Contents

Chapter 1

Elementary Functions

In our calculus course, we are going to deal mostly with **elementary functions**. They are

- Power functions $\left(x^2,\ \sqrt{x},\ x^{\frac{1}{3}}, \cdots\right)$,
- Exponential functions $(2^x,\ e^x,\ \pi^x, \cdots)$,
- Logarithmic functions $(\ln x,\ \log_2 x, \cdots)$,
- Trigonometric functions $(\sin x,\ \cos x,\ \tan x, \cdots)$,
- Inverse trigonometric functions $(\arcsin x,\ \arccos x,\ \arctan x, \cdots)$,
- Hyperbolic functions $(chx,\ shx,\ thx, \cdots)$,

and their sums, differences, products, quotients, and compositions. For example

$$f(x) = \frac{\arcsin\sqrt{x^2-3}}{\ln(x^4+3) - \tan e^{\cos x}} \text{ is an elementary function.}$$

1.1 Power functions

1.1.1 *Review of exponents*

We start at the beginning. For a number a and a positive integer n,

$$a^n = \underbrace{a.a.a.\cdots.a}_{n\ times}.$$

1.1.2 *Basic laws of exponents*

$$a^1 = a, \qquad (ab)^n = a^n b^n, \qquad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n},$$
$$a^m a^n = a^{m+n}, \qquad \frac{a^m}{b^n} = a^{m-n}, \qquad (a^m)^n = a^{mn}.$$

1.1.3 *Definition of power functions*

Definition 1.1.1. *Let $a \in \mathbb{R}$, we name power function of exponent a, the function defined by*

$$\forall x \in]0, +\infty[, \quad x^a = e^{a \ln(x)}.$$

For example, $y = x$, $y = x^4$, $y = x^{\frac{2}{3}}$ are power functions.

In a power function $f(x) = x^a$, the base x is a variable, and the exponent a is a constant.

The appearance of the graph of a power function depends on the constant a .

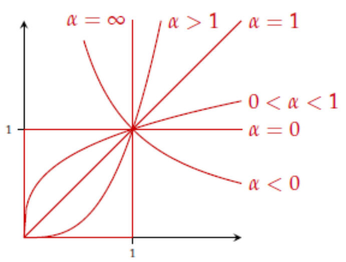

Figure 1.1: Power function with real exponents.

Definition 1.1.2. ***(Power functions*** $y = x^n$ ***)***

If n is an integer greater than 1, then the overall shape of the graph of $y = x^n$ is determined by the parity of n (whether n is even or odd).

- *If n is even, then the graph has a shape similar to the parabola $y = x^2$.*
- *If n is odd, then the graph has a shape similar to the cubic parabola $y = x^3$.*

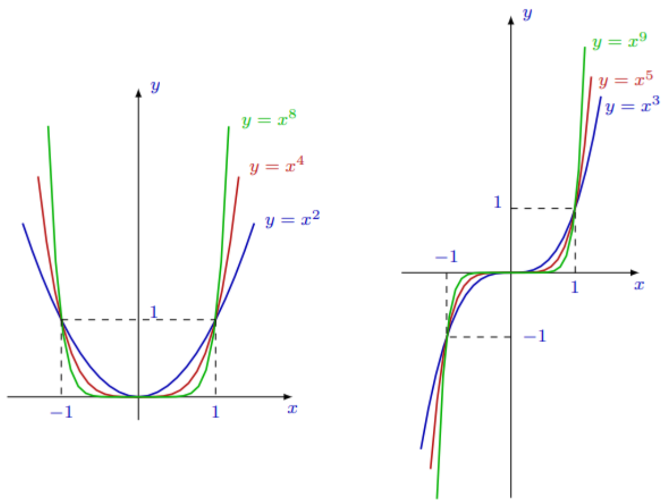

Figure 1.2: Power function with integer exponents.

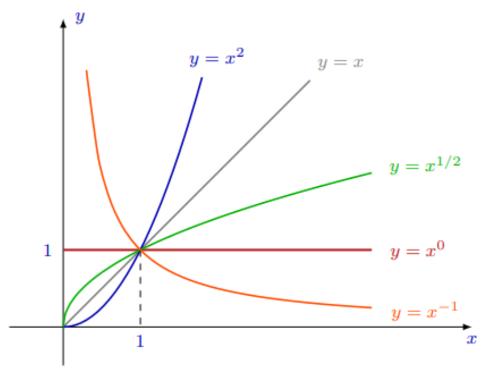

Figure 1.3: The graphs of $y = x^n$ for some rational n and $x > 0$.

