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DISCRETE DYNAMICAL SYSTEM .

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CONTENTS

CHAPTER 1

THE STABILITY OF ONE-DIMENSIONAL MAPS

Introduction

Difference equations have been increasingly used as mathematical models in many disciplines including genetics, eipdemiology, ecology, physiology, neural networks, psychology, engineering, physics, chemistry and social sciences. Their amenability to computerization and their mathematical simplicity have attracted researchers from a wide range of disciplines. As we will see in Section 1.2, difference equations are generated by maps (functions). Section 1.3 illustrates how discretizing a differential equation would yeild a difference equation. Discretization algorithms are part of a discipline called numerical analysis which belong to both mathematics and computer science. As most differential equations are unsolvable, one needs to resort to computers for help. However, computers understand only recursions or difference equations; thus the need to discretize differential equations.

1.1 Maps vs. Difference Equations

Consider a map $f:\mathbb{R}\to\mathbb{R}$ where $\mathbb{R}$ is the set of real numbers. Then the (positive) orbit $O(x_0)$ of a point $x_0\in\mathbb{R}$ is defined to be the set of points

$$O(x_0)=\left\{x_0, f(x_0), f^2(x_0), f^3(x_0), \ldots\right\}$$

where $f^2 = f \circ f, f^3 = f \circ f \circ f$, etc.

Since most maps that we deal with are noninvertible, positive orbits will be called orbits, unless otherwise stated.

If we let $x(n) := f^n(x_0)$, then we obtain the first-order difference equation

$$x(n+1) = f(x(n)) \tag{1.1}$$

with $x(0) = x_0$.

In population biology, $x(n)$ may represent a population size in generation n. Equation (1.1) models a simple population system with seasonal breeding whose generations do not overlap (e.g., orchard pests and temperate zone insects). It simply states that the size $x(n+1)$ of a population in generation $n+1$ is related to the size $x(n)$ of the population in the preceding generation n by the function f.

In epidemiology, $x(n)$ represents the fraction of the population infected at time n. In economics, $x(n)$ may be the price per unit in time n for a certain commodity. In the social sciences, $x(n)$ may be the number of bits of information that can be remembered after a period n.

Example 1.1.1 ***(The Logistic Map).*** *Let $x(n)$ be the size of a population of a certain species at time n. Let μ be the rate of growth of the population from one generation to another. Then a mathematical model that describes the size of the population take the form*

$$x(n+1) = \mu x(n), \mu > 0 \tag{1.2}$$

If the initial population $x(0) = x_0$, then by a simple iteration we find that

$$x(n) = \mu^n x_0 \tag{1.3}$$

is the solution of Equation (1.2).

If $\mu > 1$, then the population $x(n)$ increases without any bound to infinity. If $\mu = 1, x(n) = x_0$ and the population stays constant forever. Finally, for $\mu < 1, \lim_{n\to\infty} x(n) = 0$, and the population eventually becomes extinct.

We observe that for most species none of the above scenarios are valid; the population increases until it reaches a certain maximum value. Then limited resources would force members of the species to fight and compete for those limited resources. This competition is proportional to the number of squabbles among them, given by $x^2(n)$. Consequently, a more reasonable model is given by

$$x(n+1) = \mu x(n) - bx^2(n) \tag{1.4}$$

where $b > 0$ is the proportionality constant of interaction among members of the species.

To simplify Equation (1.4), we let $y(n) = \frac{b}{\mu}x(n)$. *Hence,*

$$y(n+1) = \mu y(n)(1 - y(n)) \tag{1.5}$$

Equation (1.5) is called the logistic equation and the map $f(y) = \mu y(1-y)$ *is called the logistic map. It is a reasonably good model for seasonably breeding populations in which generations do not overlap.*

This equation/map will be the focus of our study throughout Chapter 1. By varying the value of μ, *this innocent-looking equation/map exhibits complicated dynamics.*

Surprisingly, a closed form solution of Equation (1.5) is not possible, except for $\mu = 2, 4$.

A map f *is called linear if it is of the form* $f(x) = ax$ *for some constant* a. *In this case, Equation (1.1) is called a first-order linear difference equation. Otherwise,* f *[or Equation (1.1)] is called nonlinear (or density-dependent in biology).*

One of the main objectives in dynamical systems theory is the study of the behavior of the orbits of a given map or a class of maps. In the language of difference equations, we are interested in investigating the behavior of solutions of Equation (1.1). By a solution of Equation (1.1), we mean a sequence $\{\varphi(n)\}, n = 0, 1, 2, \ldots$, *with* $\varphi(n+1) = f(\varphi(n))$ *and* $\varphi(0) = x_0$, *i.e., a sequence that satisfies the equation.*

1.2 Maps vs. Differential Equations

1.2.1 Euler's Method

Consider the differential equation

$$x'(t) = g(x(t)), x(0) = x_0 \tag{1.6}$$

where $x'(t) = \dfrac{dx}{dt}$.

For many differential equations such as Equation (1.6), it may not be possible to find a "closed form" solution. In this case, we resort to numerical methods to approximate the solution of Equation (1.6). In the Euler algorithm, for example, we start with a discrete set of points $t_0, t_1, \ldots, t_n, \ldots$, with $h = t_{n+1} - t_n$ as the step size. Then, for $t_n \le t < t_{n+1}$, we approximate $x(t)$ by $x(t_n)$ and $x'(t)$ by $\frac{x(t_{n+1}) - x(t_n)}{h}$. Equation (1.6) now yields the difference equation

$$x(t_{n+1}) = x(t_n) + hg(x(t_n))$$

which may be written in the simpler form

$$x(n+1) = x(n) + hg(x(n)) \tag{1.7}$$

where $x(n) = x(t_n)$.

Note that Equation (1.7) is of the form of Equation (1.1) with

$$f(x) = f(x, h) = x + hg(x)$$

Now given the initial data $x(0) = x_0$, we may use Equation (1.7) to generate the values $x(1), x(2), x(3), \ldots$ These values approximate the solution of the differential Equation (1.6) at the "grid" points $t_1, t_2, t_3, \ldots$, provided that h is sufficiently small.

Example 1.2.1 *Let us now apply Euler's method to the differential equation:*

$$x'(t) = 0.7x^2(t) + 0.7, \quad x(0) = 1, \quad t \in [0, 1]. \quad (DE)^1$$

n	t	(ΔE) Euler ($h = 0.2$) $x(n)$	(ΔE) Euler ($h = 0.1$) $x(n)$	Exact (DE) $x(t)$
0	0	1	1	1
1	0.1		1.14	1.150
2	0.2	1.28	1.301	1.328
3	0.3		1.489	1.542
4	0.4	1.649	1.715	1.807
5	0.5		1.991	2.150
6	0.6	2.170	2.338	2.614
7	0.7		2.791	3.286
8	0.8	2.969	3.406	4.361
9	0.9		4.288	6.383
10	1	4.343	5.645	11.681

Table 1.1

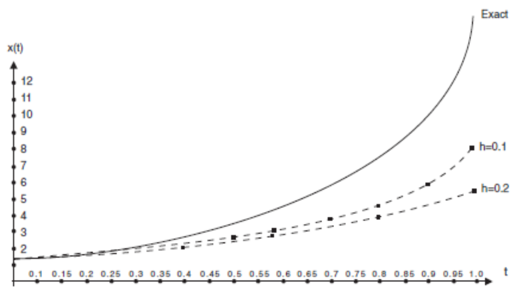


FIGURE 1.1

Comparison of exact and approximate numerical solutions for Example 1.2.

Using the separation of variable method, we obtain

$$\frac{1}{0.7} \int \frac{dx}{x^2 + 1} = \int dt$$

Hence

$$\tan^{-1}(x(t)) = 0.7t + c$$

Letting $x(0) = 1$, *we get* $c = \frac{\pi}{4}$. *Thus, the exact solution of this equation is given by* $x(t) = \tan\left(0.7t + \frac{\pi}{4}\right)$.

The corresponding difference equation using Euler's method is

$$x(n+1) = x(n) + 0.7h\left(x^2(n) + 1\right), \quad x(0) = 1. \quad (\Delta E)^2$$

Table 1.1 shows the Euler approximations for $h = 0.2$ *and 0.1 , as well as the exact values. Figure 1.1 depicts the* $n - x(n)$ *diagram or the "time series." Notice that the smaller the step size we use, the better the approximation we have.*

Note that discretization schemes may be applied to nonlinear and higher order differential equations.

Example 1.2.2 ***(An Insect Population).*** *Let us contemplate a population of aphids. These are plant lice, soft bodied, pear shaped insects which are commonly found on nearly all indoor and outdoor plants, as well as vegetables, field crops, and fruit trees.*

Let

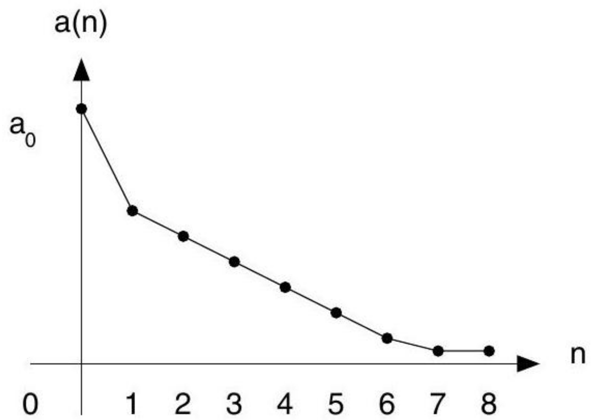
$$\begin{aligned} a(n) &= \text{number of adult females in the nth generation,} \\ p(n) &= \text{number of progeny (offspring) in the nth generation,} \\ m &= \text{fractional mortality in the young aphids,} \\ q &= \text{number of progeny per female aphid,} \\ r &= \text{ratio of female aphids to total adult aphids.} \end{aligned}$$

Since each female produces q progeny, it follows that

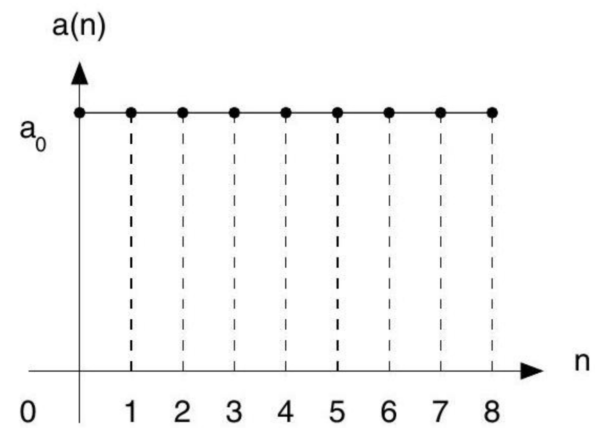
$$p(n+1) = qa(n) \tag{1.8}$$

Now of these $p(n+1)$ *progeny,* $rp(n+1)$ *are female young aphids of which* $(1-m)rp(n+1)$ *survives to adulthood. Thus*

