Solution of series $N°4$

Exercise 1:

1. $f(x) = \begin{cases} x^2 \cos\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, $x_0 = 0$.

   we have:

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x^2 \cos\frac{1}{x}}{x-0} = \lim_{x\to 0} x \cos\frac{1}{x} = 0$$

   because $\left(-x \leq x\cos\dfrac{1}{x} \leq x,\ \textit{and}\ \lim\limits_{x\to 0} x = 0\right)$. So the function f is differentiable in x_0 and $f'(0) = 0$.

2. $f(x) = \begin{cases} \sin x \sin\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, $x_0 = 0$. we have:

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{\sin x \sin\frac{1}{x}}{x-0} = \lim_{x\to 0} \frac{\sin x}{x} \sin\frac{1}{x} = \lim_{x\to 0} \sin\frac{1}{x}, \text{ does not exist.}$$

   $\left(\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1\right)$, therefore f is not differentiable at $x_0 = 0$.

3. $f(x) = \begin{cases} \exp(\frac{1}{x^2-a^2}), & |x| < a \\ 0, & |x| \geq a \end{cases}$, $|x_0| = a,\ a \in \mathbb{R}_+$.

   We have

$$f(x) = \begin{cases} \exp(\frac{1}{x^2-a^2}), & -a < x < a \\ 0, & x \in\ ]-\infty, -a] \cup [a, +\infty[\end{cases}$$

   the differentiability of f in x_0=a: $\lim\limits_{x\to a^+} \dfrac{f(x)-f(a)}{x-a} = \lim\limits_{x\to a^+} \dfrac{0-0}{x-a} = 0 = f'_r(a)$

$$\lim_{x\to a^-} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a^-} \frac{\exp(\frac{1}{x^2-a^2}) - 0}{x-a} = 0 = f'_l(a)$$

   We have $f'_r(a) = f'_l(a)$, then f is differentiable at $x_0 = a$, and $f'(a) = 0$.

The differentiability of f in $x_0 = -a$:

$$\lim_{x\to -a^-} = \frac{f(x)-f(-a)}{x+a} = \lim_{x\to -a^-} \frac{0-0}{x+a} = 0 = f'_r(-a).$$

$$\lim_{x\to -a^+} = \frac{f(x)-f(-a)}{x+a} = \lim_{x\to -a^+} \frac{\exp(\frac{1}{x^2-a^2})-a}{x+a} = 0 = f'_l(-a).$$

We have: $f'_r(-a) = f'_l(-a)$, then f is differentiable at $x_0 = -a$, and $f'(-a) = 0$.

Exercise 2:

$$f(x) = \begin{cases} ax^2+bx+1, & 0 \le x < 1 \\ \sqrt{x}, & x \ge 1 \end{cases}$$

We determine a and b such that f is differentiable on $\mathbb{R}^*_+$, we have $\sqrt{x}$ is differentiable on $]0,1[$, and ax^2+bx+1 is differentiable on $]1,+\infty[$, so f is differentiable on $]0,1[\cup]1,+\infty[$.

The differentiability of f in $x_0 = 1$: $(f(1) = 1)$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \sqrt{x} = 1 = f(1)$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} ax^2+bx+1 = a+b+1.$

f is continuous at $x_0 = 1 \iff \lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = f(1) \Longrightarrow a+b+1 = 1 \iff a = -b$. Therefore f is continuous for $a = -b$.

$$\lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-} \frac{\sqrt{x}-1}{x-1} = \lim_{x\to 1^-} \frac{\sqrt{x}-1}{(\sqrt{x}-1)(\sqrt{x}+1)} = \lim_{x\to 1^-} \frac{1}{\sqrt{x}+1} = \frac{1}{2} = f'_l(1).$$

$$\lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+} \frac{ax^2+bx+1-1}{x-1} = \lim_{x\to 1^+} \frac{ax^2-ax}{x-1} = \lim_{x\to 1^+} \frac{ax(x-1)}{x-1} = a = f'_r(1).$$

f is differentiable on $x_0 = 1 \Leftrightarrow f'_r(1) = f'_l(1) \Rightarrow a = \frac{1}{2}$, *and* $b = -a = -\frac{1}{2}$. So f is differentiable on $x_0 = 1$ for $a = \frac{1}{2}$, and $b = -\frac{1}{2}$.

