Contents

Chapter 1

Real-Valued Functions of a Real Variable

1.1 Basics

1.1.1 *Definition*

Definition 1.1.1. *Let $D \subseteq \mathbb{R}$. A function f from D into $\mathbb{R}$ is a rule which associates with each $x \in D$ one and only one $y \in \mathbb{R}$. We denote*

$$\begin{aligned} f : D &\longrightarrow \mathbb{R}, \\ x &\longmapsto f(x). \end{aligned}$$

D is called the domain of the function. If $x \in D$, then the element $y \in \mathbb{R}$ which is associated with x is called the value of f at x or ***the image*** *of x under f, y is denoted by $f(x)$.*

1.1.2 *Graph of a function*

Definition 1.1.2. *Each couple $(x, f(x))$ corresponds to a point in the $xy-$plane. The set of all these points forms a curve called the graph of the function f.*

$$G_f = \{(x, y)|\ x \in D,\ y = f(x)\}.$$

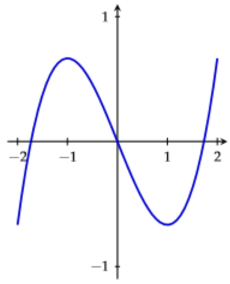


Figure 1.1: Graph of function $f(x) = 1/3x^3 - x$ in interval $[-2, 2]$.

1.1.3 *Operations on functions*

Arithmetic

Let $f, g : D \longrightarrow \mathbb{R}$ be tow functions, then:

1. $(f \pm g)(x) = f(x) \pm g(x), \quad \forall x \in D,$

2. $(f.g)(x) = f(x).g(x), \quad \forall x \in D,$

3. $0.9\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0, \quad \forall x \in D,$

4. $(\lambda.f)(x) = \lambda.f(x), \quad \forall x \in D, \quad \lambda \in \mathbb{R}.$

Composition

Let $f : D \longrightarrow \mathbb{R}$ and let $g : E \longrightarrow \mathbb{R}$, if $f(D) \subseteq E$, then g composed with f is the function $g \circ f : D \longrightarrow \mathbb{R}$ defined by $g \circ f = g[f(x)]$.

Restriction

We say that g is a restriction of the function f if:

$$g(x) = f(x) \text{ and } D(g) \subseteq D(f).$$

Example 1.1.3. *$f(x) = \ln|x|$, and $g(x) = \ln x$, $\forall x \in]0, +\infty[$: $g(x) = f(x)$, and $D(g) \subseteq D(f)$.*

1.1.4 *Bounded functions*

Definition 1.1.4. *Let $f : D \longrightarrow \mathbb{R}$ be a function, then:*

- *We say that f is **bounded from below** on its domain $D(f)$ if*

$$\forall x \in D(f),\ \exists\ m \in \mathbb{R} : \quad m \leq f(x).$$

- *We say that f is **bounded from above** on its domain $D(f)$ if*

$$\forall x \in D(f),\ \exists\ M \in \mathbb{R} : \quad f(x) \geq M.$$

- *Function is **bounded** if it is bounded from below and above.*

Definition 1.1.5. *Let $f, g : D \longrightarrow \mathbb{R}$ be two functions, then:*

- $f \geq g$ *si* $\forall x \in D : \quad f(x) \geq g(x).$
- $f \geq 0$ *si* $\forall x \in D : \quad f(x) \geq 0.$
- $f > 0$ *si* $\forall x \in D : \quad f(x) > 0.$
- f *is said to be constant over D if* $\exists a \in \mathbb{R}, \quad \forall x \in D : \quad f(x) = a.$
- f *is said to be zero over D if* $\forall x \in D : \quad f(x) = 0.$

1.1.5 *Monotone functions*

Definition 1.1.6. *Consider $f : D(f) \subseteq \mathbb{R} \longrightarrow \mathbb{R}$. For all $x, y \in D$, we have:*

- *f is **increasing** (or **strictly increasing**) over D if: $x \leq y \Rightarrow f(x) \leq f(y)$, (or $x < y \Rightarrow f(x) < f(y)$).*
- *f is **decreasing** (or **strictly decreasing**) over D if: $x \leq y \Rightarrow f(x) \geq f(y)$, (or $x < y \Rightarrow f(x) > f(y)$).*

- *f is **monotone** (or **strictly monotone**) over D if f is increasing or decreasing (strictly increasing or strictly decreasing).*

Proposition 1.1.7. *A sum of two increasing (decreasing) functions is an increasing (decreasing) function.*

