Contents

Chapter 1

Differentiable Functions

1.1 The Derivative

1.1.1 *Definition and basic properties*

Definition 1.1.1. *Let I be an interval, and $c \in I$, let $f : I \longrightarrow \mathbb{R}$ be a function defined in the neighborhood of c. If the limit*

$$l = \lim_{x \longrightarrow c} \frac{f(x) - f(c)}{x - c},$$

exists in $\mathbb{R}$, then we say that f is differentiable at c. When this limit exists, it is denoted by $f'(c)$ and called the derivative of f at c.
If f is differentiable at all $c \in I$, then we simply say that f is differentiable. The derivative is sometimes written as $\frac{df}{dx}$ or $\frac{d}{dx}(f(x))$. The expression $\frac{f(x) - f(c)}{x - c}$ is called the difference quotient.

The graphical interpretation of the derivative is depicted in Figure 1.1. The left-hand plot gives the line through $(c, f(c))$ and $(x, f(x))$ with slope $\frac{f(x) - f(c)}{x - c}$, that is, the so-called secant line. When we take the limit as x goes to c, we get the right-hand plot, where we see that the derivative of the function at the point c is the slope of the line tangent to the graph of f at the point $(c, f(c))$.

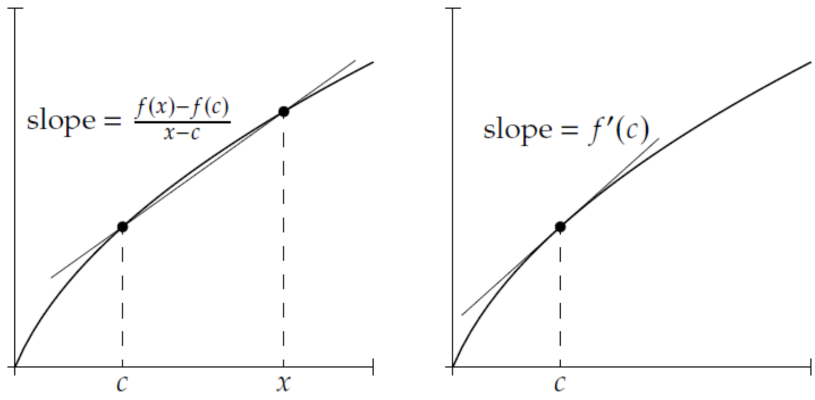

Figure 1.1: Graphical interpretation of the derivative

Example 1.1.2. *Let $f(x) = x^2$ defined on the whole real line, and let $c \in \mathbb{R}$ be arbitrary. We find that if $x \neq c$,*

$$\frac{x^2 - c^2}{x - c} = \frac{(x+c)(x-c)}{x-c} = x + c.$$

Therefore,

$$f'(c) = \lim_{x \longrightarrow c} \frac{x^2 - c^2}{x - c} = \lim_{x \longrightarrow c} (x + c) = 2c.$$

Example 1.1.3. *The function $f(x) = \sqrt{x}$ is differentiable for $x > 0$. To see this fact, fix $c > 0$, and suppose $x \neq c$ and $x > 0$. Compute*

$$\frac{\sqrt{x} - \sqrt{c}}{x - c} = \frac{\sqrt{x} - \sqrt{c}}{(\sqrt{x} - \sqrt{c})(\sqrt{x} + \sqrt{c})} = \frac{1}{\sqrt{x} + \sqrt{c}}.$$

Therefore,

$$f'(c) = \lim_{x \longrightarrow c} \frac{\sqrt{x} - \sqrt{c}}{x - c} = \lim_{x \longrightarrow c} \frac{1}{\sqrt{x} + \sqrt{c}} = \frac{1}{2\sqrt{c}}.$$

Remark 1.1.4. *If we put $x - c = h$, the quantity $\frac{f(x) - f(c)}{x - c}$ becomes $\frac{f(c+h) - f(c)}{h}$. So we can define the notion of differentiability of f at c in the following way:*

$$f \text{ is differentiable at } c \Leftrightarrow \lim_{h \longrightarrow 0} \frac{f(c+h) - f(c)}{h} \text{ exists in } \mathbb{R}.$$

Proposition 1.1.5. *Let $f : I \longrightarrow \mathbb{R}$ be differentiable at $c \in I$, then it is continuous at c.*

Proof 1.1.6. *We know the limits*

$$\lim_{x\longrightarrow c}\frac{f(x)-f(c)}{x-c}=f'(c),\qquad \textit{and}\qquad \lim_{x\longrightarrow c}(x-c)=0.$$

exists. Furthermore,

$$f(x)-f(c)=\left(\frac{f(x)-f(c)}{x-c}\right)(x-c),$$

Therefore, the limit of $f(x)-f(c)$ exists and

$$\lim_{x\longrightarrow c}(f(x)-f(c))=\left(\lim_{x\longrightarrow c}\frac{f(x)-f(c)}{x-c}\right)\left(\lim_{x\longrightarrow c}(x-c)\right)=f'(c).0=0.$$

Hence $\lim\limits_{x\longrightarrow c}f(x)=f(c)$, and f is continuous at c.