Proposition 1.1.3. *1. For $a \in \mathbb{R}^*$, the power function with exponent a is a continuous* *function on $]0, +\infty[$, and strictly monotonic (strictly increasing if $a > 0$ and strictly decreasing if $a < 0$).*

2. *It is differentiable on $]0, +\infty[$ with derivative: $(x^a)' = ax^{a-1}, \ \forall x \in]0, +\infty[$.*

3. *We have:*

$$\lim_{x \to +\infty} x^a = \begin{cases} 0 & : \ a < 0 \\ 1 & : \ a = 0 \\ +\infty & : \ a > 0 \end{cases} \quad \text{and} \quad \lim_{x \to 0^+} x^a = \begin{cases} +\infty & : \ a < 0 \\ 1 & : \ a = 0 \\ 0 & : \ a > 0 \end{cases}$$

1.2 Logarithm and Exponential Functions

1.2.1 *Logarithm*

Definition 1.2.1. *The function that satisfies the following two conditions is called the neperian logarithm function and is denoted by* ln

- $\forall x \in \mathbb{R}_+^*, \quad \ln'(x) = \dfrac{1}{x}$.
- $\ln(1) = 0$.

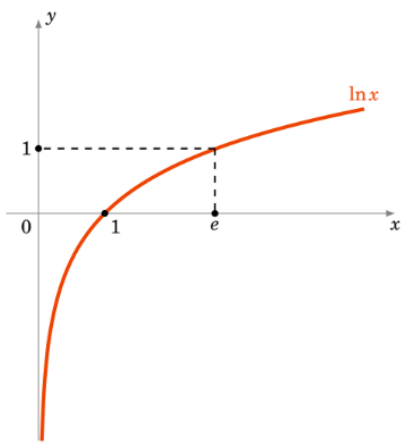

Figure 1.4: Logarithm function

Remark 1.2.2. *(Properties of derivatives)*

1. *According to the previous definition, the function* $\ln(x)$ *is differentiable on* $\mathbb{R}_+^*$ *and* $\forall x \in \mathbb{R}_+^*$ $(\ln(x))' = \frac{1}{x}$.

2. *The function* $ln(|x|)$ *is differentiable on* $\mathbb{R}^*$ *and* $\forall x \in \mathbb{R}^*$ $(\ln|x|)' = \frac{1}{x}$.

3. *Let* g *be a function differentiable and non-zero on* I *then the function* $\ln(|g(x)|)$ *is differentiable on* I *and its derivative:* $\ln(|g(x)|)' = \frac{g'(x)}{g(x)}$.

Proposition 1.2.3. *(Algebraic properties of the function* $\ln(x)$*)*

The logarithm function satisfies the following properties: (for all $a,\ b > 0$ *):*

1. $\ln(a \times b) = \ln a + \ln b$,

2. $\ln(\frac{a}{b}) = \ln a - \ln b$,

3. $\ln(\frac{1}{a}) = -\ln a$,

4. $\ln(a^n) = n \ln a$, *for all* $n \in \mathbb{N}$.

Proposition 1.2.4. *(Limits and classical inequalities)*

1. $\lim_{x \to +\infty} \ln(x) = +\infty$, *and* $\lim_{x \to 0^+} \ln(x) = -\infty$.

2. $\lim_{x \to +\infty} \frac{\ln(x)}{x} = 0$.

3. $\lim_{x \to +\infty} \frac{\ln(x)}{x^p} = 0,\ p \in \mathbb{R}_+^*$.

4. $\lim_{x \to 0^+} \frac{\ln(x+1)}{x} = 1$.

5. $\lim_{x \to 0^+} x \ln(x) = 0$.

6. $\forall x \in]-1, +\infty[, \quad \ln(x+1) \leq x$.

Remark 1.2.5. *Let $a \in]0,1[\cup]1,+\infty[$, we call the logarithm function with base a and denote $\log_a$, the function defined by:*

$$\log_a = \frac{\ln x}{\ln a}, \quad \forall x > 0.$$

- *We have: $\ln(x) = \log_e(x)$ i.e., the neperian logarithm function is the logarithm function with base e.*
- $\log_a(a) = 1$.

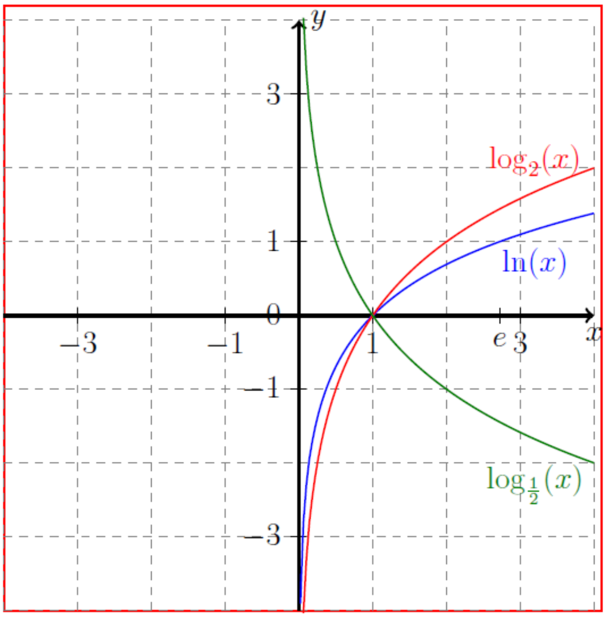

Figure 1.5: Graphical representation of the logarithmic functions and logarithms with base a for $a = \frac{1}{2}$, $a = 2$

1.2.2 *Exponential*

Definition 1.2.6. *The inverse function of the function $\ln(x)$ is called the exponential function and is denoted by: $\exp(x)$ or e^x, and satisfies the following properties:*

1. $\forall x > 0, \quad x = e^{\ln(x)}$.
2. $\forall\, y \in \mathbb{R}, \quad y = \ln(e^y)$.