Proof 1.1.8. *By induction on $N \geq 1$, for any reals $a_1,\ a_2,\ \cdots,\ a_N,\ b_1,\ b_2, \cdots,\ b_N$ with $a_i < b_1$ for all $i = 1, \cdots, N$, we have:*

$$\sum_{i=1}^{N} a_i < \sum_{i=1}^{N} b_i.$$

Assume first that the f_i are all monotone increasing (and that this means strictly). In any case we assume that they're all "the same kind of monotone".
Given reals $x,\ y$ with $x < y$, letting $a_i = f_i(x)$, and $b_i = f_i(y)$, we have $a_i < b_i$ for all i, so:

$$g(x) = \sum_{i=1}^{N} a_i < \sum_{i=1}^{N} b_i = g(y),$$

so g is monotone increasing too. Similarly if the f_i are monotone decreasing.

Corollary 1.1.9. *If f is strictly monotone on D, then f is injective.*
Indeed:

$$\begin{pmatrix} x \neq y \\ \\ x < y \end{pmatrix} \Longrightarrow \begin{pmatrix} f(x) < f(y) \\ \\ or \\ \\ f(x) > f(y) \end{pmatrix} \Longrightarrow f(x) \neq f(y).$$

Example 1.1.10. *Consider the function $f = 2x + 1$. We have*

$$\forall\ x,\ y \in \mathbb{R},\ x < y \Longrightarrow 2x < 2y \Longrightarrow 2x + 1 < 2y + 1 \Longrightarrow f(x) < f(y)$$

so f is strictly increasing then f is injective.

1.1.6 *Even and odd functions*

Definition 1.1.11. • *We say that function $f : D(f) \longrightarrow \mathbb{R}$ is **even** if*

$$\forall x \in D(f): \; f(-x) = f(x).$$

• *We say that function $f : D(f) \longrightarrow \mathbb{R}$ is **odd** if*

$$\forall x \in D(f): \; f(-x) = -f(x).$$

Remark 1.1.12. 1. *Graph of an even function is symmetric with, respect to the y axis.*

2. *Graph of an odd function is symmetric with, respect to the origin.*

3. *Domain of an even or odd function is always symmetric with respect to the origin.*

1.1.7 *Periodic functions*

Definition 1.1.13. *A function $f : D(f) \longrightarrow \mathbb{R}$ is called **periodic** if $\exists \; T \in \mathbb{R}_+^*$ such that:*

1. $x \in D(f) \Rightarrow x \pm T \in D(f)$,

2. $x \in D(f) : f(x \pm T) = f(x)$.

Number T is called a period of f.

1.2 Limits of Functions

1.2.1 *Definition*

Definition 1.2.1. *A set* $U \subset \mathbb{R}$ *is a neighborhood of a point* $x \in \mathbb{R}$ *if:*

$$]x-\delta, x+\delta[\subset U,$$

for some $\delta > 0$. *The open interval* $]x-\delta, x+\delta[$ *is called a* $\delta-$*neighborhood of* x.

Example 1.2.2. *If* $a < x < b$ *then the closed interval* $[a, b]$ *is a neighborhood of* x, *since it contains the interval* $]x-\delta, x+\delta[$ *for sufficiently small* $\delta > 0$. *On the other hand,* $[a, b]$ *is not a neighborhood of the endpoints* a, b *since no open interval about* a *or* b *is contained in* $[a, b]$.

Definition 1.2.3. *Let* f *be a function defined in the neighborhood of* x_0 *except perhaps at* x_0. *A number* $l \in \mathbb{R}$ *is the limit of* f *at* x_0 *if:*

$$\forall \varepsilon > 0,\ \exists\ \delta > 0,\ \forall x \neq x_0 :\ |x - x_0| < \delta \Rightarrow |f(x) - l| < \varepsilon.$$

Notation: $\lim_{x \to x_0} f(x) = l$.

Example 1.2.4. *Let*

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longrightarrow 5x - 3 \end{aligned}$$

Show that $\lim_{x \to 1} f(x) = 2$.
By definition: $\forall \varepsilon > 0,\ \exists\ \delta > 0,\ \forall x \neq 1 :\ |x - 1| < \delta \Rightarrow |f(x) - l| < \varepsilon$. *So we have:*

$$\forall \varepsilon > 0,\ |5x - 3 - 2| < \varepsilon \Rightarrow |5x - 5| < \varepsilon \Rightarrow 5\,|x - 1| < \varepsilon.$$

Then: $|x - 1| < 0.9\frac{\varepsilon}{5}$, *so* $\exists\ \delta = 0.9\frac{\varepsilon}{5} > 0$ *such that* $\lim_{x \to 1} f(x) = 2$.