Proposition 1.1.7. *If f is differentiable over I, then f is continuous over I.*

Proposition 1.1.8. *Let I be an interval, let $f : I \longrightarrow \mathbb{R}$ and $g : I \longrightarrow \mathbb{R}$ be a differentiable functions at $c \in I$, and let $\alpha \in \mathbb{R}$, then:*

1. ***The linearity:***

   - *Define $h : I \longrightarrow \mathbb{R}$ by $h(x) = \alpha.f(x)$. Then h is differentiable at c and $h'(c) = \alpha.f'(c)$.*
   - *Define $h : I \longrightarrow \mathbb{R}$ by $h(x) = f(x) + g(x)$. Then h is differentiable at c and $h'(c) = f'(c) + g'(c)$.*

2. ***Product rule:***

   If $h : I \longrightarrow \mathbb{R}$ is defined by $h(x) = g(x)f(x)$, then h is differentiable at c and

   $$h'(c) = f(c)g'(c) + f'(c)g(c).$$

   Proof 1.1.9. *We have:*

$$\begin{aligned}
\lim_{h\to 0}\frac{(f.g)\,(c+h)-(f.g)\,(c)}{h} &= \lim_{h\to 0}\frac{f\,(c+h)\,.g\,(c+h)-f\,(c)\,.g\,(c)}{h}\\
&= \lim_{h\to 0}\left[\frac{f(c+h)[g(c+h)-g(c)]}{h}+\frac{[f(c+h)-f(c)]g(c)}{h}\right]\\
&= \lim_{h\to 0}f(c+h)\lim_{h\to 0}\frac{g(c+h)-g(c)}{h}+\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}\lim_{h\to 0}g(c)\\
&= f'(c)g(c)+f(c)g'(c).
\end{aligned}$$

3. ***Quotient rule:***

   If $g(x)\neq 0$ for all $x\in I$, and if $h:I\longrightarrow\mathbb{R}$ is defined by $h(x)=\dfrac{f(x)}{g(x)}$, then h is differentiable at c and

$$h'(c)=\frac{f'(c)g(c)-f(c)g'(c)}{(g(c))^2}.$$

1.1.2 *Chain rule*

Proposition 1.1.10. *Let I, and J be an intervals, let $g:I\longrightarrow J$ be a differentiable at $c\in I$, and $f:J\longrightarrow\mathbb{R}$ be differentiable at $g(c)$. If $h:I\longrightarrow\mathbb{R}$ is defined by*

$$h(x)=(f\circ g)(x)=f(g(x)),$$

then h is differentiable at c and

$$h'(c)=f'(g(c))g'(c).$$

1.1.3 *Inverse function*

Proposition 1.1.11. *Let $I\subset\mathbb{R}$ be an interval, and let f be an injective and continuous function on I. If f is differentiable at a point c with $f'(c)\neq 0$, then the inverse function: $f^{-1}:f(I)\longrightarrow\mathbb{R}$ is differentiable at $f(c)$ and*

$$(f^{-1}(f(c)))' = \frac{1}{f'(c)}.$$

1.2 Left and Right Derivatives

Definition 1.2.1. *Let $f : [a,b] \longrightarrow \mathbb{R}$ be a function, we say that f is right-differentiable at $a \le c < b$ with right derivative $f'(c^+)$ if*

$$\lim_{x \overset{>}{\longrightarrow} c} \frac{f(x) - f(c)}{x - c} = f'(c^+),$$

exists, and f is left-differentiable at $a < c \le b$ with left derivative $f'(c^-)$ if

$$\lim_{x \overset{<}{\longrightarrow} c} \frac{f(x) - f(c)}{x - c} = f'(c^-) \text{ exists.}$$

A function is differentiable at $a < c < b$ if and only if the left and right derivatives exist at c and are equal.

