Solution of series $N°3$

Exercise 1:

1. $f(x) = \sqrt{\frac{x+1}{x-1}}$.

   $D_f = \left\{ x \in \mathbb{R} | \ \frac{x+1}{x-1} \geq 0, \ and \ x-1 \neq 0 \right\}$

   $\frac{x+1}{x-1} \geq 0 \Longrightarrow x \in]-\infty, -1] \cup [1, +\infty[$, and $x-1 \neq 0 \Longrightarrow x \neq 1$, so

   $D_f =]-\infty, -1] \cup]1, +\infty[$.

2. $g(x) = \sqrt{x^2 + x - 2}$.

   $D_g = \{x \in \mathbb{R} | \ x^2 + x - 2 \geq 0\} =]-\infty, -2] \cup [1, +\infty[$.

3. $h(x) = \ln\left(\frac{2+x}{2-x}\right)$.

   $D_h = \left\{ x \in \mathbb{R} | \ \frac{2+x}{2-x} > 0, \ and \ 2-x \neq 0 \right\}$, so $D_h =]-2, 2[$.

4. $k(x) = \frac{\sin x - \cos x}{x - \pi}$.

   $D_k = \{x \in \mathbb{R} | \ x \neq \pi\} =]-\infty, \pi[\cup]\pi, +\infty[$.

5. $p(x) = (1+x)^{\frac{1}{x}} = e^{\frac{1}{x}\ln(1+x)}$.

   $D_p = \{x \in \mathbb{R} | \ x \neq 0, \ and \ 1 + x > 0\} =]-1, 0[\cup]0, +\infty[$.

6. $\phi(x) = \begin{cases} \frac{\sin x . \cos x}{x - \pi} & if \ \ x \neq \pi \\ 1 & Otherwise \end{cases}$

   $D_\phi = \mathbb{R}$.

Exercise 2:

It is necessary to show that $x_1 < x_2 \Longrightarrow f(x_1) < f(x_2)$

We have

$$f(x) = \begin{cases} \frac{x}{1+x} & if \ \ x \geq 0 \\ \frac{x}{1-x} & if \ \ x < 0 \end{cases}$$

- If $x_1 < 0 < x_2$, then it is obvious that $f(x_1) < 0 < f(x_2)$ (if one of the two is zero it is also obvious).

- If $0 < x_1 < x_2$, we note that: $f(x) = \dfrac{x}{x+1} = 1 - \dfrac{1}{1+x}$, so:

$$\begin{aligned} x_1 < x_2 \Longrightarrow \quad x_1 + 1 &< x_2 + 1 \\ \Longrightarrow \quad \frac{-1}{x_1+1} &< \frac{-1}{x_2+1} \\ \Longrightarrow \quad 1 - \frac{1}{x_1+1} &< 1 - \frac{1}{x_2+1} \end{aligned}$$

Therefore, $f(x_1) < f(x_2)$, and f is strictly increasing.

- If $x_1 < x_2 < 0$, in the same way and take $f(x) = \dfrac{x}{1-x} = -1 + \dfrac{1}{1-x}$.

Exercise 3:

1. $\lim\limits_{x \to +\infty} e^{x - \sin x}$, we have:

$$\begin{aligned} \forall x \in \mathbb{R}, \qquad & -1 \leq \sin x \leq 1 \\ \Longrightarrow \quad & -1 \leq -\sin x \leq 1 \\ \Longrightarrow \quad & x - 1 \leq x - \sin x \leq x + 1 \end{aligned}$$

therefore: $x - \sin x \geq x - 1 \Longrightarrow e^{x - \sin x} \geq e^{x-1}$, and because $\lim\limits_{x \to +\infty} e^{x-1} = +\infty$ then $\lim\limits_{x \to +\infty} e^{x - \sin x} = +\infty$.

2. $\lim\limits_{x \to 0} \dfrac{(\tan x)^2}{\cos(2x) - 1}$.
We have $\cos(2x) = 2\cos^2 x - 1$, then

$$\cos(2x) - 1 = 2\cos^2 x - 2 = -2(1 - \cos^2 x) = -2\sin^2 x.$$

So

$$\frac{(\tan x)^2}{\cos(2x) - 1} = \frac{\dfrac{\sin^2 x}{\cos^2 x}}{-2\sin^2 x} = \frac{-\sin^2 x}{2\cos^2 x \sin^2 x} = \frac{-1}{2\cos^2 x}$$

whene $x \longrightarrow 0$ then $\cos^2 x \longrightarrow 1$, therefore, $\lim_{x\to 0} \frac{\tan^2 x}{\cos(2x)-1} = \frac{-1}{2}$.