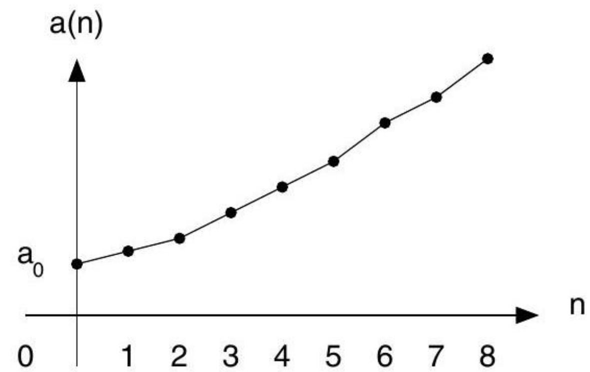
$$a(n+1) = r(1-m)p(n+1) \tag{1.9}$$



(i)



(ii)



(iii)

FIGURE 1.2

(i) $a(n)$ *goes to extinction.*

(ii) $a(n) = a_0$, *constant population.*

(iii) $a(n) \to \infty$ *as* $n \to \infty$, *exponential growth.*

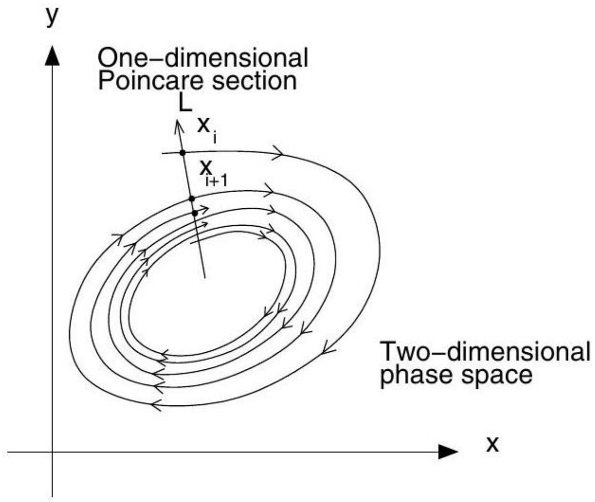


FIGURE 1.3 *The Poincaré map is defined by* $P(x_i) = x_{i+1}$.

Substituting from Equation (1.8) yields

$$a(n+1) = rq(1-m)a(n) \tag{1.10}$$

Hence

$$a(n) = [rq(1-m)]^n a(0) \tag{1.11}$$

There are three cases to consider.

(i) If $rq(1-m) < 1$, *then* $\lim_{n\to\infty} a(n) = 0$ *and the population of aphids goes to extinction.*

(ii) If $rq(1-m) = 1$, *then* $a(n) = a_0$, *and we have a constant population size.*

(iii) If $rq(1-m) > 1$, *then* $\lim_{n\to\infty} a(n) = \infty$, *and the population grows exponentially to* ∞.

1.2.2 Poincaré Map

One of the most interesting ways on which a differential equation leads to a map, called a Poincaré map, is through the study of periodic solutions of a system of two differential equations

$$\frac{dx}{dt} = f(x,y)$$
$$\frac{dy}{dt} = g(x,y)$$

which has a periodic orbit (closed curve) in the plane. Now choose a line L that intersects this periodic orbit at a right angle. For any x_0 on the line L, $x_1 = P(x_0)$ is the point of intersection of the orbit starting at x_0 after it returns to the line L for the first time. Consequently, x_i is the intersection point of the orbit starting at x_0 after it returns to the line L for the i th time. This defines the Poincaré map associated with our differential equation (Figure 1.3).

1.3 Linear Maps/Difference Equations

The simplest maps to deal with are the linear maps and the simplest difference equations to solve are the linear ones. Consider the linear map

$$f(x) = ax$$

then

$$f^n(x) = a^n x$$

In other words, the solution of the difference equation

$$x(n+1) = ax(n), x(0) = x_0 \tag{1.12}$$

is given by

$$x(n) = a^n x_0 \tag{1.13}$$

We can make the following conclusions about the limiting behavior of the orbits of f or the solutions of Equation (1.12):

1. If $|a| < 1$, then $\lim_{n\to\infty} \left|f^n(x_0)\right| = 0$ (or $\lim_{n\to\infty} |x(n)| = 0$) [see Fig. 1.4 (b) and (c)].
2. If $|a| > 1$, then $\lim_{n\to\infty} \left|f^n(x_0)\right| = \infty$ (or $\lim_{n\to\infty} |x(n)| = \infty$) if $x_0 \neq 0$ [see Fig. 1.4 (a) and (d)].
3. (a) If $a = 1$, then f is the identity map where every point is a fixed point of f.

(b) If $a = -1$, then $f^n(x_0) = \begin{cases} x_0 & \text{if } n \text{ is even} \\ -x_0 & \text{if } n \text{ is odd} \end{cases}$ and the solution $x(n) = (-1)^n x_0$ of Equation (1.12) is said to be periodic of period 2 .

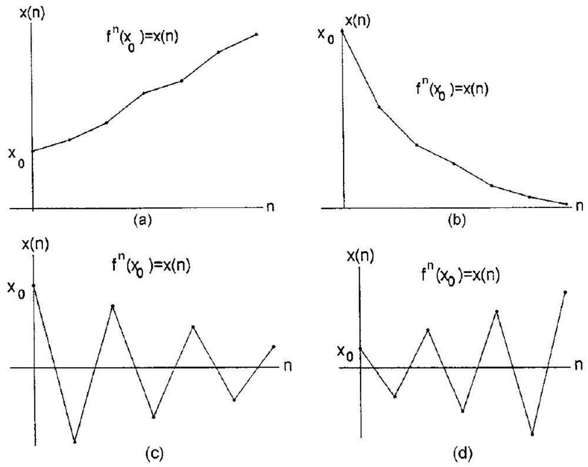


FIGURE 1.4 Time series $[n - x(n)]$ graphs (a) $a = 1.2$, (b) $a = 0.7$, (c) $a = -0.7$, (d) $a = -1.2$. Solutions of Eqs. (1.12) for different values of the parameter a.

Next, let us look at the affine map $f(x) = ax + b$. By successive iteration, we get

$$\begin{aligned} f^2(x) &= a^2x + ab + b \\ f^3(x) &= a^3x + a^2b + ab + b \\ &\vdots \\ f^n(x) &= a^nx + \sum_{j=0}^{n-1} a^{n-j-1}b \end{aligned}$$

In other words, the solution of the difference equation

$$x(n+1) = ax(n) + b, x(0) = x_0 \tag{1.14}$$

is given by

$$\begin{aligned} x(n) &= a^nx_0 + \sum_{j=0}^{n-1} a^{n-j-1}b \\ &= a^nx_0 + b\left(\frac{a^n - 1}{a - 1}\right), \text{ if } a \neq 1 \end{aligned} \tag{1.15}$$

$$x(n) = \left(x_0 + \frac{b}{a-1}\right)a^n + \frac{b}{1-a}, \qquad \text{if } a \neq 1. \tag{1.16}$$

Using the formula of Equation (1.16), the following conclusions can be easily verified:

1. If $|a| < 1$, then $\lim_{n\to\infty} f^n(x_0) = \frac{b}{1-a}$ (or $\lim_{n\to\infty} x(n) = \frac{b}{1-a}$).
2. If $|a| > 1$, then $\lim_{n\to\infty} f^n(x_0) = \pm\infty$, depending on whether $x_0 + \frac{b}{a-1}$ is positive or negative, respectively.

3. (a) If $a = 1$, then $f^n(x_0) = x_0 + nb$, which tends to ∞ or $-\infty$ as $n \to \infty$ (or $x(n) = x_o + nb$).

(b) If $a = -1$, then $f^n(x_0) = (-1)^n x_0 + \begin{cases} b & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$

$$\left(\text{or} \quad x(n) = (-1)^n x_0 + \begin{cases} b & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}\right).$$

Notice that the solution of the differential equation

$$\frac{dx}{dt} = ax(t), \quad x(0) = x_0$$

is given by

$$x(t) = e^{at} x_0 \tag{1.17}$$

Comparing (1.14) and (1.17) we see that the exponential e^{at} in the differential equation corresponds to a^n, the nth power of a, in the difference equation. The solution of the nonhomogeneous differential equation

$$\frac{dx}{dt} = ax(t) + b, \quad x(0) = x_0 \tag{1.18}$$

is given by

$$\begin{aligned} x(t) &= e^{at} x_0 + \int_0^t e^{a(t-s)} b\,ds \\ &= e^{at} x_0 + \frac{b}{a}\left(e^{at} - 1\right) \\ &= \left(x_0 + \frac{b}{a}\right) e^{at} - \frac{b}{a}. \end{aligned} \tag{1.19}$$

In cases 1, 2, 3, the behavior of the difference equation (1.15) depends on whether a is inside the interval $(-1, 1)$, on its boundary, or outside it. However for differential equations, the behavior of the solution of Equation (1.18) depends on whether $a < 0, a = 0$, or $a > 0$, respectively. Consequently,

1. $a < 0, \lim_{t\to\infty} x(t) = -\frac{b}{a}$ as $e^{at} \to 0$ as $t \to \infty$,

2. $a = 0, x(t) = x_0$ since $\frac{dx}{dt} = 0$,

3. $a > 0, \lim_{t\to\infty} x(t) = \infty$ since $e^{at} \to \infty$ since $t \to \infty$.

Example 1.3.1 *A drug is administered every six hours. Let $D(n)$ be the amount of the drug in the blood system at the nth interval. The body eliminates a certain fraction p of the drug during each time interval. If the amount administered is D_0, find $D(n)$ and $\lim_{n\to\infty} D(n)$.*

SOLUTION The first step in solving this example is to write down a difference equation that relates the amount of drug in the patient's system $D(n+1)$ at the time interval $(n+1)$ with $D(n)$. Now, the amount of drug $D(n+1)$ is equal to the amount $D(n)$ minus the fraction p of $D(n)$ that has been eliminated from the body plus the new dose D_0. This yields

$$D(n+1) = (1-p)D(n) + D_0.$$

From Equations (1.14) and (1.15), we obtain

$$\begin{aligned} D(n) &= (1-p)^n D_0 + D_0\left(\frac{1-(1-p)^n}{p}\right) \\ &= \left(D_0 - \frac{D_0}{p}\right)(1-p)^n + \frac{D_o}{p}. \end{aligned}$$

Thus,

$$\lim_{n\to\infty} D(n) = \frac{D_o}{p}.$$

1.4 Fixed (Equilibrium) Points

In Section 1.4, we were able to obtain closed form solutions of first-order linear difference equations. In other words, it was possible to write down an explicit formula for points $f^n(x_0)$ in the orbit of a point x_0 under the linear or

affine map f. However, the situation changes drastically when the map f is nonlinear. For example, one cannot find a closed form solution for the simple difference equation $(\Delta E): x(n+1) = \mu x(n)(1-x(n))$, except when $\mu = 2$ or 4 . For those of you who are familiar with first-order differential equations, this may be rather shocking. We may solve the corresponding differential equation $\left(DE^4 : x'(t) = \lambda x(t)(1-x(t))\right)$ by simply separating the variables x and t and then integrating both sides of the equation. The solution of (DE) may be written in the form

$$x(t) = \frac{x_0 e^{\lambda t}}{1 + x_0\left(e^{\lambda t} - 1\right)}.$$

Note that the behavior of this solution is very simple: for $\lambda > 0$, $\lim_{t\to\infty} x(t) = 1$ and for $\lambda < 0$, $\lim_{t\to\infty} x(t) = 0$. Unlike those of (DE), the behavior of solutions of (ΔE) is extremely complicated and depends very much on the values of the parameter μ. Since we cannot, in general, solve (ΔE), it is important to develop qualitative or graphical methods to determine the behavior of their orbits. Of particular importance is finding orbits that consist of one point. Such points are called **fixed points**, or **equilibrium points**

(steady states).