Calculate: $f'(x)$:

$$f'(x) = \begin{cases} x - \dfrac{1}{2} & : \ 0 \leq x < 1, \\ \dfrac{1}{2\sqrt{x}} & : \ x \geq 1. \end{cases}$$

Exercise 3:

1. Calculate derivatives:

- $y_1(x) = \sqrt{\ln x + 1} + \ln(\sqrt{x} + 1) \Longrightarrow y_1' = \dfrac{1}{2x\sqrt{\ln x + 1}} + \dfrac{1}{2(x + \sqrt{x})}$.
- $y_2(x) = \dfrac{\sqrt{\cos x}}{1 - e^x} \Longrightarrow y_2' = \dfrac{-\sin x + \sin x e^{-x} - \sqrt{\cos x} e^{-x}}{(1 - e^{-x})^2}$.
- $y_3(x) = e^{\cos \sqrt{x}} \Longrightarrow y_3' = \dfrac{-1}{2\sqrt{x}} \sin(\sqrt{x}) e^{\cos \sqrt{x}}$.

2. Calculate $n - th$ derivatives:

- $y_1(x) = \ln(1 + x)$

$$y_1'(x) = \frac{1}{1+x} \qquad y_1''(x) = \frac{-1}{(1+x)^2}$$
$$y_1^{(3)}(x) = \frac{2}{(1+x)^3} \qquad y_1^{(4)}(x) = -\frac{2 \times 3}{(1+x)^4}$$
$$y_1^{(5)}(x) = \frac{2 \times 3 \times 4}{(1+x)^5} \qquad y_1^{(6)}(x) = -\frac{2 \times 3 \times 4 \times 5}{(1+x)^6}.$$
$$\vdots$$
$$y_1^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$$

- $y_2(x) = \dfrac{1+x}{1-x}$

$$y_2' = \frac{2}{(1-x)^2} \qquad y_2'' = \frac{2 \times 2}{(1-x)^3}$$
$$y_2^{(3)} = \frac{2 \times 2 \times 3}{(1-x)^4} \qquad y_2^{(4)} = \frac{2 \times 2 \times 3 \times 4}{(1-x)^5}$$
$$\vdots$$
$$y_2^{(n)} = \frac{2n!}{(1-x)^{n+1}}$$

- $y_3(x) = (x+1)^3 e^{-x}$

  Assume that $g(x) = (x+1)^3$, and $f(x) = e^{-x}$, so

$g'(x) = 3(x+1)^2, \quad g''(x) = 6(x+1), \quad g^{(3)}(x) = 6,\ g^{(n)}(x) = 0,\ \forall n \geq 4$

$f'(x) = -e^{-x}, \quad f''(x) = e^{-x}, \quad f^{(n)}(x) = (-1)^n e^{-x},\ \forall n \in \mathbb{N}$

then,

$$\begin{aligned} (y_3)^{(n)} &= \sum_{k=0}^{n} C_n^k\ (e^{-x})^{(n-k)}\ ((1+x)^3)^{(k)} \\ &= \sum_{k=0}^{3} C_3^k\ (e^{-x})^{(3-k)}\ ((1+x)^3)^{(k)} \\ &= -e^{-x}(1+x)^3 + 9e^{-x}(1+x)^2 - 18e^{-x}(1+x) + 6e^{-x}. \end{aligned}$$

- $y_4(x) = x^2 \sin 3x$, according to Leibniz

  $f^{(n)} = \sum_{k=0}^{n} C_n^k\ (\sin\ 3x)^{(n-k)}\ (x^2)^{(k)}$

  $= 3^n x^2 \sin\left(3x + \frac{n\pi}{2}\right) + 2xn3^{n-1} \sin\left(3x + \frac{(n-1)\pi}{2}\right) + n\,(n-1)\,3^{n-2} \sin\left(3x + \frac{(n-2)\pi}{2}\right)$

Exercise 4:

Determine the extrema:

x_0 is extremum $\Longleftrightarrow f'(x_0) = 0$ and $f''(x_0) \neq 0$.