1.2.2 *Right and left limits*

Definition 1.2.5. *Let f be a function defined in the neighborhood of x_0.*

- *We say that f has a limit l to the right of x_0 if:*

$$\forall \varepsilon > 0,\ \exists\ \delta > 0,\ \forall x_0 < x < x_0 + \delta \Rightarrow |f(x) - l| < \varepsilon.$$

  We write $\lim_{x \to x_0^+} f(x) = \lim_{x \overset{>}{\longrightarrow} x_0} f(x) = l.$

- *We say that f has a limit l to the left of x_0 if:*

$$\forall \varepsilon > 0,\ \exists\ \delta > 0,\ \forall x_0 - \delta < x < x_0 \Rightarrow |f(x) - l| < \varepsilon.$$

  We write $\lim_{x \to x_0^-} f(x) = \lim_{x \overset{<}{\longrightarrow} x_0} f(x) = l.$

- *If f admits a limit at the point x_0 then:*

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = l.$$

Example 1.2.6. *Consider the integer part function at the point $x = 2$.*

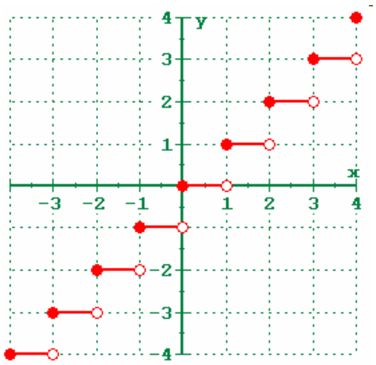


Figure 1.2: Graph of function $f(x) = E(x)$.

- *Since* $x \in]2, 3[$, *we have:* $E(x) = 2$, *and* $\lim_{x \to 2^+} E(x) = 2$.

- *Since* $x \in]1, 2[$, *we have:* $E(x) = 1$, *and* $\lim_{x \to 2^-} E(x) = 1$.

Since these two limits are different, we deduce that the function $f(x) = E(x)$ has no limit at $x = 2$.

Theorem 1.2.7. *If $\lim_{x\to x_0} f(x)$ exists, then it is unique. That is, f can have only one limit at x_0.*

Proof 1.2.8. *We assume that f has two different limits at point x_0; l and l' $(l \neq l')$. We have*

$$\lim_{x\to x_0} f(x) = l \iff \forall\varepsilon > 0,\ \exists\ \delta_1 > 0,\ \forall\ x \neq x_0,\ |x - x_0| < \delta_1 \Longrightarrow |f(x) - l| < \frac{\varepsilon}{2}$$

$$\lim_{x\to x_0} f(x) = l' \iff \forall\varepsilon > 0,\ \exists\ \delta_2 > 0,\ \forall\ x \neq x_0,\ |x - x_0| < \delta_2 \Longrightarrow |f(x) - l'| < \frac{\varepsilon}{2}$$

We pose $\delta = \min(\delta_1,\ \delta_2)$, and $\varepsilon < |l - l'|$, then

$$\forall\ \varepsilon > 0,\ \exists\ \delta > 0,\ \forall\ x \neq x_0,\ |x - x_0| < \delta \Longrightarrow \begin{cases} |f(x) - l| < \dfrac{\varepsilon}{2} \\ and \\ |f(x) - l'| < \dfrac{\varepsilon}{2} \end{cases}$$

we have

$$\begin{aligned} |l - l'| &= |l - l' + f(x) - f(x)| \\ &\leq |f(x) - l| + |f(x) - l'|\ . \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Hence the contradiction with $\varepsilon < |l - l'|$. So $l = l'$.

Proposition 1.2.9. *If $\lim_{x\to x_0} f(x) = l$, and $\lim_{x\to x_0} g(x) = l'$, $l,\ l' \in \mathbb{R}$, then:*

1. $\lim_{x\to x_0}(\lambda.f)(x) = \lambda.\lim_{x\to x_0} f(x) = \lambda.l$, $\forall\lambda \in \mathbb{R}$.

2. $\lim_{x\to x_0}(f+g)(x) = l + l'$, *and* $\lim_{x\to x_0}(f \times g)(x) = l \times l'$.

3. *If $l \neq 0$, then* $\lim_{x\to x_0}\left(0.9\frac{1}{f(x)}\right) = 0.9\frac{1}{l}$.

4. $\lim_{x\to x_0} g \circ f = l'$.

5. $\lim_{x\to x_0}\left(0.9\frac{f(x)}{g(x)}\right) = 0.9\frac{l}{l'}$, $l' \neq 0$.

6. $\lim_{x\to x_0} |f(x)| = |l|$.