3. $\lim_{x\to 0^+} \frac{x}{b}\left[\frac{c}{x}\right]$. We have:

$$\begin{aligned} \left[\frac{c}{x}\right] \leq \frac{c}{x} \leq \left[\frac{c}{x}\right] + 1 \\ \Longrightarrow \quad \frac{x}{b}\left[\frac{c}{x}\right] \leq \frac{x}{b}\frac{c}{x} \leq \frac{x}{b}\left[\frac{c}{x}\right] + \frac{x}{b} \\ \Longrightarrow \quad 0 \leq \frac{c}{b} - \frac{x}{b}\left[\frac{c}{x}\right] \leq \frac{x}{b} \end{aligned}$$

$\lim_{x\to 0} \frac{x}{b} = 0 \Longrightarrow \lim_{x\to 0^+} \frac{c}{b} - \frac{x}{b}\left[\frac{c}{x}\right] = 0$, so $\lim_{x\to 0^+} \frac{x}{b}\left[\frac{c}{x}\right] = \frac{c}{b}$.

4. $\lim_{x\to 0} \frac{\ln(1+x^2)}{\sin^2 x}$. We use the L'Hpital's rule, we set $f(x) = \ln(1+x^2)$, and $g(x) = \sin^2 x$, then: $f'(x) = \frac{2x}{1+x^2}$, and $g'(x) = 2\sin x \cos x$.
$\frac{f'(x)}{g'(x)} = \frac{x}{\sin x}.\frac{1}{\cos x(1+x^2)}$, we note that $\lim_{x\to 0} \frac{x}{\sin x} = 1 \left(\lim_{x\to 0} \frac{\sin x}{x} = 1\right)$, and $\lim_{x\to 0} \frac{1}{(1+x^2)\cos x} = 1$, so $\lim_{x\to 0} \frac{\ln(1+x^2)}{\sin^2 x} = 1$.

5. $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$. we have:

$$\begin{aligned} \lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} &= \lim_{x\to 0} \frac{(1+x)-(1-x)}{x(\sqrt{1+x}-\sqrt{1-x})} \\ &= \lim_{x\to 0} \frac{2x}{x(\sqrt{1+x}-\sqrt{1-x})} \\ &= 1. \end{aligned}$$

6. $\lim_{x\to +\infty} \frac{x\ln x + 5}{x^2+4} = \lim_{x\to +\infty} \frac{x\ln x\left(1+\frac{5}{x\ln x}\right)}{x^2\left(1+\frac{4}{x^2}\right)} = \lim_{x\to +\infty} \frac{\ln x}{x}\left(\frac{1+\frac{5}{x\ln x}}{1+\frac{4}{x^2}}\right) = 0.$

Exercise 4:

1. We have:

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longrightarrow f(x) = \begin{cases} \frac{\sin ax}{x} & : x < 0 \\ 1 & : x = 0 \\ 2be^x - x & : x > 0 \end{cases} \end{aligned}$$

we note that for $x > 0$, and $x < 0$ the function f is continuous. For f to be continuous on $\mathbb{R}$, it must be continuous on the right and left of 0.

we have $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 2be^x - x = 2b = f(0) = 1$, so $b = \frac{1}{2}$.

And $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin ax}{x} = a \lim_{x\to 0^-} \frac{\sin ax}{ax} = a = f(0) = 1$, so $a = 1$.

$$2.\ \ g(x) = \begin{cases} \sqrt{x} - \frac{1}{x}, & x \geq 4 \\ (x+a)^2, & x < 4 \end{cases}$$

For the function g to be continuous on $\mathbb{R}$, it is enough to study the continuity at point 4.

$\lim_{x\to 4^+} g(x) = \lim_{x\to 4^+} \sqrt{x} - \frac{1}{x} = \frac{7}{4}$.

$\lim_{x\to 4^-} g(x) = \lim_{x\to 4^-} (x+a)^2 = (4+a)^2$.

g is continuous in 4, i.e.

$$\lim_{x\to 4^+} g(x) = \lim_{x\to 4^-} g(x) \Leftrightarrow (4+a)^2 = \frac{7}{4} \Leftrightarrow |4+a| = \frac{\sqrt{7}}{2}.$$

$$\Longleftrightarrow \begin{cases} 4+a &= \frac{\sqrt{7}}{2} \\ -4-a &= \frac{\sqrt{7}}{2} \end{cases} \Longleftrightarrow \begin{cases} a &= \frac{\sqrt{7}}{2} - 4 \\ a &= \frac{-\sqrt{7}}{2} - 4 \end{cases}$$

Exercise 5:

$$1.\ \ f(x) = \begin{cases} x + \frac{\sqrt{x^2}}{x} & : x \neq 0 \\ 0 & : x = 0 \end{cases}$$

We note that the function f is continuous on $\mathbb{R}^*$, for the continuity at 0 we have:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x+1) = 1.$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x-1) = -1.$$

$\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$, so f is not continuous at 0.