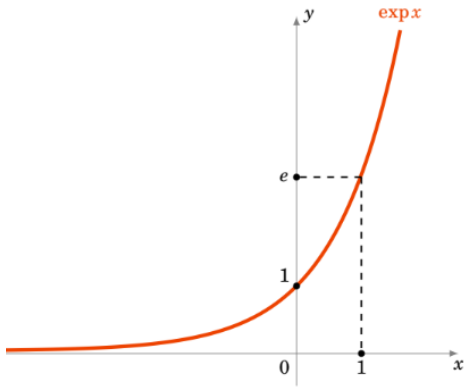

Figure 1.6: Exponential function

Proposition 1.2.7. *1. The function e^x is continuous and strictly increasing on $\mathbb{R}$.*

2. *The function e^x is differentiable on $\mathbb{R}$ and we have: $\forall x \in \mathbb{R},\ (e^x)' = e^x$.*

3. *If u is differentiable on I then: the function $e^{u(x)}$ is differentiable on I and its derivative defined by: $\forall x \in I,\ (e^{u(x)})' = u'(x).e^{u(x)}$.*

Proposition 1.2.8. *(Algebraic properties of the function e^x):*

1. $e^{x+y} = e^x \times e^y,\ \forall\ x,\ y \in \mathbb{R}$.

2. $e^{-x} = \frac{1}{x},\ \forall\ x \in \mathbb{R}$.

3. $e^{x-y} = \frac{x}{y},\ \forall\ x,\ y \in \mathbb{R}$.

4. $e^{nx} = (e^x)^n$,

Proposition 1.2.9. *(Limits and inequalities):*

1. $\lim_{x \to -\infty} e^x = 0$.

2. $\lim_{x \to +\infty} e^x = +\infty$.

3. $\lim_{x \to +\infty} xe^{-x} = 0,\quad \lim_{x \to +\infty} \frac{e^x}{x^a} = +\infty,\quad \lim_{x \to +\infty} \frac{x^a}{e^x} = 0,\quad a \in \mathbb{R}$.

4. $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$.

5. $\forall x \in \mathbb{R}, \quad e^x \geq 1 + x$.

Remark 1.2.10. *Let* $a \in]0,1[\cup]1,+\infty[$. *The inverse function of the function* $log_a(x)$ *is called the exponential function with base* a *and is denoted* a^x*:*

- $\forall x \in \mathbb{R}, \ a^x = e^{x\ln(a)}$.
- $\forall x \in \mathbb{R}, \ log_a(a^x) = log_a(e^{x\ln(a)}) = \frac{\ln(e^{x\ln(a)})}{\ln(a)} = x$.

1.3 Trigonometric Functions

1.3.1 *Sine function*

Definition 1.3.1. *The sine function* $y = \sin x$ *is defined as follows*

$$\begin{aligned} \sin : \ \mathbb{R} &\longrightarrow [-1,1] \\ x &\longrightarrow \sin x. \end{aligned}$$

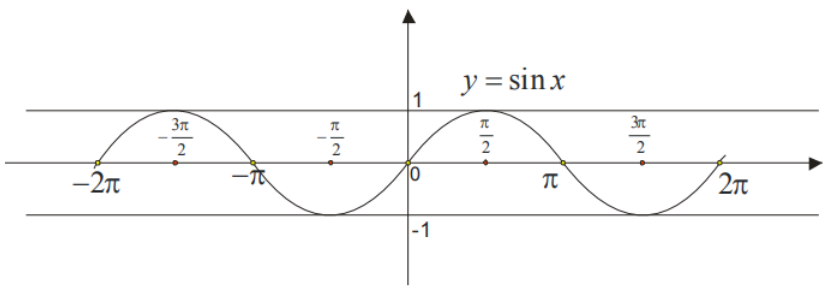

Figure 1.7: Sine function

1.3.2 *Cosine function*

Definition 1.3.2. *The cosine function* $y = \cos x$ *is defined as follows*

$$\begin{aligned} \cos: \quad \mathbb{R} &\longrightarrow [-1,1] \\ x &\longrightarrow \cos x. \end{aligned}$$