Let us consider again the difference equation

$$x(n+1) = f(x(n)). \tag{1.20}$$

Definition 1.4.1 *A point x^* is said to be a fixed point of the map f or an equilibrium point of Equation (1.20) if $f(x^*) = x^*$.*

Note that for an equilibrium point x^*, the orbit is a singleton and consists of only the point x^*. Moreover, to find all equilibrium points of Equation (1.20), we must solve the equation $f(x) = x$. Graphically speaking, a fixed point of a map f is a point where the curve $y = f(x)$ intersects the diagonal line $y = x$. For example, the fixed points of the cubic map $f(x) = x^3$ can be obtained by solving the equation $x^3 = x$ or $x^3 - x = 0$. Hence, there are three fixed points -1, 0, 1 for this map (see Fig. 1.5).

Closely related to fixed points are the eventually fixed points. These are the points that reach a fixed point after finitely many iterations. More explicitly, a point x is said to be an eventually fixed point of a map f if there exists a positive integer r and a fixed point x^* of f such that $f^r(x) = x^*$, but $f^{r-1}(x) \neq x^*$.

We denote the set of all fixed points by Fix (f), the set of all eventually fixed points by EFix (f), and the set of all eventually fixed points of the fixed points x^* by $EFix_x \cdot (f)$.

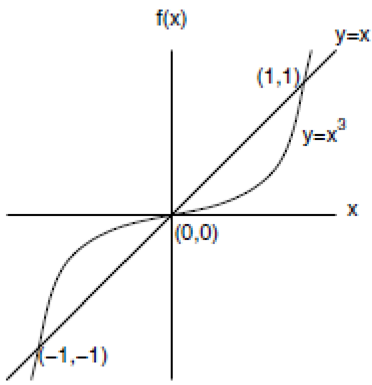


***FIGURE 1.5** The fixed points of $f(x) = x^3$ are the intersection points with the diagonal line.*

Given a fixed point x^* of a map f, then one can easily construct eventually fixed points by computing the ancestor set $f^{-1}(x^*) = \{x \neq x^* : f(x) = x^*\}$, $f^{-2}(x^*) = \left\{x : f^2(x) = x^*\right\}, \ldots, f^{-n}(x^*) = \{x : f^n(x) = x^*\}, \ldots$. Thus one may show that

$$EFix_{x^*}(f) = \{x : f^n(x) = x^*, \quad n \in \mathbb{Z}^+\}. \tag{1.21}$$

Note that the set $EFix(f) \setminus \{x^*\}$ may be empty, finite, or infinite as demonstrated by the following example.

Example 1.4.1 ***(i)*** *Consider the logistic map $f(x) = 2x(1-x)$. Then there are two fixed points $x^* = 0$ and $y^* = \frac{1}{2}$.*

A simple computation reveals that

$$f^{-1}(x) = \frac{1}{2}[1 \pm \sqrt{1-2x}].$$

Thus $f^{-1}\left(\frac{1}{2}\right) = \frac{1}{2}$ *and* $EFix_{y^*}(f)\backslash\left\{\frac{1}{2}\right\} = \emptyset$. *Moreover,* $f^{-1}(0) = \{0,1\}$, *and* $EFix_x * (f) = \{0,1\}$. *We conclude that we have only one "genuine" eventually fixed point, namely* $x = 1$.

(ii) *Let us now contemplate a more interesting example,* $f(x) = 4x(1-x)$. *There are two fixed points,* $x^* = 0$, *and* $y^* = \frac{3}{4}$. *Clearly* $EFix_{x^*}(f) = \{0,1\}$. *Notice that* $f^{-1}(x) = \frac{1}{2}[1 \pm \sqrt{1-x}]$. *Hence*

$$f^{-1}\left(\frac{3}{4}\right) = \frac{1}{2}\left[1 \pm \sqrt{1-\frac{3}{4}}\right] = \frac{1}{2}\left[1 \pm \frac{1}{2}\right]$$

which equals either $\frac{3}{4}$ *or* $\frac{1}{4}$. *Now* $f^{-1}\left(\frac{1}{4}\right) = \frac{1}{2}\left[1 \pm \sqrt{1-\frac{1}{4}}\right]$ *which equals either* $\frac{1}{2}\left[1+\frac{\sqrt{3}}{2}\right]$ *or* $\frac{1}{2}\left[1-\frac{\sqrt{3}}{2}\right]$. *Repeating this process we may generate an infinitely many eventually fixed point, that is the set* $EFix_{y^*}(f)$ *is infinite. The following diagram shows some of the eventually fixed points.*

$$\begin{aligned} 1 &\rightarrow 0 \\ \frac{1}{4} &\rightarrow \frac{3}{4} \\ \left(\frac{1}{2}-\frac{\sqrt{3}}{4}\right) \rightarrow \frac{1}{4} &\rightarrow \frac{3}{4} \\ \left(\frac{1}{2}+\frac{\sqrt{3}}{4}\right) \rightarrow \frac{1}{4} &\rightarrow \frac{3}{4} \\ \left[\frac{1}{2}-\frac{1}{2}\sqrt{\frac{1}{2}+\frac{\sqrt{3}}{2}}\right] \rightarrow \left[\frac{1}{2}-\frac{\sqrt{3}}{2}\right] \rightarrow \frac{1}{4} &\rightarrow \frac{3}{4} \end{aligned}$$

It is interesting to npte that the phenomenon of eventually fixed points does not have a counterpart in differential equations, since no solution can reach an equilibrium point in a finite time.

Next we introduce one of the most interesting examples in discrete dynamical systems: the tent map T.

Example 1.4.2 *(The Tent Map). The tent map* T *is defined as*

$$T(x) = \begin{cases} 2x, & \text{for } 0 \le x \le \frac{1}{2} \\ 2(1-x), & \text{for } \frac{1}{2} < x \le 1. \end{cases}$$

This map may be written in the form

$$T(x) = 1 - 2\left|x - \frac{1}{2}\right|.$$

Note that the tent map is a piecewise linear map (see Fig. 1.6). The tent map possesses a rich dynamics and in Chapter 3 we show it is in fact "chaotic." There are two equilibrium points $x_1^* = 0$ *and* $x_2^* = \frac{2}{3}$. *Moreover, the point* $\frac{1}{4}$ *is an eventual equilibrium point since* $T\left(\frac{1}{4}\right) = \frac{1}{2}, T^2\left(\frac{1}{4}\right) = T\left(\frac{1}{2}\right) = 1, T^3\left(\frac{1}{4}\right) \overset{4}{=} T(1) = 0$. *It is left to you* *to show that if $x = \frac{k}{2^n}$, where k, and n are positive*

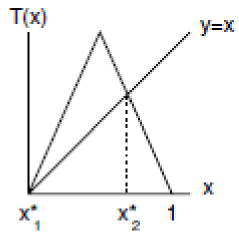


FIGURE 1.6 *The tent map has two fixed points $x_1^* = 0$ and $x_2^* = \frac{2}{3}$.*

*integers with $0 < \frac{k}{2^n} \leq 1$, then x is an eventually fixed point (Problem 9). Numbers of this form are called **dyadic rationals.***

Remark 1.4.1 *Note that not every map has a fixed point. For example, the map $f(x) = x + 1$ has no fixed points since the equation $x + 1 = x$ has no solution.*

Now, our mathematical curiosity would lead to the following question: under what conditions does a map have a fixed point. Well, for continuous maps, there are two simple and interesting results that guarantee the presence of fixed points.

Theorem 1.4.1 *Let $f : I \to I$ be a continuous map, where $I = [a, b]$ is a closed interval in $\mathbb{R}$. Then, f has a fixed point.*

Proof. Define $g(x) = f(x) - x$. Then, $g(x)$ is also a continuous map. If $f(a) = a$ or $f(b) = b$, we are done. So assume that $f(a) \neq a$ and $f(b) \neq b$. Hence, $f(a) > a$ and $f(b) < b$. Consequently, $g(a) > 0$ and $g(b) < 0$. By the intermediate value theorem, there exists a point $c \in (a, b)$ with $g(c) = 0$. This implies that $f(c) = c$ and c is thus a fixed point of f.

The above theorem says that for a continuous map f if $f(I) \subset I$, then f has a fixed point in I. The next theorem gives the same assertion if $f(I) \supset I$. ■

Theorem 1.4.2 *Let $f : I = [a, b] \to \mathbb{R}$ be a continuous map such that $f(I) \supset I$. Then f has a fixed point in I.*

Proof. The proof is left to the reader as Problem 10. Even if fixed points of a map do exist, it is sometimes not possible to compute them algebraically. For example, to find the fixed points of the map $f(x) = 2 \sin x$, one needs to solve the transcendental equation $2 \sin x - x = 0$. Clearly $x = 0$ is a root of this equation and thus a fixed point of the map f. However, the other two fixed points may be found by graphical or numerical methods. They are approximately ± 1.944795452. ■

1.5 Graphical Iteration and Stability

One of the main objectives in the theory of dynamical systems is the study of the behavior of orbits near fixed points, i.e., the behavior of solutions of a difference equation near equilibrium points. Such a program of investigation is called stability theory, which henceforth will be our main focus. We begin our exposition by introducing the basic notions of stability. Let $\mathbb{Z}^+$denote the set of nonnegative integers.

Definition 1.5.1 *Let $f : I \to I$ be a map and x^* be a fixed point of f, where I is an interval in the set of real numbers $\mathbb{R}$. Then*

1. *x^* is said to be **stable** if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x_0 \in I$ with $|x_0 - x^*| < \delta$ we have $\left|f^n(x_0) - x^*\right| < \varepsilon$ for all $n \in \mathbb{Z}^+$. Otherwise, the fixed point x^* will be called unstable (see Figs. 1.7 and 1.8).*

2. *x^* is said to be **attracting** if there exists $\eta > 0$ such that $|x_0 - x^*| < \eta$ implies $\lim_{n\to\infty} f^n(x_0) = x^*$ (see Fig. 1.9).*

3. *x^* is **asymptotically stable** if it is both stable and attracting (see Fig. 1.10). If in (2) $\eta = \infty$, then x^* is said to be **globally asymptotically stable.***

Henceforth, unless otherwise stated, "stable" (asymptotically stable) always means "locally stable" (asymptotically stable).

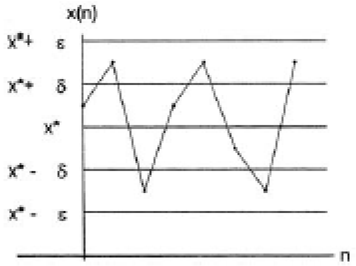


FIGURE 1.7
Stable fixed point x^*.

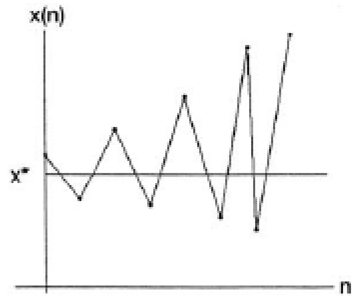


FIGURE 1.8
Unstable fixed point x^*.

