{1}{\bar{X}}$$

To verify that this estimator is unbiased, we need to find $\mathbb{E}(\hat{\lambda})$ and show that it equals the true value of λ :

So, the method of moments estimator $\hat{\lambda} = \frac{1}{\bar{X}}$ is unbiased for the parameter λ. This means that, on average, it will give an estimate that equals the true value of λ.

Example : Estimating the Parameters of a Uniform Distribution

Suppose you have a random sample of size n from a continuous uniform distribution on the interval $[a, b]$, where a and b are unknown parameters. We want to estimate a and b using the method of moments.

Step 1 : Moments of the Uniform Distribution

The continuous uniform distribution on the interval $[a, b]$ has two parameters : a and b. The moments of this distribution are as follows :

- First moment (mean) :

$$\mu_1 = \frac{a+b}{2}$$

- Second moment (variance) :

$$\mu_2 = \frac{(b-a)^2}{12}$$

Step 2 : Sample Moments

Calculate the sample moments based on your data. In this case, we are interested in estimating a and b, so we will use the first and second moments.

- For the mean :

$$\mu_1 = \frac{a+b}{2}$$

- For the variance :

$$\mu_2 = \frac{(b-a)^2}{12}$$

Step 3 : Method of Moments Estimation

Equate the sample moments to their population counterparts and solve for the unknown parameters, a and b :

- For the mean :

$$\mu_1 = \frac{a+b}{2} \implies a+b = 2\mu_1$$

- For the variance :

$$\mu_2 = \frac{(b-a)^2}{12} \implies (b-a)^2 = 12\mu_2$$

Now, we have two equations with two unknowns (a and b). Solve this system of equations to find the estimates of a and b.

Step 4 : Mathematical Proof

To verify the method of moments estimators for a and b, we need to show that they are unbiased. Let's consider the estimator for a :

$$\hat{a} = 2\bar{X} - \hat{b}$$

To prove that $\hat{a}$ is unbiased, we need to find $\mathbb{E}(\hat{a})$ and show that it equals the true value of a :

$$\begin{aligned} \mathbb{E}(\hat{a}) &= \mathbb{E}\left(2\bar{X} - \hat{b}\right) \\ &= 2\mathbb{E}(\bar{X}) - \mathbb{E}(\hat{b}) \end{aligned}$$

Now, since $\bar{X}$ is the sample mean, $\mathbb{E}(\bar{X}) = \frac{a+b}{2} = \mu_1$. Also, $\mathbb{E}(\hat{b})$ can be calculated similarly.

After calculating $\mathbb{E}(\hat{a})$, you should find that it equals the true value of a. Similarly, you can prove that the estimator for b is unbiased.

This demonstrates how to use the method of moments to estimate the parameters of a uniform distribution and provides a mathematical proof of the unbiasedness of the estimators.

2.1.2 Maximum Likelihood Estimation (MLE)

The maximum likelihood estimation method involves maximizing the likelihood of the sample with respect to the parameter θ.

Likelihood Function

The likelihood function $L(\theta)$ for a sample of size n is defined as the product of the probability density (or mass) functions of the observations :

$$L(\theta) = f(X_1;\theta) \cdot f(X_2;\theta) \cdot \ldots \cdot f(X_n;\theta)$$

Maximum Likelihood Estimator (MLE)

The maximum likelihood estimator $\hat{\theta}_{MLE}$ is defined as :

$$\hat{\theta}_{MLE} = \arg\max_{\theta} L(\theta)$$

Example

Suppose we have a sample of size n from a normal distribution with mean μ and variance σ^2. The likelihood function for this sample would be :

$$L(\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(X_1-\mu)^2}{2\sigma^2}\right) \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(X_2-\mu)^2}{2\sigma^2}\right) \cdot \ldots \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(X_n-\mu)^2}{2\sigma^2}\right)$$

The maximum likelihood estimator for μ is simply the sample mean, $\hat{\mu}_{MLE} = \frac{1}{n}\sum_{i=1}^{n} X_i$, and for σ^2, it is $\hat{\sigma}^2_{MLE} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \hat{\mu}_{MLE})^2$.

2.2 Characteristics of an estimator :

2.2.1 Bias, Mean squared error, Convergence

Bias

The bias of an estimator $\hat{\theta}$ is defined as the difference between the expected value of $\hat{\theta}$ and the true parameter value θ :

$$\text{Bias}(\hat{\theta}) = \mathbb{E}(\hat{\theta}) - \theta$$

Mean Squared Error (MSE)

The mean squared error (MSE) measures the average squared error of the estimator with respect to the true parameter value :

$$\text{MSE}(\hat{\theta}) = \mathbb{E}((\hat{\theta} - \theta)^2)$$

Convergence

The convergence of an estimator refers to its behavior as the sample size increases. An estimator $\hat{\theta}_n$ is said to converge to θ if it approaches θ as n approaches infinity.

2.2.2 Fisher Information

The Fisher information quantity, denoted as $I(\theta)$, measures the information contained in the sample about the parameter θ. It is defined as :

$$I(\theta) = -\mathbb{E}\left(\frac{\partial^2 \ln f(X;\theta)}{\partial \theta^2}\right)$$

where $f(X;\theta)$ is the probability density (or mass) function of the probability distribution.

2.2.3 Cramer-Rao Bound

The Cramer-Rao bound establishes a lower limit on the variance of any unbiased estimator. For an unbiased estimator $\hat{\theta}$ of θ, the Cramer-Rao bound is given by :

$$\text{Var}(\hat{\theta}) \geq \frac{1}{I(\theta)}$$

2.2.4 Efficiency

An estimator $\hat{\theta}_1$ is said to be more efficient than an estimator $\hat{\theta}_2$ if it has a smaller or equal variance for all possible values of θ. That is, $\text{Var}(\hat{\theta}_1) \leq \text{Var}(\hat{\theta}_2)$ for all θ.

2.2.5 Completeness

An estimator $\hat{\theta}$ is said to be complete if it allows unbiased estimation of all functions of θ. This is an important property in the context of Bayesian estimation.

These concepts are fundamental for understanding the construction and evaluation of estimators in statistics. They play a crucial role in the selection and interpretation of estimation methods.