Remark 1.2.2. *If $f'(c^+)$ and $f'(c^-)$ exist but $f'(c^+) \neq f'(c^-)$ then f is not differentiable at c and point $(c, f(c))$ is an angular point.*

Example 1.2.3. *The absolute value function $f(x) = |x|$ is left and right differentiable at 0 with left and right derivatives*

$$f'(0^+) = 1 \quad \text{and} \quad f'(0^-) = -1.$$

These are not equal, and f is not differentiable at 0.

1.3 Successive Derivatives and Leibnitz's Rule

1.3.1 *Successive derivatives*

Let f be a function differentiable on I, then f' is called the first order derivative of f, if f' is differentiable on I, then its derivative is called the second order derivative of f and is denoted by f'' or $f^{(2)}$. Recursively, we define the derivative of order n of f as follows: $f^{(n)}(x) = (f^{(n-1)}(x))'$.

Example 1.3.1. *1). Let* $f(x) = \sin(x)$*. Calculate* $f^{(n)}(x)$*. We have:*

$$\begin{aligned} f^{(0)}(x) &= \sin(x), \\ f'(x) &= f^{(1)}(x) = \cos(x) = \sin(x + \frac{\pi}{2}), \\ f^{(2)}(x) &= -\sin(x) = \sin(x + \pi), \\ f^{(3)}(x) &= -\cos(x) = \sin(x + \frac{3\pi}{2}), \\ f^{(4)}(x) &= \sin(x) = \sin(x + 2\pi), \\ &\vdots \\ f^{(n)}(x) &= \sin(x + \frac{n\pi}{2}). \end{aligned}$$

2). $f(x) = \ln x$*. Calculate* $f^{(n)}(x)$*. We have:*

$$f^{(0)}(x) = \ln x, \qquad f'(x) = \frac{1}{x},$$

$$f^{(2)}(x) = \frac{-1}{x^2}, \qquad f^{(3)}(x) = \frac{2}{x^3},$$

$$f^{(4)}(x) = \frac{-2 \times 3}{x^4}, \qquad f^{(5)}(x) = \frac{2 \times 3 \times 4}{x^5} = \frac{4!}{x^5},$$

$$\vdots$$

$$f^{(n)}(x) = (-1)^{n+1} \frac{(n-1)!}{x^n}, \quad n \in \mathbb{N}^*.$$

Definition 1.3.2. *(Class Functions:* C^n*)*

Let n *be a non-zero natural number. A function* f *defined on* I *is said to be of class* C^n *or* n *times continuously differentiable if it is* n *times differentiable and* $f^{(n)}$ *is continuous on* I*, and we note* $f \in C^n(I)$*.*

Remark 1.3.3. *A function* f *is said to be of class* C^0 *if it is continuous on* I*.*

Definition 1.3.4. *(Class Functions:* C^∞*)*

A function f *is said to be of class* C^∞ *on* I *if it is in the class* C^n*.* $\forall n \in \mathbb{N}$*. For example* $f(x) = e^x$*.*

1.3.2 *Leibnitz formula*

Theorem 1.3.5. *Let f and g be two functions n times differentiable on I, then $f \times g$ is $n-$times differentiable on I, and we have:*

$$(f \times g)^{(n)} = \sum_{k=0}^{n} C_n^k \ f^{(n-k)} \ g^{(k)}, \qquad C_n^k = \frac{n!}{k!(n-k)!}.$$

Example 1.3.6. *For $n = 2$, we have:*

$$\begin{aligned}(f \times g)^{(2)} &= C_2^0 f''g + C_2^1 f'g' + C_2^2 fg'' \\ &= f''g + 2f'g' + fg''.\end{aligned}$$

For $n = 6$, we have:

$$\begin{aligned}(f \times g)^{(6)} &= C_6^0 f^{(6)}g + C_6^1 f^{(5)}g' + C_6^2 f^{(4)}g'' + C_6^3 f^{(3)}g^{(3)} + C_6^4 f''g^{(4)} + C_6^5 f'g^{(5)} + C_6^6 fg^{(6)} \\ &= f^{(6)}g + 6f^{(5)}g' + 15f^{(4)}g'' + 20f^{(3)}g^{(3)} + 15f''g^{(4)} + 6f'g^{(5)} + fg^{(6)}.\end{aligned}$$

If $h(x) = (x^3 + 5x + 1)\, e^x = f(x)g(x)$, then:

$$\begin{aligned}&f'(x) = 3x^2 + 5, && g'(x) = e^x, \\ &f''(x) = 6x, && g''(x) = e^x, \\ &f^{(3)}(x) = 6, && g^{(3)}(x) = e^x, \\ &f^{(4)}(x) = 0, && g^{(4)}(x) = e^x, \\ &f^{(n)}(x) = 0, \quad \forall n \geq 4, && g^{(n)}(x) = e^x.\end{aligned}$$