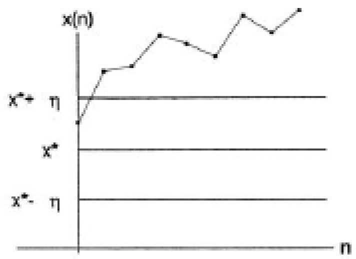


FIGURE 1.9
Unstable nonoscillating fixed point x^*.

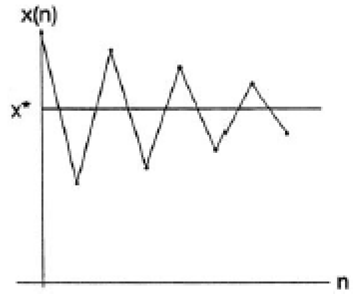


FIGURE 1.10
Asymptotically stable fixed point x^*.

The Cobweb Diagram:

One of the most effective graphical iteration methods to determine the stability of fixed points is the **cobweb diagram**. On the $x - y$ plane, we draw the curve $y = f(x)$ and the diagonal line $y = x$ on the same plot (see Fig. 1.11).

We start at an initial point x_0. Then we move vertically until we hit the graph of f at the point $(x_0, f(x_0))$. We then travel horizontally to meet the line $y = x$ at the point $(f(x_0), f(x_0))$. This determines $f(x_0)$ on the x axis. To find $f^2(x_0)$, we move again vertically until we strike the graph of f at the point $\left(f(x_0), f^2(x_0)\right)$; and then we move horizontally to meet the line $y = x$ at the point $\left(f^2(x_0), f^2(x_0)\right)$. Continuing this process, we can evaluate all of the points in the orbit of x_0, namely, the set $\left\{x_0, f(x_0), f^2(x_0), \ldots, f^n(x_0), \ldots\right\}$ (see Fig. 1.11).

Example 1.5.1 *Use the cobweb diagram to find the fixed points for the quadratic map $Q_c(x) = x^2 + c$ on the interval* $[-2, 2]$, *where* $c \in [-2, 0]$. *Then determine the stability of all fixed points.*

SOLUTION To find the fixed point of Q_c, we solve the equation $x^2 + c = x$ or $x^2 - x + c = 0$. This yields the two fixed points $x_1^* = \frac{1}{2} - \frac{1}{2}\sqrt{1-4c}$ and $x_2^* = \frac{1}{2} + \frac{1}{2}\sqrt{1-4c}$. Since we have not developed enough machinery to treat the general case for arbitrary c, let us examine few values of c. We begin with $c = -0.5$ and an initial point $x_0 = 1.1$. It is clear from Fig. 1.12 that the fixed point $x_1^* = \frac{1}{2} - \frac{\sqrt{3}}{2} \approx -0.366$ is asymptotically stable, whereas the second fixed point $x_2^* = \frac{1}{2} + \frac{\sqrt{3}}{2} \approx 1.366$ is unstable.

Example 1.5.2 *Consider again the tent map of Example 1.4.2. Find the fixed points and determine their stability.*

SOLUTION The fixed points are obtained by putting $2x = x$ and $2(1 - x) = x$. From the first equation, we obtain the first fixed point $x_1^* = 0$; and from the second equation, we obtain the second fixed point $x_2^* = \frac{2}{3}$. Observe from the cobweb diagram (Fig. 1.13) that both fixed points are unstable.

Remark 1.5.1 *If one uses the language of difference equations, then in the Cobweb diagrams, the x-axis is labeled $x(n)$ and the y-axis is labeled $x(n+1)$.*

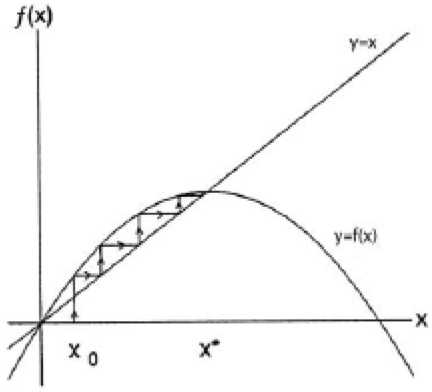


FIGURE 1.11
The Cobweb diagram: asymptotically stable fixed point x^*, $\lim_{n\to\infty} f^n(x_0) = x^*$.

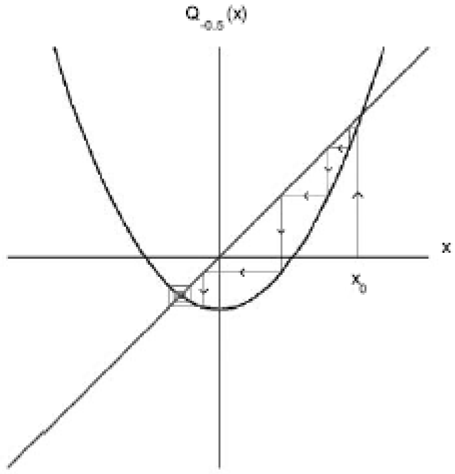


FIGURE 1.12
The Cobweb diagram of $Q_{-0.5}$: x_1^* is asymptotically stable but x_2^* is unstable.

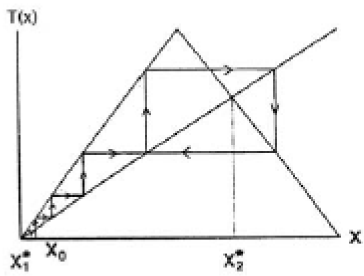


FIGURE 1.13
Both equilibrium points $x_1^* = 0$ and $x_2^* = \frac{2}{3}$ are unstable.

1.6 Criteria for Stability

In this section, we will establish some simple but powerful criteria for local stability of fixed points. Fixed (equilibrium) points may be divided into two types: **hyperbolic** and **nonhyperbolic**. A fixed point x^* of a map f is said to be hyperbolic if $\left|f'(x^*)\right| \neq 1$. Otherwise, it is nonhyperbolic. We will treat the stability of each type separately.

1.6.1 Hyperbolic Fixed Points

The following result is the main tool in detecting local stability.

Theorem 1.6.1 *Let x^* be a hyperbolic fixed point of a map f, where f is continuously differentiable at x^*. The following statements then hold true:*

1. *If $\left|f'(x^*)\right| < 1$, then x^* is asymptotically stable.*
2. *If $\left|f'(x^*)\right| > 1$, then x^* is unstable.*

Proof. Suppose that $\left|f'(x^*)\right| < M < 1$ for some $M > 0$. Then, there is an open interval $I = (x^* - \varepsilon, x^* + \varepsilon)$ such that $\left|f'(x)\right| \leq M < 1$ for
all $x \in I$ (Why? Problem 10). By the mean value theorem, for any $x_0 \in I$, there exists c between x_0 and x^* such that

$$\left|f(x_0) - x^*\right| = \left|f(x_0) - f(x^*)\right| = \left|f'(c)\right| |x_0 - x^*| \leq M\,|x_0 - x^*|. \tag{1.22}$$

Since $M < 1$, inequality (1.22) shows that $f(x_0)$ is closer to x^* than x_0. Consequently, $f(x_0) \in I$. Repeating the above argument on $f(x_0)$ instead of x_0, we can show that

$$\left|f^2(x_0) - x^*\right| \leq M\left|f(x_0) - x^*\right| \leq M^2\,|x_0 - x^*|. \tag{1.23}$$

By mathematical induction, we can show that for all $n \in \mathbb{Z}^+$,

$$\left|f^n(x_0) - x^*\right| \leq M^n\,|x_0 - x^*|. \tag{1.24}$$

To prove the stability of x^*, for any $\varepsilon > 0$, we let $\delta = \min(\varepsilon, \tilde{\varepsilon})$. Then, $|x_0 - x^*| < \delta$ implies that $\left|f^n(x_0) - x^*\right| \leq M^n\,|x_0 - x^*| < \varepsilon$, which establishes stability. Furthermore, from Inequality (1.24) $\lim_{n\to\infty} \left|f^n(x_0) - x^*\right| = 0$ and thus $\lim_{n\to\infty} f^n(x_0) = x^*$, which yields asymptotic stability. The proof of part 2 is left to you as Problem 14 .
■ The following examples illustrate the applicability of the above theorem.

Example 1.6.1 *Consider the map $G_\lambda(x) = 1 - \lambda x^2$ defined on the interval $[-1, 1]$, where $\lambda \in (0, 2]$. Find the fixed points of $G_\lambda(x)$ and determine their stability.*

SOLUTION To find the fixed points of $G_\lambda(x)$ we solve the equation $1 - \lambda x^2 = x$ or $\lambda x^2 + x - 1 = 0$. There are two fixed points:

$$x_1^* = \frac{-1 - \sqrt{1 + 4\lambda}}{2\lambda} \text{ and } x_2^* = \frac{-1 + \sqrt{1 + 4\lambda}}{2\lambda}.$$

Observe that $G'_\lambda(x) = -2\lambda x$. Thus, $\left|G'_\lambda\left(x_1^*\right)\right| = 1 + \sqrt{1 + 4\lambda} > 1$, and hence, x_1^* is unstable for all $\lambda \in (0, 2$. Furthermore, $\left|G'_\lambda\left(x_2^*\right)\right| = \sqrt{1 + 4\lambda} - 1 < 1$ if and only if $\sqrt{1 + 4\lambda} < 2$. Solving the latter inequality for λ, we obtain $\lambda < \frac{3}{4}$. This implies by Theorem 1.3 that the fixed point x_2^* is asymptotically stable if $0 < \lambda < \frac{3}{4}$ and unstable if $\lambda > \frac{3}{4}$ (see Fig. 1.15). When $\lambda = \frac{3}{4}$, $G'_\lambda\left(x_2^*\right) = -1$.

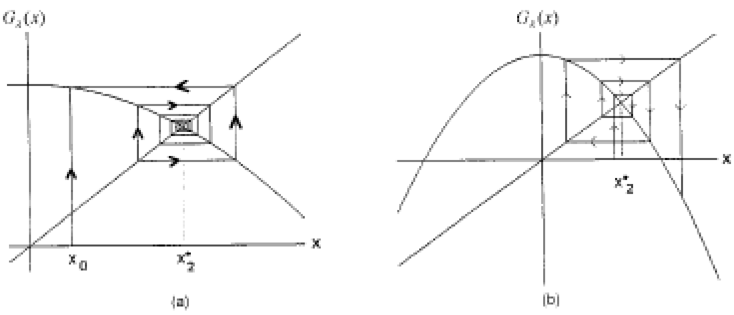


FIGURE 1.15
(a) $\lambda = \frac{1}{2}, x_2^*$ is asymptotically stable while (b) $\lambda = \frac{3}{2}, x_2^*$ is unstable.

Example 1.6.2 ***(Raphson-Newton's Method).*** *Raphson-Newton's method is one of the simplest and oldest numerical methods for finding the roots of the equation $g(x) = 0$. The Newton algorithm for finding a zero r of $g(x)$ is given by the difference equation*

$$x(n+1) = x(n) - \frac{g(x(n))}{g'(x(n))} \tag{1.25}$$

where $x(0) = x_0$ is our initial guess of the root r. Equation (1.25) is of the form of Equation (1.20) with

$$f_N(x) = x - \frac{g(x)}{g'(x)} \tag{1.26}$$

where f_N is called Newton's function.