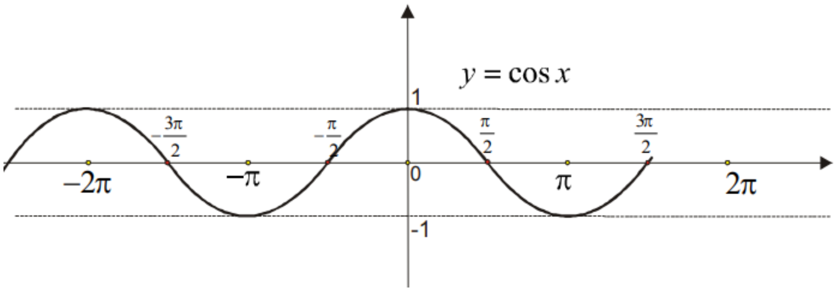

Figure 1.8: Cosine function

Properties: For all $x, \in \mathbb{R}$, we have

- $|\cos(x)| \leq 1, \quad and \quad |\sin(x)| \leq 1.$
- $\sin^2 x + \cos^2 x = 1.$
- $\cos(x)$ and $\sin(x)$ are $2\pi-$periodic, and

$$\begin{cases} \cos(x+2\pi) = \cos(x) \\ \sin(x+2\pi) = \sin(x) \end{cases}$$

- The function $\cos(x)$ is even and the function $\sin(x)$ is odd.
- The functions $\cos(x)$ and $\sin(x)$ belong to $C^{+\infty}(\mathbb{R})$ and we have:

$$\forall x \in \mathbb{R}, \begin{cases} (\cos(x))' = -\sin(x) \\ \quad and \\ (\sin(x))' = \cos(x) \end{cases}$$

$$\forall x \in \mathbb{R}, \ \forall n \in \mathbb{N}, \begin{cases} \cos(x)^{(n)} = \cos(x + \frac{n\pi}{2}) \\ \quad and \\ \sin(x)^{(n)} = \sin(x + \frac{n\pi}{2}) \end{cases}.$$

Properties: For all $(x, y) \in \mathbb{R}^2$, we have the following formulas:

- $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$.
- $\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$.
- $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$.
- $\sin(x - y) = \sin(x)\cos(y) - \cos(x)\sin(y)$.
- $\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$.
- $\sin(2x) = 2\sin(x)\cos(x)$.
- $\sin(x) + \sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$.
- $\sin(x) - \sin(y) = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$.
- $\cos(x) + \cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$.
- $\cos(x) - \cos(y) = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$.

1.3.3 *Tangent function*

Definition 1.3.3. *The tangent function is one of the main trigonometric functions and defined by:*

$$\begin{aligned} \tan: \quad \mathbb{R}| \left\{\frac{\pi}{2} + k\pi\right\} &\longrightarrow \mathbb{R} \\ x &\longrightarrow \tan x = \frac{\sin x}{\cos x} \end{aligned}, \quad k \in \mathbb{Z}$$

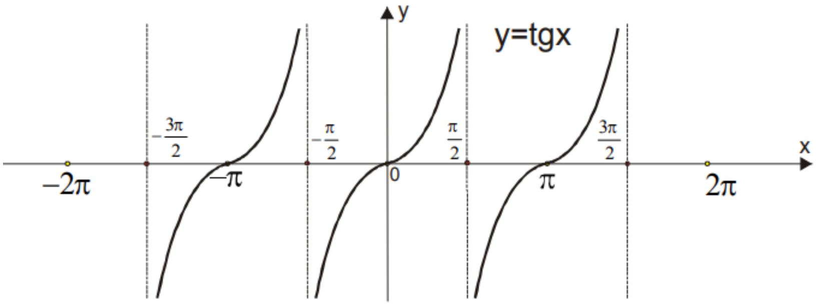

Figure 1.9: Tangent function

Proposition 1.3.4. *The function* $\tan(x)$ *checks the following properties:*

- *The function* $\tan(x)$ *is differentiable on* $\mathbb{R}|\left\{\frac{\pi}{2}+k\pi\right\},\ k\in\mathbb{Z}$ *and we have:*

$$(\tan(x))' = \frac{1}{\cos^2(x)} = 1+\tan^2(x).$$

- *The function* $\tan(x)$ *is* $\pi-$*periodic i.e:* $\tan(x+\pi)=\tan(x)$.

- *For any* $x,\ y\in\mathbb{R}|\left\{\frac{\pi}{2}+k\pi\right\},\ k\in\mathbb{Z}$ *we have:*

$$\begin{cases} \tan(x+y) = \dfrac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)} \\ \qquad\qquad \textit{and} \\ \tan(x-y) = \dfrac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)} \end{cases}$$

- $x\in\mathbb{R}|\left\{\frac{\pi}{2}+k\pi\right\},\ k\in\mathbb{Z}$, *we have* $\tan(2x)=\dfrac{2\tan(x)}{1-\tan^2(x)}$.