So:

$$\begin{aligned}h^{(n)}(x) &= C_n^0 fg^{(n)} + C_n^1 f'g^{(n-1)} + C_n^2 f''g^{(n-2)} + C_n^3 f^{(3)}g^{(n-3)} + C_n^4 f^{(4)}g^{(n-4)} + \cdots \\ &= (x^3 + 5x + 1)e^x + n(3x^2 + 5)e^x + \frac{n(n-1)}{2}(6x)e^x + \frac{n(n-1)(n-2)}{6}6e^x.\end{aligned}$$

1.4 The Mean Value Theorem

1.4.1 *Extreme values*

Definition 1.4.1. *A critical point of a function $f(x)$, is a value c in the domain of f where f is not differentiable or its derivative is 0 (i.e. $f'(c) = 0$).*

Definition 1.4.2. *A function f is said to have a local maximum (local minimum) at c if f is defined on an open interval I containing c and $f(x) \leq f(c)$ $(f(x) \geq f(c))$ for all $x \in I$. In either case, f is said to have a local extremum at c.*

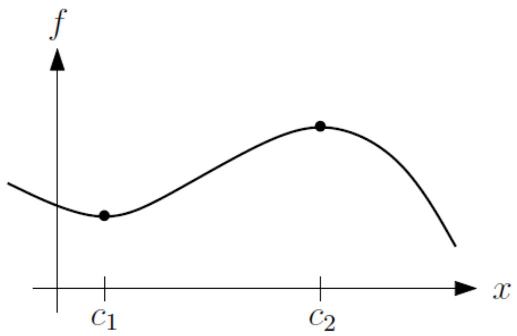

Figure 1.2: Local extrema of f

1.4.2 *Local extremum theorem*

Theorem 1.4.3. *If f has a local extremum at c and if f is differentiable at c, then $f'(c) = 0$.*

Proof. Suppose that f has a local maximum at c. Let I be an open interval containing c such that $f(x) \leq f(c)$ for all $x \in I$. Then:

$$\frac{f(x) - f(c)}{x - c} = \begin{cases} \geq 0, & if \;\; x \in I \;\; and \;\; x < c, \\ \leq 0, & if \;\; x \in I \;\; and \;\; x > c. \end{cases}$$

It follows that the left-hand derivative of f at c is ≥ 0 and the right-hand derivative is ≤ 0, hence $f'(c) = 0$. The proof for the local minimum case is similar. □

1.4.3 *Rolle's theorem*

Theorem 1.4.4. *Let f be a continuous function on $[a,b]$ and differentiable on $]a,b[$. If $f(a) = f(b)$, then there exists a point $c \in]a,b[$ such that $f'(c) = 0$.*

Proof. By the extreme value theorem there exist $x_m, x_M \in [a,b]$ such that $f(x_m) \leq f(x) \leq f(x_M)$ for all $x \in [a,b]$. If $f(x_m) = f(x_M)$, then f is a constant function and the assertion of the theorem holds trivially. If $f(x_m) \neq f(x_M)$, then either $x_m \in]a,b[$ or $x_M \in]a,b[$, and the conclusion follows from the local extremum theorem. □

1.4.4 *Mean value theorem*

Theorem 1.4.5. *If f is continuous on $[a,b]$ and differentiable on $]a,b[$, then there exists $c \in]a,b[$ such that:*

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$

Proof. The function $g : [a,b] \longrightarrow \mathbb{R}$ defined by:

$$g(x) = f(x) - f(a) - \left[\frac{f(b)-f(a)}{b-a}\right](x-a),$$

is continuous on $[a,b]$ and differentiable on $]a,b[$ with

$$g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}.$$

Moreover, $g(a) = g(b) = 0$. Rolle's theorem implies that there exists $a < c < b$ such that $g'(c) = 0$, which proves the result. □

1.4.5 *Mean value inequality*

Let f be a continuous function on $[a,b]$, and differentiable on $]a,b[$. If there exists a constant M such that: $\forall\, x \in]a,b[$: $|f'(x)| \leq M$, then

$$\forall\, x,\ y \in [a,b] :\ |f(x) - f(y)| \leq M\,|x-y|.$$

According to the Mean value theorem on $[x, y]$, $\exists\ c \in]x, y[$: $f'(c) = \dfrac{f(x) - f(y)}{x - y}$. Then

$$|f'(c)| \leq M \Longrightarrow \left|\frac{f(x) - f(y)}{x - y}\right| \leq M \Longrightarrow M\,|x - y|\,.$$