Theorem 1.6.2 ***(Taylor's Theorem)***
Let f be differentiable of all orders at x_0. Then

$$f(x) = f(x_0) + (x - x_0) f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \ldots$$

for all x in a small open interval containing x_0.
Formula (1.25) may be justified using Taylor's Theorem. A linear approximation of $f(x)$ is given by the equation of the tangent line to $f(x)$ at x_0 :

$$f(x) = f(x_0) + (x - x_0) f'(x_0)$$

. The intersection of this tangent line with the x-axis produces the next point x_1 in Newton's algorithm (Fig. 1.16). Letting $f(x) = 0$ and $x = x_1$ yields

$$x_1 = x_0 - \frac{f(x_0)}{f' x_0)}$$

By repeating the process, replacing x_0 by x_1, x_1 by $x_2, \ldots$, we obtain formula (1.25).

We observe first that if r is a root of $g(x)$*, i.e.,* $g(r) = 0$*, then from Equation (1.26) we have* $f_N(r) = r$ *and thus r is a fixed point of* f_N *(assuming that* $g'(r) \neq 0$*). On the other hand, if* x^* *is a fixed point of* f_N*, then from Equation (1.26) again we get* $\frac{g(x^*)}{g'(x)} = 0$*. This implies that* $g(x^*) = 0$*, i.e.,* x^* *is a zero of* $g(x)$*. Now, starting with a point* x_0 *close to a root r of* $g(x) = 0$*, then Algorithm (1.25) gives the next approximation* $x(1)$ *of the root r. By applying the algorithm repeatedly, we obtain the sequence of approximations*

$$x_0 = x(0), x(1), x(2), \dots, x(n), \dots$$

(see Fig. 1.16). The question is whether or not this sequence converges to the root r. In other words, we need to check the asymptotic stability of the fixed point $x^* = r$ *of* f_N*. To do so, we evaluate* $f_N'(r)$ *and then use Theorem 1.6.1,*

$$\left|f_N'(r)\right| = \left|1 - \frac{[g'(r)]^2 - g(r)g''(r)}{[g'(r)]^2}\right| = 0, \quad \textit{since } g(r) = 0$$

Hence, by Theorem 1.6.1, $\lim_{n\to\infty} x(n) = r$*, provided that* x_0 *is sufficiently close to r.*

For $g(x) = x^2 - 1$*, we have two zero's* $-1, 1$*. In this case, Newton's function is given by* $f_N(x) = x - \frac{x^2-1}{2x} = \frac{x^2+1}{2x}$*. The cobweb diagram of* f_N *shows that Newton's algorithm converges quickly to both roots (see Fig. 1.17).*

1.6.2 Nonhyperbolic Fixed Points

The stability criteria for nonhyperbolic fixed points are more involved. They will be summarized in the next two results, the first of which treats the case when $f'(x^*) = 1$ and the second for $f'(x^*) = -1$.

Theorem 1.6.3 *Let* x^* *be a fixed point of a map f such that* $f'(x^*) = 1$*. If* $f'(x)$*,* $f''(x)$*, and* $f'''(x)$ *are continuous at* x^**, then the following statements hold:*

1. *If* $f''(x^*) \neq 0$*, then* x^* *is unstable (semistable).*
2. *If* $f''(x^*) = 0$ *and* $f'''(x^*) > 0$*, then* x^* *is unstable.*
3. *If* $f''(x^*) = 0$ *and* $f'''(x^*) < 0$*, then* x^* *is asymptotically stable.*

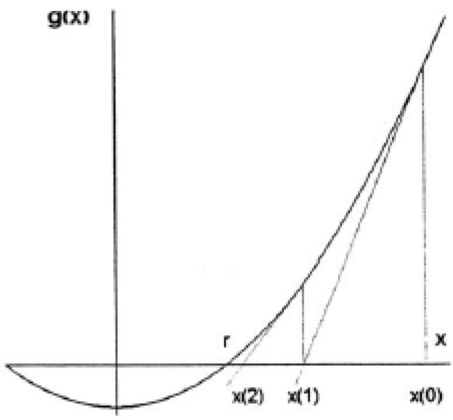


FIGURE 1.16
Newton's method for $g(x) = x^2 - 1$.

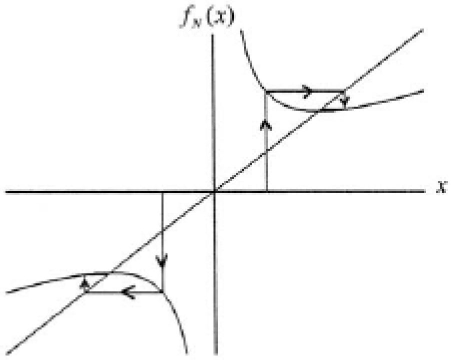


FIGURE 1.17
Cobweb diagram for Newton's function f_N when $g(x) = x^2 - 1$.

Proof. Assume that $f'(x^*) = 1$ and $f''(x^*) \neq 0$. Then, the curve $y = f(x)$ is either concave upward ($f''(x^*) > 0$) or concave downward ($f''(x^*) < 0$), as shown in Fig. 1.18(a) and (b). Now, if $f''(x^*) > 0$, then $f'(x)$ is increasing in a small interval containing x^*. Hence, $f'(x) > 1$ for all $x \in (x^*, x^* + \delta)$, for some small $\delta > 0$ [see Fig. 1.18(a)]. Using the same proof as in Theorem 1.3, we conclude that x^* is unstable. Similarly, if $f''(x^*) < 0$ then $f'(x)$ is decreasing in a small neighborhood of x^*. Therefore, $f'(x) > 1$ for all $x \in (x^* - \delta, x^*)$, for some small $\delta > 0$, and again we conclude that x^* is unstable [see Fig. 1.18(b)]. ■

Example 1.6.3 *Let $f(x) = -x^3 + x$. Then $x^* = 0$ is the only fixed point of f. Note that $f'(0) = 1, f''(0) = 0, f'''(0) < 0$. Hence by Theorem 1.5, 0 is asymptotically stable.*

The preceding theorem may be used to establish stability criteria for the case when $f'(x^*) = -1$. But before doing so, we need to introduce the notion of the ***Schwarzian derivative****.*

Definition 1.6.1 *The Schwarzian derivative, Sf, of a function f is defined by*

$$\mathrm{S}f(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left[\frac{f''(x)}{f'(x)}\right]^2 \tag{1.27}$$

And if $f'(x^*) = -1$, then

$$\mathrm{S}f(x^*) = -f'''(x^*) - \frac{3}{2}\left[f''(x^*)\right]^2 \tag{1.28}$$

Theorem 1.6.4 *Let x^* be a fixed point of a map f such that $f'(x^*) = -1$. If $f'(x)$, $f''(x)$, and $f'''(x)$ are continuous at x^*, then the following statements hold:*

1. *If $\mathrm{S}f(x^*) < 0$, then x^* is asymptotically stable.*

2. *If $\mathrm{S}f(x^*) > 0$, then x^* is unstable.*

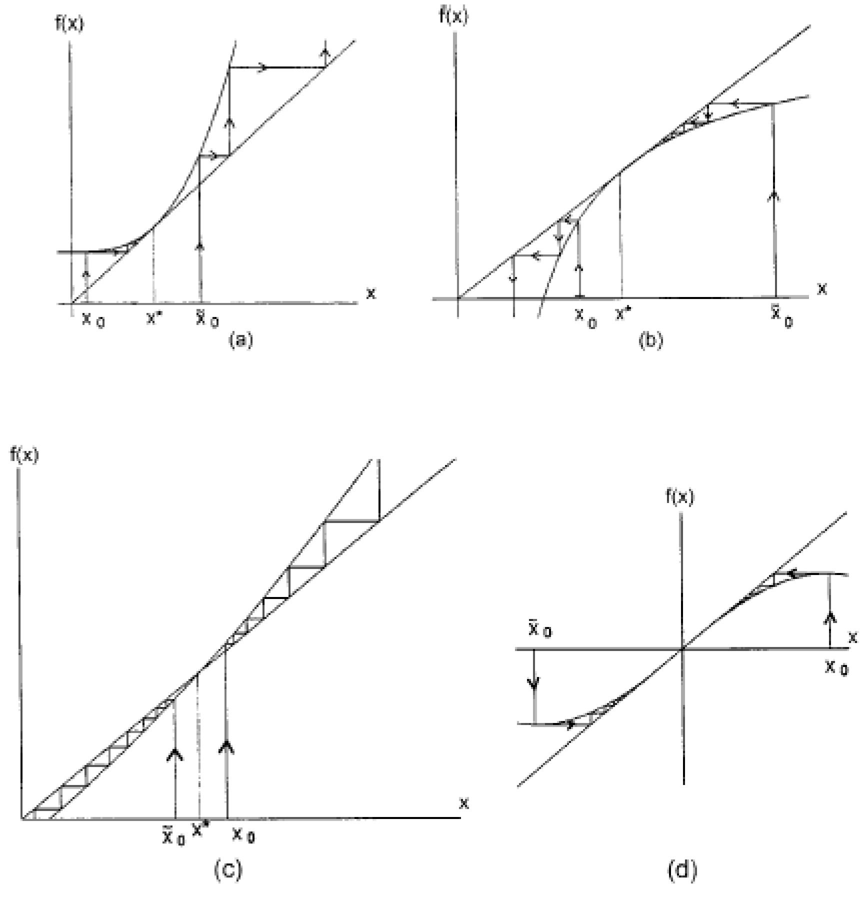


FIGURE 1.18
(a) $f'(x^*) = 1$, $f''(x^*) > 0$, unstable fixed point, semi-stable from the left.
(b) $f'(x^*) = 1$, $f''(x^*) < 0$, unstable fixed point, semi-stable from the right.
(c) $f'(x^*) = 1$, $f''(x^*) = 0$, $f'''(x^*) > 0$, unstable fixed point.
(d) $f'(x^*) = 1$, $f''(x^*) = 0$, $f'''(x^*) < 0$, asymptotically stable fixed point.

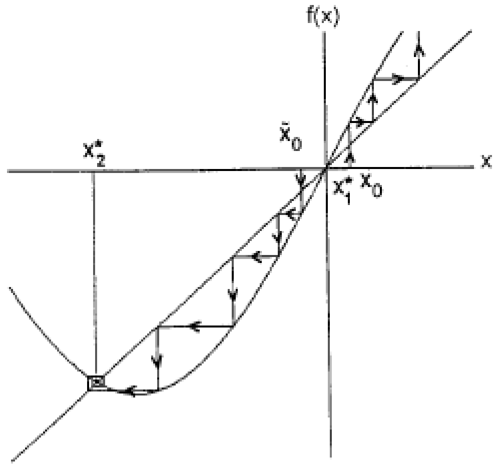


FIGURE 1.19
An asymptotically stable nonhyperbolic fixed point x_2^*.