Proposition 1.3.5. *(Some usual limits)*

1. $\lim\limits_{x\to 0}\dfrac{\sin(x)}{x}=1.$
2. $\lim\limits_{x\to 0}\dfrac{1-\cos(x)}{x^2}=\dfrac{1}{2}.$
3. $\lim\limits_{x\to 0}\dfrac{\cos(x)-1}{x}=0.$

4. $\lim\limits_{x \to -\frac{\pi}{2}} \tan(x) = -\infty$.

5. $\lim\limits_{x \to \frac{\pi}{2}} \tan(x) = +\infty$.

6. $\lim\limits_{x \to 0} \dfrac{\tan(x)}{x} = 1$.

1.3.4 *Cotangent function*

Definition 1.3.6. *The cotangent function $y = \cot x$ is defined by:*

$$\begin{aligned} \cot :\ \mathbb{R}| \{k\pi\} &\longrightarrow \mathbb{R} \\ x &\longrightarrow \cot x = \frac{\cos x}{\sin x} \end{aligned}, \quad k \in \mathbb{Z}$$

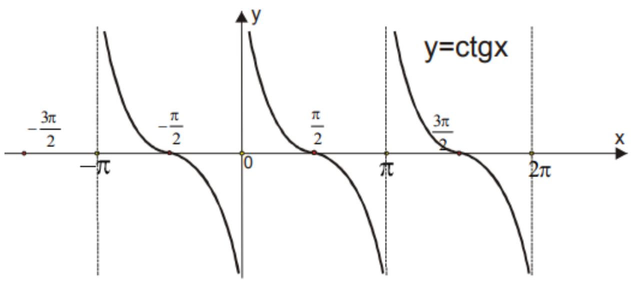

Figure 1.10: Cotangent function

1.4 Inverse Trigonometric Functions

1.4.1 *The function arcsine*

According to the variation table below, we have: the function $\sin(x)$ is continuous and strictly increasing on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, then the function $\sin(x)$ represents a bijection from $[-\frac{\pi}{2}, \frac{\pi}{2}]$ to $[-1, 1]$.

x	$-\frac{\pi}{2}$	0	$+\frac{\pi}{2}$
$\sin(x)' = \cos(x)$		$+$	
$\sin(x)$	-1 ↗	0 ↗	1

Definition 1.4.1. *The inverse function of the restriction of* $\sin(x)$ *on* $[-\frac{\pi}{2}, \frac{\pi}{2}]$ *is called the arcsine function and is denoted by* $\arcsin(x)$ *or* $\sin^{-1}(x)$*:*

$$\begin{aligned} \arcsin: \ [-1,1] &\longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}] \\ x &\longrightarrow \arcsin(x) \end{aligned}$$

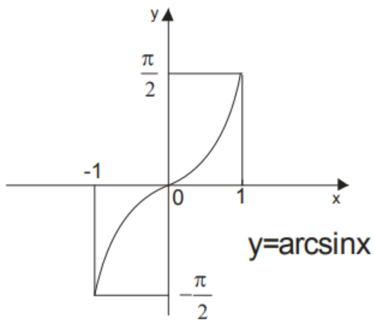

Figure 1.11: Arcsine function

Proposition 1.4.2. *The function* $\arcsin(x)$ *has the following properties:*

1. *The function* $\arcsin(x)$ *is continuous and strictly increasing on* $[-1,1]$.
2. $\arcsin(\sin x) = x, \quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
3. $\sin(\arcsin(x)) = x, \quad x \in [-1,1]$.
4. *The function* $\arcsin(x)$ *is odd.*

5. *Thehe* arcsin *function is indefinitely differentiable on* $]-1,1[$, *and*

$$\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}.$$

More general

$$\arcsin'(f(x)) = \frac{f'(x)}{\sqrt{1-f(x)^2}}.$$

Remark 1.4.3. *some usual values for the function* $\arcsin(x)$*:*

$$\arcsin(-1) = -\frac{\pi}{2} \qquad \arcsin(0) = 0 \qquad \arcsin(1) = \frac{\pi}{2}$$
$$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6} \qquad \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6} \qquad \arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$$
$$\arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4} \qquad \arcsin\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3} \qquad \arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

1.4.2 *The Arccosine Function*

In the variation table below, we have, the function $\cos(x)$ is continuous and strictly decreasing on $[0,\pi]$, so the function $\cos(x)$ makes a bijection from $[0,\pi]$ into $[-1,1]$.

x	0 $\qquad\qquad\qquad$ π
$(\cos(x))' = -\sin(x)$	$-$
$\cos(x)$	$1 \searrow -1$

Definition 1.4.4. *The inverse function of the restriction of* $\cos(x)$ *on* $[0,\pi]$ *is called the arccosine function and is denoted by* $\arccos(x)$ *or* $\cos^{-1}(x)$*:*

$$\begin{aligned} \arccos:\ [-1,1] &\longrightarrow [0,\pi] \\ x &\longrightarrow \arccos(x) \end{aligned}$$