1.5 Variation of a Functions

Let f be a continuous function on $[a, b]$, and differentiable on $]a, b[$ then:

1. $\forall\ x \in]a, b[$: $\quad f'(x) > 0 \Longleftrightarrow f$ is strictly increasing on $[a, b]$.
2. $\forall\ x \in]a, b[$: $\quad f'(x) < 0 \Longleftrightarrow f$ is strictly decreasing on $[a, b]$.
3. $\forall\ x \in]a, b[$: $\quad f'(x) = 0 \Longleftrightarrow f$ is a constant.

1.6 L'Hpital's Rule

Let f and g be two continuous functions on I (I is a neighborhood of c), differentiable on $I - \{c\}$, and satisfying the following conditions:

- $\lim\limits_{x \longrightarrow c} f(x) = \lim\limits_{x \longrightarrow c} g(x) = 0$ or $\pm\infty$.
- $g'(x) \neq 0, \quad \forall x \in I - \{c\}$.

then:

$$\text{if } \lim_{x \longrightarrow c} \frac{f'(x)}{g'(x)} = l \Longrightarrow \lim_{x \longrightarrow c} \frac{f(x)}{g(x)} = l.$$

Example 1.6.1. *Using L'Hopital's rule:*

1. $\lim\limits_{x \longrightarrow 0} \dfrac{3x - \sin x}{x} = \lim\limits_{x \longrightarrow 0} \dfrac{3 - \cos x}{1} = 2.$

2. $\lim\limits_{x \longrightarrow 0} \dfrac{\sqrt{1 + x} - 1}{x} = \lim\limits_{x \longrightarrow 0} \dfrac{\frac{1}{2\sqrt{1+x}}}{1} = \dfrac{1}{2}.$

Remark 1.6.2. *The converse is generally false. For example:* $f(x) = x^2 \cos(\frac{1}{x})$*, and* $g(x) = x$*, so we have* $\lim_{x \longrightarrow 0} \frac{f(x)}{g(x)} = \lim_{x \longrightarrow 0} x \cos(\frac{1}{x}) = 0$ *while* $\lim_{x \longrightarrow 0} \frac{f'(x)}{g'(X)} = \lim_{x \longrightarrow 0} (2x \cos(\frac{1}{x}) + \sin(\frac{1}{x}))$ *does not exists because* $(\lim_{x \longrightarrow 0} \sin(\frac{1}{x})$ *does not exists).*

1.7 Convex Functions

Definition 1.7.1. *A function* f *is said to be convex on an interval* I *if*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y), \quad \forall\ t \in [0.1],\ x,\ y \in I.$$

f *is concave if* $-f$ *is convex.*

Example 1.7.2.

1. *The function* $x \longrightarrow |x|$ *is convex on* $\mathbb{R}$ *because* $|tx + (1+t)y| \leq t\,|x| + (1-t)\,|y|$.
2. *The affine functions* $f : x \longrightarrow \alpha x + \beta$ *are both convex and concave on* $\mathbb{R}$*, because they indeed satisfy* $f(xt + (1-t)y) = tf(x) + (1-t)f(y)$*. Conversely, if a function is both convex and concave then it is affine.*

Theorem 1.7.3. *If* $f :]a, b[\longrightarrow \mathbb{R}$ *has an increasing derivative, then* f *is convex. In particular,* f *is convex if* $f'' \geq 0$.

Example 1.7.4. *Consider the function* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *given by* $f(x) = \sqrt{x^2+1}$*. We have* $f'(x) = \frac{x}{\sqrt{x^2+1}}$*, and* $f''(x) = \frac{1}{(x^2+1)^{\frac{3}{2}}}$*. Since* $f''(x) \geq 0$ *for all* x*, it follows from the corollary that* f *is convex.*

Remark 1.7.5. *If* $f : I \longrightarrow \mathbb{R}$ *is convex then:*

- f *differentiable on the left and right (therefore continues) and* $f'_l \leq f'_r$.
- *The functions* f'_l, f'_r *are increasing.*
- f *is continuous at every interior point of* I.

- *Let* $f : I \longrightarrow \mathbb{R}$ *a differentiable function. Then* f *is convex* $\Longleftrightarrow$ f' *is increasing on* I.

- *A concave function on* I *is continuous at all points interior to* I.

- *If* f *is differentiable and concave* $\Longleftrightarrow$ f *is decreasing.*