Proof. The main idea of the proof is to create an associated function g with the property that $g'(x^*) = 1$, so that we can use Theorem 1.5. This function is indeed $g = f \circ f = f^2$. Two important facts need to be observed here. First, if x^* is a fixed point of f, then it is also a fixed point of g. Second, if x^* is asymptotically stable (unstable) with respect to g, then it is also asymptotically stable (unstable) with respect to f. By the chain rule:

$$g'(x) = \frac{d}{dx} f(f(x)) = f'(f(x))f'(x) \tag{1.29}$$

Hence,

$$g'(x) = (f''(x^*))^2$$

and Theorem now applies. For this reason we compute $g''(x^*)$. From Equation (1.29) , we have

$$g''(x) = f'(f(x))f''(x) + f''(f(x))(f'(x))^2 \tag{1.30}$$

$$\begin{aligned} g''(x^*) &= f'(x^*)\, f''(x^*) + f''(x^*)\,(f'(x^*))^2 \\ &= 0 \quad (\text{ since } f'(x^*) = -1)\,. \end{aligned} \tag{1.31}$$

Computing $g'''(x)$ from Equation (1.31), we get

$$g'''(x^*) = -2f'''(x^*) - 3(f''(x^*))^2 \tag{1.32}$$

It follows from Equation (1.29)

$$g'''(x^*) = 2\, Sf(x^*) \tag{1.33}$$

Statements 1 and 2 now follow immediately from Theorem 1.6.3 ■

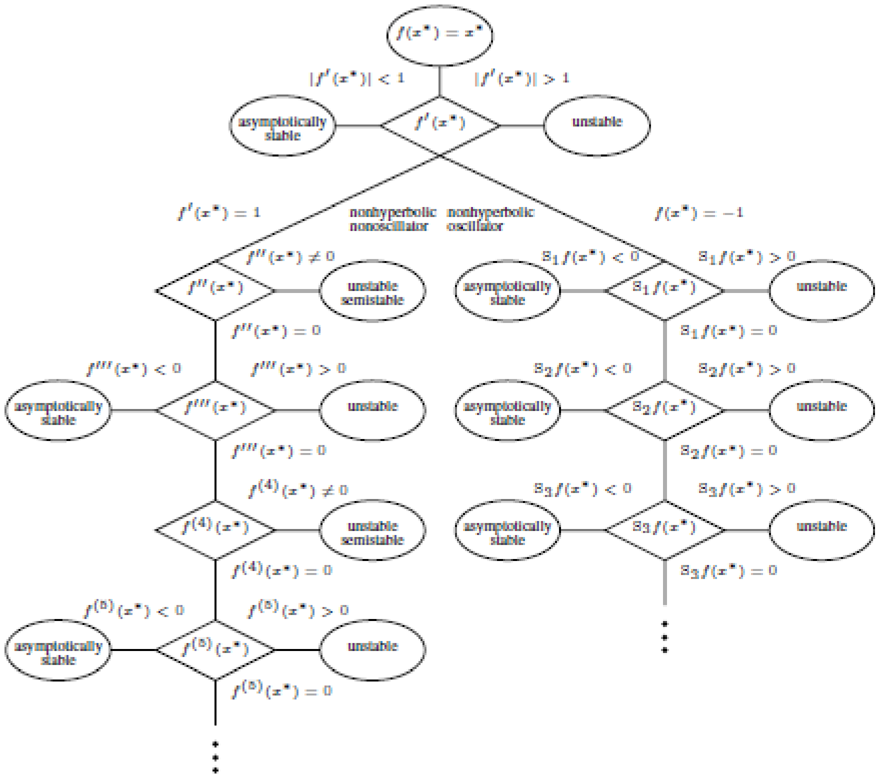


FIGURE 1.20
Classification of fixed points.

Remark 1.6.1 *Note that if $f'(x^*) = -1$ and $g = f \circ f$, then from (1.31) we have*

$$Sf(x^*) = \frac{1}{2} g'''(x^*) * \tag{1.34}$$

Furthermore,

$$g''(x^*) = 0 \tag{1.35}$$

We are now ready to give an example of a nonhyperbolic fixed point.

Example 1.6.4 *Consider the map $f(x) = x^2 + 3x$ on the interval $[-3, 3]$. Find the equilibrium points and then determine their stability.*

SOLUTION The fixed points of f are obtained by solving the equation $x^2 + 3x = x$. Thus, there are two fixed points: $x_1^* = 0$ and $x_2^* = -2$. So for x_1^*, $f'(0) = 3$, which implies by Theorem 1.3 that x_1^* is unstable. For x_2^*, we have $f'(-2) = -1$, which requires the employment of Theorem 1.6. We observe that

$$Sf(-2) = -f'''(-2) - \frac{3}{2}\left[f''(-2)\right]^2 = -6 < 0$$

Hence, x_2^* is asymptotically stable (see Fig. 1.19).

Diagram 1.20 provides a complete classification of fixed points which goes beyond the material in this section. Detailed analysis of the contents in the diagram may be found in [22]. In the cases when $Sf(x^*) = 0$ and $f'''(x^*) = 0$ were investigated. In the diagram, we have $S_1 f(x) = Sf(x), S_2 f(x) = \frac{1}{2}g^{(5)}(x)$, where $g = f^2$, and more generally $S_k f(x) = \frac{1}{2}g(2k+1)(x)$.

1.7 Periodic Points and their Stability

The notion of periodicity is one of the most important notion in the field of dynamical systems. Its importance stems from the fact that many physical phenomena have certain patterns that repeat themselves. These patterns produce cycles (or periodic cycles), where a cycle is understood to be the orbit of a periodic point. In this section, we address the questions of existence and stability of periodic points.

Definition 1.7.1 *Let $\bar{x}$ be in the domain of a map f. Then,*

1. *$\bar{x}$ is said to be a* ***periodic point*** *of f with period k if $f^k(\bar{x}) = \bar{x}$ for some positive integer k. In this case $\bar{x}$ may be called k-periodic. If in addition $f^r(\bar{x}) \neq \bar{x}$ for $0 < r < k$, then k is called the* ***minimal period*** *of $\bar{x}$. Note that $\bar{x}$ is k-periodic if it is a fixed point of the map f^k.*

2. *$\bar{x}$ is said to be an* ***eventually*** *periodic point of a period k and delay m if $f^{k+m}(\bar{x}) = f^m(\bar{x})$ for some positive integer k and $m \in \mathbb{Z}^+$(see Fig. 1.21). Notice that if $k = 1$, then $f(f^m(\bar{x})) = f^m(\bar{x})$ and $\bar{x}$ is then an eventually fixed point, and if $m = 0$, then $\bar{x}$ is k-periodic. In other words, $\bar{x}$ is eventually periodic if $f^k(\bar{x})$ is periodic, for some positive integer k.*

The orbit of a k-periodic point is the set

$$O(\bar{x}) = \left\{\bar{x}, f(\bar{x}), f^2(\bar{x}), \ldots, f^{k-1}(\bar{x})\right\}$$

and is often called a k-periodic cycle. Graphically, a k-periodic point is the x coordinate of a point at which the graph of the map f^k meets the diagonal line $y = x$.
Next we turn our attention to the question of stability of periodic points.

Definition 1.7.2 *Let $\bar{x}$ be a periodic point of f with minimal period k. Then,*

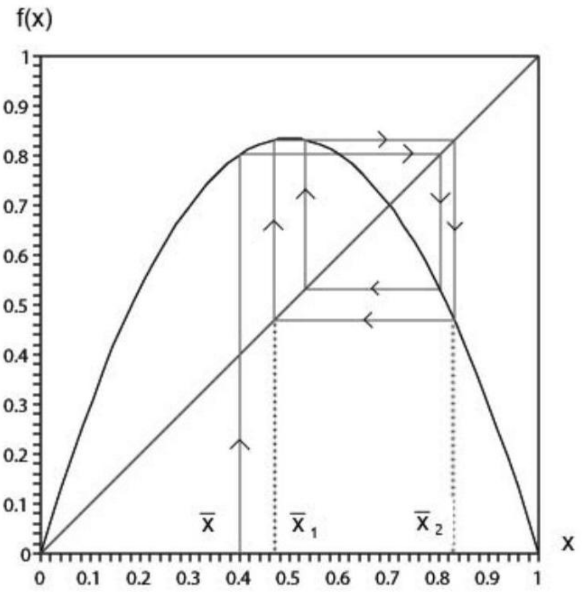


FIGURE 1.21 *An eventually periodic point* $\bar{x}$ *: The orbit of* $\bar{x}$ *goes into a 2-periodic cycle* $\{\bar{x}_1, \bar{x}_2\}$.

1. $\bar{x}$ *is* ***stable*** *if it is a stable fixed point of* f^k.

2. $\bar{x}$ *is* ***asymptotically stable*** *if it is an asymptotically stable fixed point of* f^k.

3. $\bar{x}$ *is* ***unstable*** *if it is an unstable fixed point of* f^k.

Thus, the study of the stability of k-periodic solutions of the difference equation

$$x(n+1) = f(x(n)) \tag{1.36}$$

reduces to studying the stability of the equilibrium points of the associated difference equation

$$y(n+1) = g(y(n)) \tag{1.37}$$

where $g = f^k$.

The next theorem gives a practical criteria for the stability of periodic points based on Theorem 1.6.1 in the preceding section.

Theorem 1.7.1 *Let* $O(\bar{x}) = \left\{\bar{x}, f(\bar{x}), \ldots, f^{k-1}(\bar{x})\right\}$ *be the orbit of the k-periodic point* $\bar{x}$*, where* f *is a continuously differentiable function at* $\bar{x}$*. Then the following statements hold true:*

1. $\bar{x}$ *is asymptotically stable if*

$$\left| f'(\bar{x}_1)\, f'(f(\bar{x}_2)) \ldots f'\left(f^{k-1}(\bar{x}_k)\right)\right| < 1 \tag{1.38}$$

2. *$\bar{x}$ is unstable if*

$$\left|f'(\bar{x})f'(f(\bar{x}))\ldots f'\left(f^{k-1}(\bar{x})\right)\right| > 1 \tag{1.39}$$

Proof. By using the chain rule, we can show that

$$\frac{d}{dx}f^k(\bar{x}) = f'(\bar{x})f'(f(\bar{x}))\ldots f'\left(f^{k-1}(\bar{x})\right)$$

Conditions (1.38) and (1.39) now follow immediately by application of Theorem 1.6.1 to the composite map $g = f^k$. ■

Example 1.7.1 *Consider the difference equation $x(n+1) = f(x(n))$ where $f(x) = 1 - x^2$ is defined on the interval* $[-1, 1]$. *Find all the 2-periodic cycles, 3 -periodic cycles, and 4 -periodic cycles of the difference equation and determine their stability.*

SOLUTION First, let us calculate the fixed points of f out of the way. Solving the equation $x^2+x-1 = 0$, we find that the fixed points of f are $x_1^* = -\frac{1}{2} - \frac{\sqrt{5}}{2}$ and $x_2^* = -\frac{1}{2} + \frac{\sqrt{5}}{2}$. Only x_2^* is in the domain of f. The fixed point x_2^* is unstable. To find the two cycles, we find f^2 and put $f^2(x) = x$. Now, $f^2(x) = 1 - \left(1 - x^2\right)^2 = 2x^2 - x^4$ and $f^2(x) = x$ yields the equation

$$x\left(x^3 - 2x + 1\right) = x(x-1)\left(x^2 + x - 1\right) = 0$$

Hence, we have the 2-periodic cycle $\{0, 1\}$; the other two roots are the fixed points of f. To check the stability of this cycle, we compute $\left|f'(0)f'(1)\right| = 0 < 1$. Hence, by Theorem 1.7, the cycle is asymptotically stable (Fig. 1.22).

Next we search for the 3 -periodic cycles. This involves solving algebraically a sixth-degree equation, which is not possible in most cases. So, we resort to graphical (or numerical) analysis. Figure 1.23 shows that there are no 3periodic cycles. Moreover, Fig. 1.24 shows that there are no 4 -periodic cycles.