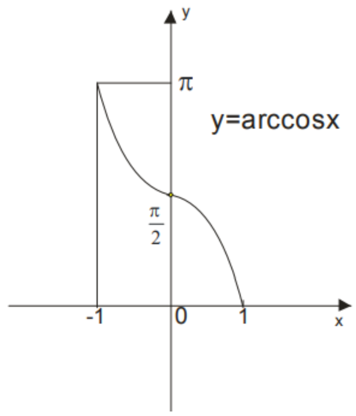

Figure 1.12: Arccosine function

Proposition 1.4.5. *The function* $\arccos(x)$ *has the following properties:*

1. *The function* $\arccos(x)$ *is continuous and strictly decreasing on* $[-1, 1]$.
2. $\arccos(\cos x) = x, \quad x \in [0, \pi]$.
3. $\cos(\arccos(x)) = x, \quad x \in [-1, 1]$.
4. *The function* $\arccos(x)$ *is neither even nor odd.*
5. *The* arccos *function is indefinitely differentiable on* $]-1, 1[$, *and*

$$\arccos'(x) = -\frac{1}{\sqrt{1-x^2}}.$$

More general

$$\arccos'(f(x)) = -\frac{f'(x)}{\sqrt{1-f(x)^2}}.$$

Remark 1.4.6. *some usual values for the function* $\arccos(x)$*:*

$$\arccos(-1) = \pi \qquad \arccos(0) = \frac{\pi}{2} \qquad \arccos(1) = 0$$
$$\arccos\left(-\frac{1}{2}\right) = -\frac{2\pi}{3} \qquad \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3} \qquad \arccos\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$$
$$\arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4} \qquad \arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6} \qquad \arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

1.4.3 *The Arctangent function*

The function $\tan(x) = \dfrac{sin(x)}{cos(x)}$ is defined on $D = \mathbb{R}|\{\frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}\}$. It is continuous and differentiable on its domain of definition and for all $x \in D$ we have:

$$(\tan(x))' = \frac{1}{\cos^2(x)} = 1 + \tan^2(x)$$

Consider the restriction of the function $\tan(x)$ on the interval $]-\frac{\pi}{2}, \frac{\pi}{2}[$, from the table of variation below we have: the function $\tan(x)$ is continuous and strictly increasing on $]-\frac{\pi}{2}, \frac{\pi}{2}[$, then the function $\tan(x)$ makes a bijection from $]-\frac{\pi}{2}, \frac{\pi}{2}[$ into $\mathbb{R}$.

x	$-\frac{\pi}{2}$		$\frac{\pi}{2}$
$(\tan(x))' = \frac{1}{\cos^2}$	||	$+$	||
$\tan(x)$	$-\infty$	$\nearrow$	$+\infty$

Definition 1.4.7. *We call the arctangent function* $\arctan(x)$ *or* $\tan^{-1}(x)$ *the inverse of the tangent function on* $]-\frac{\pi}{2}, \frac{\pi}{2}[$ *defined by:*

$$\begin{aligned} \arctan :]-\infty, +\infty[\ &\longrightarrow \ ]-\frac{\pi}{2}, \frac{\pi}{2}[\\ x \ &\longrightarrow \ \arctan(x) \end{aligned}$$

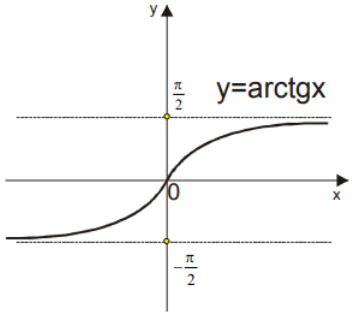

Figure 1.13: Arctan function

Proposition 1.4.8. *The function* $\arctan(x)$ *has the following properties:*

1. *The function* $\arctan(x)$ *is continuous and strictly increasing on* $\mathbb{R}$*, with values in* $]-\frac{\pi}{2}, \frac{\pi}{2}[$.

2. $\arctan(\tan x) = x, \quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$

3. $\tan(\arctan(x)) = x, \quad x \in \mathbb{R}.$

4. *The function* $\arctan(x)$ *is odd.*

5. *The function* $\arctan \in C^{\infty}(\mathbb{R})$*, and we have*

$$\arctan'(x) = \frac{1}{1+x^2}.$$

More general

$$\arctan'(f(x)) = \frac{f'(x)}{1+f^2(x)}.$$

Remark 1.4.9. *The table below shows some usual values for the function* $\arctan(x)$*.*

$\tan(0) = 0$	$\arctan(0) = 0$
$\tan(\frac{\pi}{6}) = \frac{1}{\sqrt{3}}$	$\arctan(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}$
$\tan(\frac{\pi}{4}) = 1$	$\arctan(1) = \frac{\sqrt{2}}{2}$
$\tan(\frac{\pi}{3}) = \sqrt{3}$	$\arctan(\sqrt{3}) = \frac{\pi}{3}$