1. $f(x) = \sin x^2$, on $[0, \pi]$, $f'(x) = 2x \cos x^2$ the critical points are:

$$f'(x) = 0 \Longleftrightarrow 2x \cos x^2 = 0 \Longleftrightarrow \begin{cases} x = 0 \\ \cos x^2 = 0 \end{cases}$$

therefore, $\begin{cases} x = 0 \\ x^2 = \dfrac{\pi}{2} + k\pi \end{cases} \Longleftrightarrow \begin{cases} x = 0 \\ x = \sqrt{\dfrac{\pi}{2} + k\pi} \end{cases}$

$f''(x) = 2\cos x^2 - 4x^2 \sin x^2$, so:

- Four $x = 0$, $f''(0) = 2 > 0$, then 0 is an extremum (minimum).
- Four $x = \sqrt{\frac{\pi}{2} + k\pi}$, $f''(\sqrt{\frac{\pi}{2} + k\pi}) = -4\left(\frac{\pi}{2} + k\pi\right) \sin\left(\frac{\pi}{2} + k\pi\right) \neq 0.$
  - if k is even: $\sin\left(\frac{\pi}{2} + k\pi\right) = 1$, and $f''(\sqrt{\frac{\pi}{2} + k\pi}) = -4\left(\frac{\pi}{2} + k\pi\right) < 0$, so $\sqrt{\frac{\pi}{2} + k\pi}$ is an extremum (maximum).

– if k is odd $\sin\left(\frac{\pi}{2}+k\pi\right)=-1$, and $f''(\sqrt{\sin\frac{\pi}{2}+k\pi})=4\left(\sqrt{\frac{\pi}{2}+k\pi}\right)>0$, so $\sqrt{\frac{\pi}{2}+k\pi}$ is an extremum (minimum).

2. $g(x)=x^4-x^3+1$, on $\mathbb{R}$

$g'(x)=4x^3-3x^2$, the critical points are:

$$g'(x)=0 \Longleftrightarrow 4x^3-3x^2=0 \Longleftrightarrow x^2(4x-3)=0 \Longleftrightarrow \begin{cases} x=0 \\ x=\frac{3}{4} \end{cases}$$

$g''(x)=12x^2-6x$

- For $x=\frac{3}{4}$, $g''(\frac{3}{4})=\frac{9}{4}>0$, so $\frac{3}{4}$ is an extremum (minimum).
- For $x=0$, $g''(0)=0 \Longrightarrow f^{(3)}(x)=24x-6 \Longrightarrow f^{(3)}(0)\neq 0$, so 0 is not an extremum.

Exercise 5:

1. (a) $f(x)=\sin^2 x$, on $[0,\pi]$

   we have f is continunous on $\mathbb{R}$, so it is continuous on $[0,\pi]$, and differentiable on $]0,\pi[$.

   $f(0)=0$, and $f(\pi)=0 \Longrightarrow f(0)=f(\pi)$, so we can apply Rolle's theorem on f.

   (b) the same for $g(x)=\frac{\sin x}{2x}$, on $[-\frac{\pi}{2},\frac{\pi}{2}]$.

2. We show that $x<\frac{y-x}{\ln y-\ln x}<y$. $\forall x,\ y\in\mathbb{R}_+^*$, we apply the Mean value theorem on the function $f(t)=\ln t$ on the iterval $[x,y]$ such that $0<x<y$.

   $f(t)=\ln t$ is continuous on $[x,y]$, and differentiable on $]x,y[$, then according to the Mean value theorem:

$\exists\ x \in]x, y[:\ \ f'(c) = \dfrac{f(y) - f(x)}{y - x}$ *so* $\dfrac{1}{c} = \dfrac{\ln y - \ln x}{y - x} \Longrightarrow c = \dfrac{y - x}{\ln y - \ln x}$, and

$c \in]x, y[$, then $x < \dfrac{y - x}{\ln y - \ln x} < y$.

Exercise 6:

1. $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{e^x - 1} = \dfrac{0}{0}$

   $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{e^x - 1} = \lim\limits_{x \to 0} \dfrac{(1 - \cos x)'}{(e^x - 1)'} = \lim\limits_{x \to 0} \dfrac{\sin x}{e^x} = 0.$

2. $\lim\limits_{x \to \pi} \dfrac{\sin x}{x^2 - \pi^2} = \dfrac{0}{0}$

   $\lim\limits_{x \to \pi} \dfrac{\sin x}{x^2 - \pi^2} = \lim\limits_{x \to \pi} \dfrac{(\sin x)'}{(x^2 - \pi^2)'} = \lim\limits_{x \to \pi} \dfrac{\cos x}{2x} = \dfrac{-1}{2\pi}.$