Since $f^{-1}(x) = \sqrt{1-x}$, it follows that the point $f^{-1}\left(x_2^*\right) = \sqrt{\frac{3-\sqrt{5}}{2}}$ is an eventually fixed point. Let $g = f^2$. Then $g^{-1}(x) = \sqrt{1 + \sqrt{1-x}}$. Now $g^{-1}(0) = \sqrt{2}$ which is outside the domain of f. Hence f has no eventually periodic points.

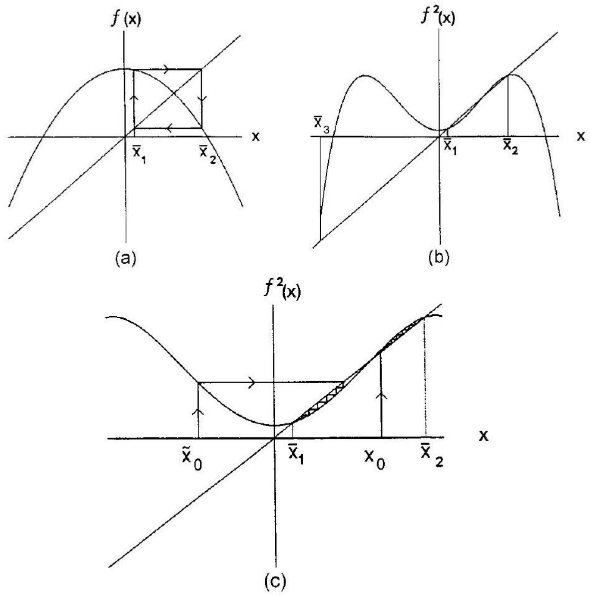


FIGURE 1.22 (a) A 2-periodic cycle $\{\bar{x}_1, \bar{x}_2\}$; (b) Periodic points of f : $\bar{x}_1$, and $\bar{x}_2$ are fixed points of f^2; (c) Periodic points of f : $\bar{x}_1$, and $\bar{x}_2$ are asymptotically stable fixed points of f^2.

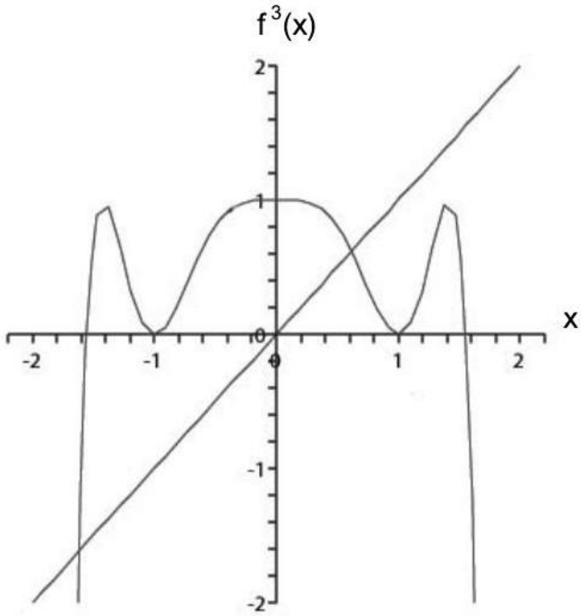


FIGURE 1.23 f^3 has no "genuine" fixed points, it has a fixed point x^* which is a fixed point of f, f has no points of period 3 .

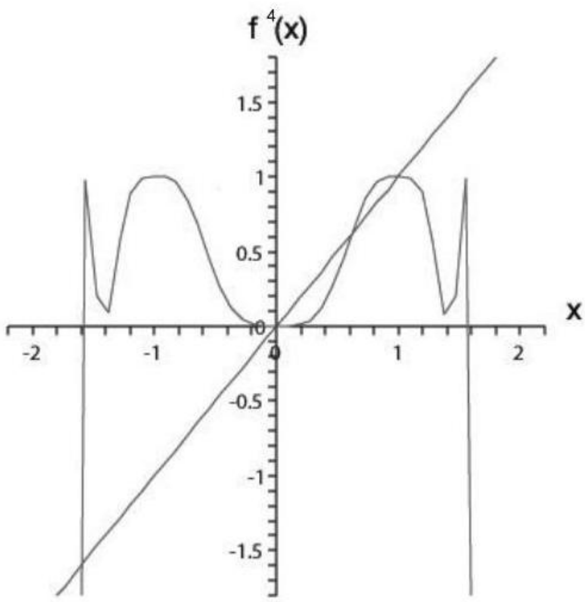


FIGURE 1.24 f^4 has no "genuine" fixed points, it has three fixed points, a fixed point x^* of f and two fixed points $\bar{x}_1, \bar{x}_2$ of f^2, f has no 4 -periodic cycles.

Example 1.7.2 ***(The Tent Map Revisited)****.*
The tent map T is defined as

$$T(x) = \begin{cases} 2x; & 0 \le x \le \frac{1}{2} \\ 2(1-x); & \frac{1}{2} < x \le 1 \end{cases}$$

It may be written in the compact form

$$T(x) = 1 - 2\left|x - \frac{1}{2}\right|$$

Find all the 2-periodic cycles and the 3-periodic cycles of T *and determine their stability.*

SOLUTION First, we observe that the fixed points of T are $x_1^* = 0$ and $x_2^* = \frac{2}{3}$; they are unstable since $|T'| = 2$. To find the 2-periodic cycles, we compute T^2. After some computation, we obtain

$$T^2(x) = \begin{cases} 4x; & 0 \le x < \frac{1}{4} \\ 2(1-2x); & \frac{1}{4} \le x < \frac{1}{2} \\ 4\left(x - \frac{1}{2}\right); & \frac{1}{2} \le x < \frac{3}{4} \\ 4(1-x); & \frac{3}{4} \le x \le 1 \end{cases}$$

There are four fixed points of $T^2 : 0, \frac{2}{5}, \frac{2}{3}, \frac{4}{5}$, two of which $\left(0, \frac{2}{3}\right)$, are fixed points of T. Thus, $\left\{\frac{2}{5}, \frac{4}{5}\right\}$ is the only 2-periodic cycle [see Fig. 1.25(b)]. Since $\left|T'\left(\frac{2}{5}\right) T'\left(\frac{4}{5}\right)\right| = 4 > 1$, this 2-periodic cycle is unstable

(Theorem 1.7.1). From Fig. 1.25(c), we observe that T^3 has eight fixed points, two of which are fixed points of T. Thus, there are two periodic cycles of period 3. It is easy to check that these cycles are $C_1 = \left\{\frac{2}{7}, \frac{4}{7}, \frac{6}{7}\right\}$ and $C_2 = \left\{\frac{2}{9}, \frac{4}{9}, \frac{8}{9}\right\}$, both of which are unstable.

Note that the point $\frac{3}{5}$ is an eventually 2-periodic point as $\frac{3}{5} \to \frac{2}{5} \to \frac{4}{5}$. Moreover, the point $\frac{3}{7}$ is an eventually 3-periodic point since $\frac{3}{7} \to \frac{6}{7} \to \frac{2}{7} \to \frac{4}{7}$. A general result characterizing periodic and eventually periodic points of the tent map will be given in Section 3.2.

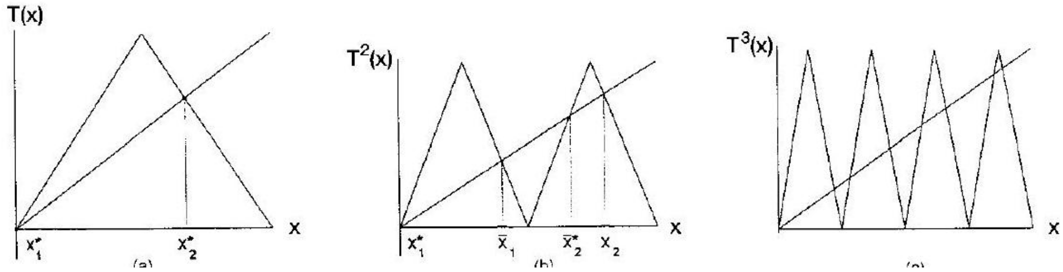


FIGURE 1.25 (a) The tent map T has two fixed points; (b) T^2 has 4 fixed points, 2 periodic points $\bar{x}_1, \bar{x}_2$, and 2 fixed points x_1^*, x_2^* of T; (c) T^3 has 8 fixed points, two cycles of period 3 and two fixed maps of T.

1.8 The Period-Doubling Route to Chaos

We end this chapter by studying in detail the logistic map:

$$F_\mu(x) = \mu x(1 - x) \tag{1.40}$$

which gives rise to the logistic difference equation

$$x(n + 1) = \mu x(n)(1 - x(n)) \tag{1.41}$$

where $x \in [0, 1]$ and $\mu \in (0, 4]$.

1.8.1 Fixed Points

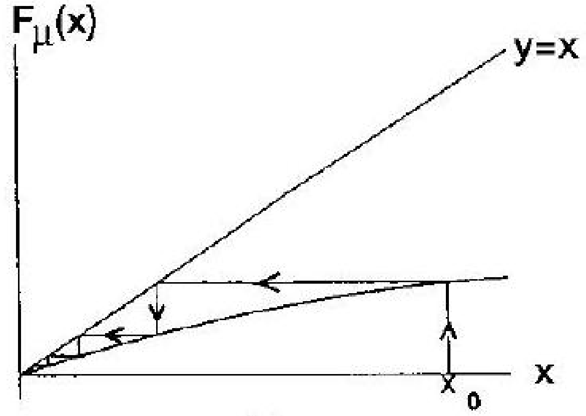
Let us begin our exposition by examining the equilibrium points of Equation (1.41). There are two fixed points of $F_\mu : x_1^* = 0$ and $x_2^* = \frac{\mu-1}{\mu}$. We now examine the stability of each fixed point separately.

1. The fixed point $x_1^* = 0$: observe that $F_\mu'(0) = \mu$. Therefore, from Theorem 1.6.1, we conclude

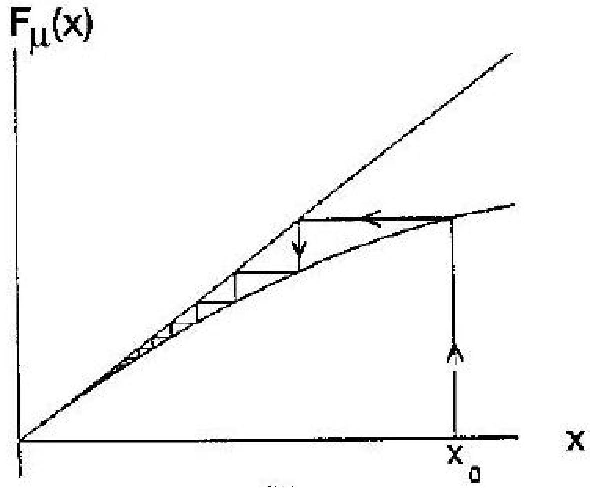
(a) x_1^* is asymptotically stable if $0 < \mu < 1$ [see Fig. 1.26(a)].

(b) x_1^* is unstable if $\mu > 1$ [see Fig. 1.26(c)].