1.4.4 *The Arccotangent function*

$k^{-1} : \mathbb{R} \longrightarrow [0, \pi]$

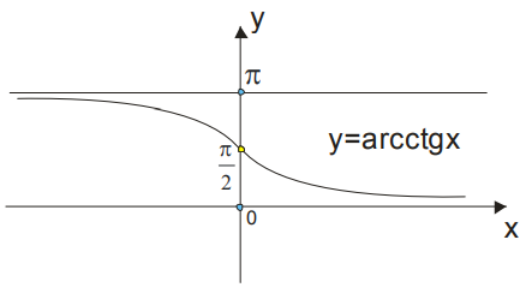

Figure 1.14: Arcctan function

Valid:

- $\operatorname{arcctan}(\cot x) = x, \quad x \in [0, \pi]$.
- $\cot(\operatorname{arcctan}(x)) = x, \quad x \in \mathbb{R}$.

The function $arcctan \in C^{\infty}(\mathbb{R})$, and we have

$$arcctan'(x) = -\frac{1}{1 + x^2}.$$

More general

$$arcctan'(f(x)) = -\frac{f'(x)}{1+f^2(x)}.$$

We have

$$arcctan\ (0) = \frac{\pi}{2}, \qquad \lim_{x\to-\infty} arcctan\ (x) = \pi, \qquad \lim_{x\to+\infty} arcctan\ (x) = 0.$$

It can easily be shown that:

$$\begin{aligned} \arctan x + arcctan\ x &= \frac{\pi}{2}, \quad \forall x \in \mathbb{R}. \\ \arctan x + \arctan\ \frac{1}{x} &= \frac{\pi}{2}, \quad \forall x > 0. \\ \arctan x + \arctan\ \frac{1}{x} &= -\frac{\pi}{2}, \quad \forall x < 0. \end{aligned}$$

1.5 Hyperbolic Functions

1.5.1 *Hyperbolic cosine*

Definition 1.5.1. *We call the hyperbolic cosine function and denoted (ch or cosh), the even part of the exponential function defined by:*

$$ch\ (x) = \frac{e^x + e^{-x}}{2}$$

1.5.2 *Hyperbolic sine*

Definition 1.5.2. *The hyperbolic sine function, denoted by (sh or sinh), is the odd part of the exponential function defined by:*

$$sh\ (x) = \frac{e^x - e^{-x}}{2}$$

1.5.3 *Hyperbolic tangent*

Definition 1.5.3. *The hyperbolic tangent function, denoted by (th or tanh), is the quotient of the hyperbolic sine function with the hyperbolic cosine function and defined by:*

$$th\ (x) = \frac{sh\ (x)}{ch\ (x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

1.5.4 *Hyperbolic cotangent*

Definition 1.5.4. *The hyperbolic tangent function, denoted by (cth or ctanh), is the quotient of the hyperbolic cosine function with the hyperbolic sine function and defined by:*

$$cth\ (x) = \frac{ch\ (x)}{sh\ (x)} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Graphs of these functions are obtained from graphics: $y = e^x$ and $y = e^{-x}$, $\left(y = \frac{1}{2}e^x, \quad and \quad y = \frac{1}{2}e^{-x}\right)$.