2. $g(x) = \begin{cases} 1 + x\cos\left(\dfrac{1}{x}\right) & : \ x \neq 0 \\ 0 & : \ x = 0 \end{cases}$

the function g is continuous on $\mathbb{R}^*$.

$$\lim_{x\to 0} g(x) = \lim_{x\to 0}\left(1 + x\cos\left(\frac{1}{x}\right)\right) = 1.$$

because $\lim_{x\to 0} x\cos\left(\dfrac{1}{x}\right) = 0$ $\left(0 < \left|x\cos\left(\dfrac{1}{x}\right)\right| < |x|\right)$. Since $\lim_{x\to 0} g(x) = 1 \neq 0 = g(0)$, then g is not continuous at 0.

Exercise 6:

1. $f(x) = \dfrac{x^3 + 2x + 3}{x^3 + 1}$, $D_f = \mathbb{R} - \{-1\}$.

   f is continuous on D_f, as f is a quotient of two continuous polynoms. We note that (-1) is a root of the numerator too so on D_f we have

$$f(x) = \frac{(x+1)(x^2 - x + 3)}{(x+1)(x^2 - x + 1)} = \frac{(x^2 - x + 3)}{(x^2 - x + 1)}$$

   so $\lim_{x\to -1} f(x) = \lim_{x\to -1} \dfrac{x^2 - x + 3}{x^2 - x + 1} = 3$ (exist), then f admits an extension by continuity at the point (-1) given by:

$$\tilde{f} = \begin{cases} f(x) & : \ x \neq -1 \\ 3 & : \ x = -1 \end{cases}$$

2. $g(x) = \dfrac{(1+x)^n - 1}{x}$, $D_g = \mathbb{R}|\{0\}$.

   - If $n = 0$, then $g(x) = 0$, so $\lim_{x\to 0} g(x) = 0$, and g admits an extension by continuity on $\mathbb{R}$ given by $\tilde{g} = 0$.

   - If $n \geq 1$, we use the Newton binomial formula

$$(1+x)^n = \sum_{k=0}^{n} C_n^k x^k \, 1^{n-k} = 1 + C_n^1 x + C_n^2 x^2 + \cdots + C_n^n x^n.$$

such that $C_n^k = \dfrac{n!}{k!(n-k)!}$, $C_n^1 = n$, $C_n^2 = \dfrac{n(n-1)}{n}, \cdots, C_n^n = 1$.

So $g(x) = \dfrac{1}{x}\left[C_n^1 x + C_n^2 x^2 + \cdots + C_n^n x^n\right] = C_n^1 + C_n^2 x + \cdots + C_n^n x^{n-1}$, and

$\lim\limits_{x\to 0} g(x) = C_n^1 = n$ (exist), then g admits extension by continuity on $\mathbb{R}$

given by:

$$\widetilde{g}(x) = \begin{cases} g(x) = \sum\limits_{k=1}^{n} C_n^k x^{k-1} & : \ x \neq 0 \\ n & : \ x = 0 \end{cases}$$

Exercise 7:

1. Let $p > 0$ such that $\forall x \in \mathbb{R},\ f(x+p) = f(x)$. By induction we can show

   $$\forall n \in \mathbb{N}: \ \forall x \in \mathbb{R} \ f(x+np) = f(x).$$

   since f is not constant, then $\exists\ a,\ b \in \mathbb{R}$ such that $f(a) \neq f(b)$. We denote $x_n = a+np$ and $y_n = b+np$, assume that f has a limit in $+\infty$, since $x_n \longrightarrow \infty$ then $f(x_n) \longrightarrow l$, but $f(x_n) = f(a+np) = f(a)$, so $l = f(a)$.

   Likewise with the sequence (y_n), $y_n \longrightarrow \infty$ then $f(y_n) \longrightarrow l$, and $f(y_n) = f(b+np) = f(b)$, so $l = f(b)$.

   Because $f(a) \neq f(b)$ we get a contradiction.

2. We consider the function $g(x) = f(x) - x$ on $[0, +\infty[$. g is continuous, and $g(0) = f(0) > 0$.

   $\lim\limits_{x\to+\infty} g(x) = \lim\limits_{x\to+\infty} (f(x) - x) = \lim\limits_{x\to+\infty} x\left(\dfrac{f(x)}{x} - 1\right) = -\infty$. (because $\lim\limits_{x\to+\infty}\left(\dfrac{f(x)}{x}\right) = a$, and $a - 1 < 0$).

   So $\exists\ b \in \mathbb{R}_+^*$ such that $g(b) < 0$ (also $g(x) < 0$ if $x \geq b$) on $[0, b]$. We have g is continuous and $g(0) > 0,\ g(b) < 0$, according to the intermediate value theorem: $\exists\ x_0 \in [0, b]$ such that $g(x_0) = 0$, so $f(x_0) = x_0$.