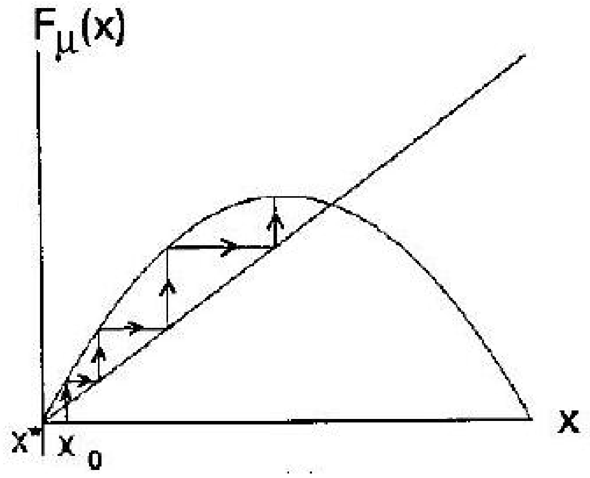
The case where $\mu = 1$ needs special attention, for we have $F_1'(0) = 1$ and $F_1''(0) = -2 \neq 0$. By applying Theorem 1.6.3, we may conclude that 0 is unstable. This is certainly true if we consider negative as well as positive initial points in the neighborhood of 0. Since negative initial points are not in the domain of F_μ, we may discard them and consider only initial points in neighborhoods of 0 of the form $(0, \delta)$. Now, Problem 17a in



(a)



(b)



(c)

FIGURE 1.26 (a) $0 < \mu < 1$: 0 is asymptotically stable; (b) $\mu = 1$: 0 is asymptotically stable; (c) $\mu > 1$: 0 is unstable.

Exercises 1.7 tells us that the fixed point is semiasymptotically stable from the right. In other words, $x_1^* = 0$ is asymptotically stable in the domain $[0, 1]$ [see Fig. 1.26(b)].

2. The fixed point $x_2^* = \frac{\mu-1}{\mu}$: Clearly x_2^* will be in the interval $(0, 1]$ if $\mu > 1$. Moreover, $F'_\mu\left(\frac{\mu-1}{\mu}\right) = \mu - 2\mu\left(\frac{\mu-1}{\mu}\right) = 2 - \mu$. Thus, by Theorem 1.6.1, x_2^* is asymptotically stable if $|2 - \mu| < 1$. Solving this inequality for μ, we obtain $1 < \mu < 3$ as the values of μ where x_2^* is asymptotically stable [see Fig. 1.27(a)]. When $\mu = 3$, we have $F'_3\left(x_2^*\right) = F'_3\left(\frac{2}{3}\right) = -1$, and x_2^* is therefore nonhyperbolic. In this case, we need to compute the Schwarzian derivative: $\text{SF}'_3\left(x_2^*\right) = -\frac{3}{2}(36) < 0$. Hence, by Theorem 1.6.4, the equilibrium point $x_2^* = \frac{2}{3}$ is asymptotically stable under F_3 [see Fig. 1.27(a)]. Furthermore, by Theorem 1.6.1, the fixed point x_2^* is unstable for $\mu > 3$. We now summarize our findings.

(a) x_2^* is asymptotically stable for $1 < \mu \leq 3$.

(b) x_2^* is unstable for $\mu > 3$ [see Fig. 1.27(b)].

Looking at Fig. 1.27(b), we observe that the orbit of x_0 flips around x_2^* and then settles bouncing between two points, which indicates the appearance of a 2-periodic cycle.

1.8.2 2-Periodic Cycles

To find the 2-periodic cycles we solve the equation $F_\mu^2(x) = x$, or

$$\mu^2 x(1 - x)[1 - \mu x(1 - x)] - x = 0 \tag{1.42}$$

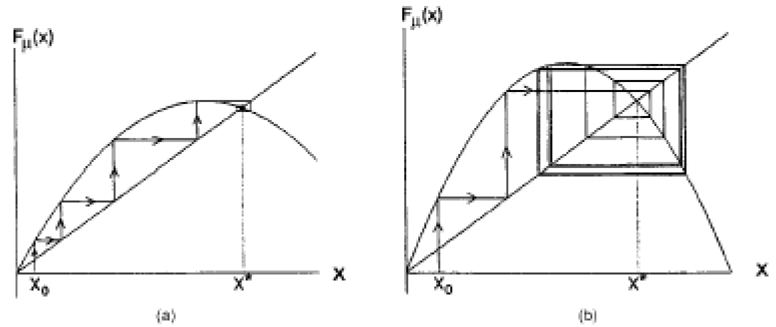


FIGURE 1.27 (a) $1 < \mu \leq 3, x_2^*$ is asymptotically stable, and (b) $\mu > 3x_2^*$ is unstable.

Discarding the equilibrium points 0 and $\frac{\mu-1}{\mu}$ by dividing the left side of Equation (1.42) by $x\left(x - \frac{\mu-1}{\mu}\right)$, we obtain

$$\mu^2 x^2 - \mu(\mu+1)x + (\mu+1) = 0$$

Solving this equation,

$$\bar{x}_1 = \frac{(1+\mu) - \sqrt{(\mu-3)(\mu+1)}}{2\mu} \text{ and } \bar{x}_2 = \frac{(1+\mu) + \sqrt{(\mu-3)(\mu+1)}}{2\mu} \tag{1.43}$$

Clearly $\bar{x}_1$ and $\bar{x}_2$ are defined only if $\mu > 3$. Next, we investigate the stability of this 2-periodic cycle. By Theorem 1.7.1, this 2-periodic cycle is asymptotically stable if

$$\left|F'_\mu(\bar{x}_1)\, F'_\mu(\bar{x}_2)\right| < 1$$

or

$$-1 < \mu^2 (1 - 2\bar{x}_1)(1 - 2\bar{x}_2) < 1$$

$$-1 < \mu^2 \left(1 - \frac{(1+\mu) - \sqrt{(\mu^2 - 2\mu - 3)}}{\mu}\right)\left(1 - \frac{(1+\mu) + \sqrt{(\mu^2 - 2\mu - 3)}}{\mu}\right) < 1$$

$$-1 < -\mu^2 + 2\mu + 4 < 1$$

Solving the last two inequalities yields the range: $3 < \mu < 1 + \sqrt{6}$ for asymptotic stability. Now, for $\mu = 1 + \sqrt{6}$,

$$F'_\mu(\bar{x}_1)\, F'_\mu(\bar{x}_2) = -1$$

In this case, we need to apply Theorem 1.6.4 on F_μ^2 to determine the stability of the periodic points $\bar{x}_1$ and $\bar{x}_2$ of F_μ. After some computation, we conclude
that $SFF_\mu^2(\bar{x}_1) < 0$ and $SF_\mu^2(\bar{x}_2) < 0$, which implies that the cycle $\{\bar{x}_1, \bar{x}_2\}$ is asymptotically stable (Problem 1). Moreover, the periodic cycle $\{\bar{x}_1, \bar{x}_2\}$ is unstable for $\mu > 1 + \sqrt{6}$.

In summary:

1. $3 < \mu \leq 1 + \sqrt{6}$: The 2 -periodic cycle $\{\bar{x}_1, \bar{x}_2\}$ is asymptotically stable.

2. $\mu > 1 + \sqrt{6}$: The 2-periodic cycle $\{\bar{x}_1, \bar{x}_2\}$ is unstable.

Thus, the positive equilibrium point is asymptotically stable for $1 < \mu \leq 3$, where it loses its stability after $\mu_1 = 3$. For $\mu > \mu_1$, an asymptotically stable 2-periodic cycle appears where it loses its stability after a second magic number $\mu_2 = 1 + \sqrt{6} \approx 3.44949\ldots$, etc.

1.8.3 2^2-Periodic Cycles

The search for 4-periodic cycles can be successful if one is able to solve the equation $F_\mu^4(x) = x$. This involves solving a twelfth-degree equation, which is not possible in general. So we turn to graphical or numerical analysis to help us find the 4 -periodic cycles (see Fig. 1.28). It turns out that there is one 2^2

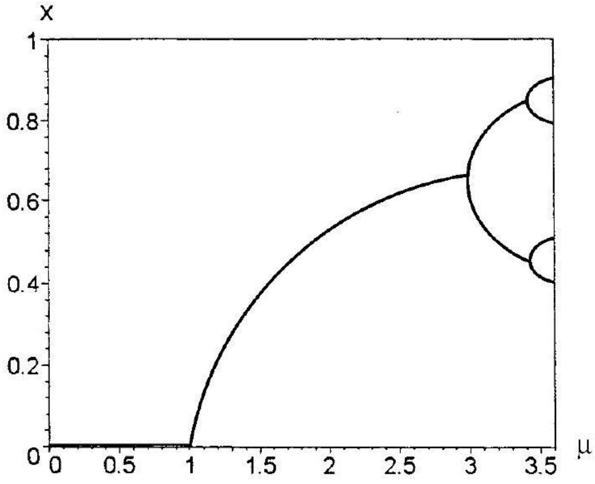


FIGURE 1.28 The appearance of a 4-periodic cycle. An exchange of stability occurs at $\mu = 1$ between $x_1^* = 0$ and $x_2^* = (\mu - 1)/\mu$.

Cycle when $\mu > 1 + \sqrt{6}$ which is asymptotically stable for $1 + \sqrt{6} < \mu \leq 3.54409$. This 2^2 cycle loses its stability when $\mu > 3.54409$. Again, the story repeats itself, when $\mu > \mu_3$, the 2^2 cycle bifurcates into an asymptotically stable 2^3
cycle. This process of double bifurcation continues indefinitely and produces a sequence $\{\mu_n\}_{n=1}^{\infty}$. Table 1.2 sheds some light on some remarkable patterns:

TABLE 1.2

n	μ_n	$\mu_n - \mu_{n-1}$	$\frac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n}$
1	3	-	-
2	3.449489 . . .	0.449489 . . .	-
3	3.544090 . . .	0.094601 . . .	4.751419 . . .
4	3.564407 . . .	0.020317 . . .	4.656248 . . .
5	3.568759 . . .	0.0043521 . . .	4.668321 . . .
6	3.569692 . . .	0.00093219 . . .	4.668683 . . .
7	3.569891 . . .	0.00019964 . . .	4.669354 . . .

From Table 1.2, we make the following observations (which can be proved, at least numerically):

1. The sequence $\{\mu_n\}$ seems to tend to a specific number, $\mu_\infty \approx 3.570$.
2. The window size $(\mu_n - \mu_{n-1})$ between successive μ_i values gets narrower and narrower, eventually approaching zero.
3. The ratio $\dfrac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n}$ approaches a constant called **Feigenbaum number** δ named after its discoverer, Mitchell Feigenbaum. In fact,

$$\delta = \lim_{n\to\infty} \frac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n} \approx 4.669201609\ldots \tag{1.44}$$

Feigenbaum discovered that the number δ is universal and does not depend on the family of maps under discussion; it is the same for a large class of maps, called unimodal maps.

Formula (1.44) may be used to generate the sequence $\{\mu_n\}$ with good accuracy. We let $\delta = \frac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n}$ and solve for μ_{n+1}. Then, we obtain

$$\mu_{n+1} = \mu_n + \frac{\mu_n - \mu_{n-1}}{\delta} \tag{1.45}$$

For example, given $\mu_1 = 3$ and $\mu_2 = 1 + \sqrt{6}$ (in Table 1.2), then from Formula (1.45) we get $\mu_3 = (1 + \sqrt{6}) + \frac{(1+\sqrt{6})-3}{4.6692} \approx 3.54575671$, which is a good approximation of the actual μ_3 in Table 1.2.

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