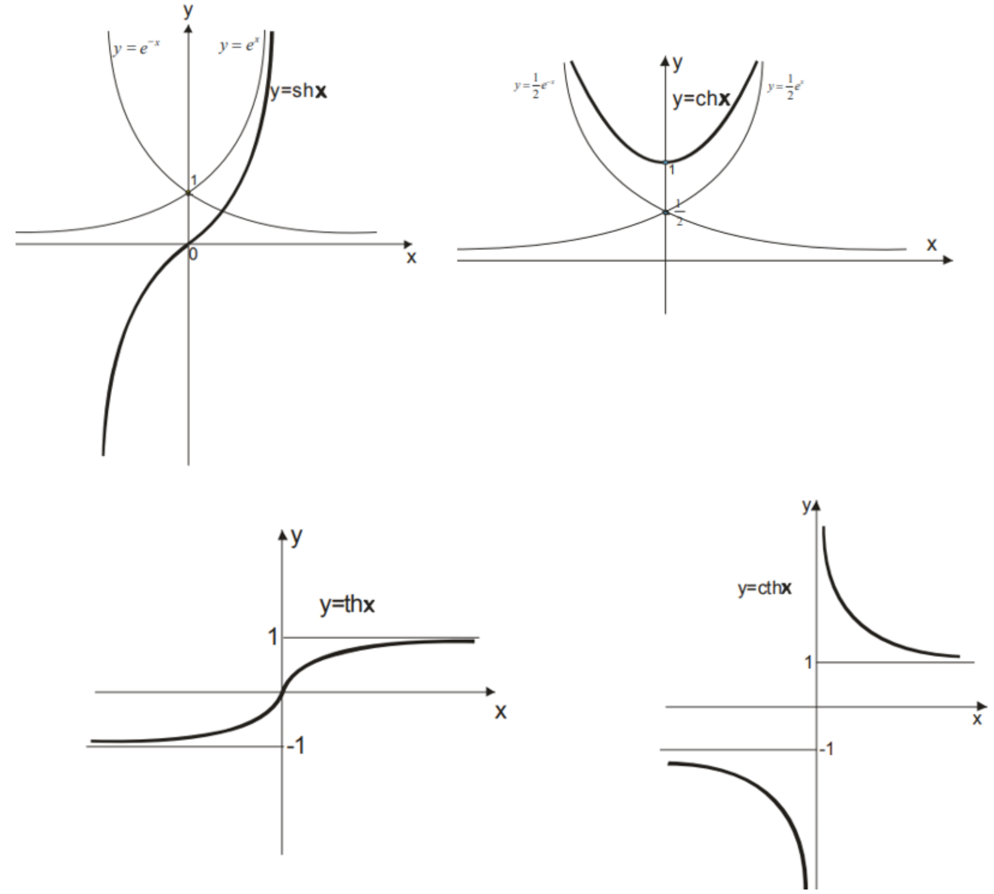

Figure 1.15: Hyperbolic functions

Proposition 1.5.5.
- *The function $ch\ (x)$ is a function defined on $\mathbb{R}$, continuous and even.*
- *The function $sh\ (x)$ is a function defined on $\mathbb{R}$, continuous and odd.*
- *The function $th\ (x)$ is a function defined on $\mathbb{R}$, continuous and odd.*

- *The function* $cth\ (x)$ *is a function defined on* $\mathbb{R}$*, continuous and odd.*

- *The functions* $ch\ (x),\ sh(x),\ th(x)$ *and* $cth\ (x)$ *are differentiable on* $\mathbb{R}$ *and their derivatives are defined by:*

$$\forall\ x \in \mathbb{R};\ \begin{cases} (ch\ (x))' = sh\ (x) \\ (sh\ (x))' = ch\ (x) \\ (th\ (x))' = \dfrac{1}{ch^2\ (x)} = 1 - th^2(x) \\ (cth\ (x))' = -\dfrac{1}{sh^2\ (x)} \end{cases}$$

Remark 1.5.6. *The hyperbolic functions have the following properties:*

1. $ch\ (0) = 1,\quad sh\ (0) = 0,\quad th(0) = 0.$

2. $\lim\limits_{x \to -\infty} ch\ (x) = +\infty,\quad \lim\limits_{x \to -\infty} sh\ (x) = -\infty,\quad \lim\limits_{x \to -\infty} th\ (x) = -1,\quad \lim\limits_{x \to -\infty} cth\ (x) = -1.$

3. $\lim\limits_{x \to +\infty} ch\ (x) = +\infty,\quad \lim\limits_{x \to +\infty} sh\ (x) = +\infty,\quad \lim\limits_{x \to +\infty} th\ (x) = 1,\quad \lim\limits_{x \to +\infty} cth\ (x) = 1.$

Proposition 1.5.7. *For every real* x*, we have:*

- $ch\ (x) + sh\ (x) = e^x,$

- $ch\ (x) - sh\ (x) = e^{-x},$

- $ch^2(x) - sh^2(x) = 1,$

- $sh\ (2x) = 2.sh\ (x).ch\ (x),$

- $ch\ (2x) = ch^2(x) + sh^2(x).$

Proposition 1.5.8. *(Addition formulas):*

For all $(x, y) \in \mathbb{R}^2$*, we have the following formulas:*

- $ch(x + y) = ch\ (x).ch\ (y) + sh\ (x).sh\ (y),$

- $ch(x - y) = ch\ (x).ch\ (y) - sh\ (x).sh\ (y),$

- $sh(x+y) = sh\ (x).ch\ (y) + ch\ (x).sh\ (y)$,
- $sh(x-y) = sh\ (x).ch\ (y) - ch\ (x).sh\ (y)$,
- $th(x+y) = \dfrac{th\ (x) + th\ (y)}{1 + th\ (x).th\ (y)}$,
- $th(x-y) = \dfrac{th\ (x) - th\ (y)}{1 - th\ (x).th\ (y)}$,

Proposition 1.5.9. *(Some usual limits of hyperbolic functions):*

1. $\lim\limits_{x\to+\infty} \dfrac{ch\ (x)}{e^x} = \dfrac{1}{2}$,
2. $\lim\limits_{x\to+\infty} \dfrac{sh\ (x)}{e^x} = \dfrac{1}{2}$,
3. $\lim\limits_{x\to 0} \dfrac{sh\ (x)}{x} = 1$,
4. $\lim\limits_{x\to 0} \dfrac{ch\ (x) - 1}{x^2} = \dfrac{1}{2}$.