7. *If $f \leq g$, then $l \leq l'$.*

8. *If $f(x) \leq g(x) \leq h(x)$, and $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x) = l \in \mathbb{R}$, then $\lim_{x\to x_0} g(x) = l$.*

1.2.3 *Relationship with limits of sequences*

Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$, and $x_0 \in \mathbb{R}$ so we have:

$\lim\limits_{x\to x_0} f(x) = l \Longleftrightarrow \forall$ a sequence (x_n) of D, $x_n \neq x_0$, and $\lim\limits_{n\to\infty} x_n = x_0 \Longrightarrow \lim\limits_{n\to\infty} f(x_n) = l$.

1.2.4 *Infinite limits*

Definition 1.2.10. *(Limits as $x \longrightarrow \pm\infty$)*

- $\lim_{x\to+\infty} f(x) = l \Leftrightarrow \forall\varepsilon > 0,\ \exists\ A > 0,\ \forall x \in \mathbb{R}:\ x > A \Rightarrow |f(x) - l| < \varepsilon$.

- $\lim_{x\to-\infty} f(x) = l \Leftrightarrow \forall\varepsilon > 0,\ \exists\ A > 0,\ \forall x \in \mathbb{R}:\ x < -A \Rightarrow |f(x) - l| < \varepsilon$.

- $\lim_{x\to+\infty} f(x) = +\infty$ *(resp:* $\lim_{x\to+\infty} f(x) = -\infty$*)* $\Leftrightarrow \forall A > 0,\ \exists\ B > 0,\ \forall x \in \mathbb{R}:\ x > B \Rightarrow f(x) > A$, *(resp:* $\forall A > 0,\ \exists\ B > 0,\ \forall x \in \mathbb{R}:\ x > B \Rightarrow f(x) < -A$*).*

- $\lim_{x\to-\infty} f(x) = +\infty$ *(resp:* $\lim_{x\to-\infty} f(x) = -\infty$*)* $\Leftrightarrow \forall A > 0,\ \exists\ B > 0,\ \forall x \in \mathbb{R}:\ x < -B \Rightarrow f(x) > A$, *(resp:* $\forall A > 0,\ \exists\ B > 0,\ \forall x \in \mathbb{R}:\ x < -B \Rightarrow f(x) < -A$*).*

1.2.5 *Indeterminate forms*

When the limits are not finite, the previous results remain true whenever the operations on the limits make sense.

In the case where we cannot calculate, we say that we are in the presence of an indeterminate form. If $x \longrightarrow x_0$.

1. $f(x) \longrightarrow +\infty$ and $g(x) \longrightarrow -\infty$ then $f + g$ is in the indeterminate form $+\infty - \infty$.
2. $f(x) \longrightarrow 0$ and $g(x) \longrightarrow$ then $\dfrac{f}{g}$ is in the indeterminate form $\dfrac{0}{0}$.

3. $f(x) \longrightarrow \infty$ and $g(x) \longrightarrow \infty$ then $\dfrac{f}{g}$ is in the indeterminate form $\dfrac{\infty}{\infty}$.

4. $f(x) \longrightarrow \infty$ and $g(x) \longrightarrow 0$ then $f \times g$ is in the indeterminate form $\infty \times 0$.

There are other cases of indeterminate forms of type: $1^\infty,\ 0^\infty,\ \infty^0$.

1.3 Continuous Functions

1.3.1 *Continuity at a point*

Definition 1.3.1. *Let $f : I \longrightarrow \mathbb{R}$, where $I \subset \mathbb{R}$, and suppose that $x_0 \in I$. Then f is continuous at x_0 if:*

$$\forall \varepsilon > 0,\ \exists\ \delta > 0,\ \forall x \in I :\ |x - x_0| < \delta \Longrightarrow |f(x) - f(x_0)| < \varepsilon.$$

In another word: $\lim_{x \to x_0} f(x) = f(x_0)$.

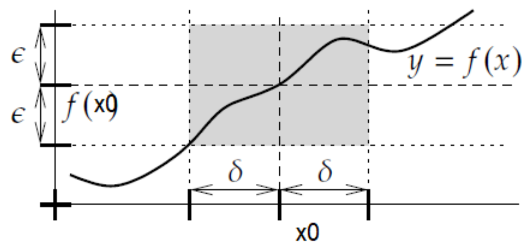


Figure 1.3: For $|x - x_0| < \delta$, the graph of $f(x)$ should be within the gray region.

A function $f : I \longrightarrow \mathbb{R}$ is continuous on a set $J \subset I$ if it is continuous at every point in J, and continuous if it is continuous at every point of its domain I.

1.3.2 *Left and right continuity*

Definition 1.3.2. *Let $f : I \longrightarrow \mathbb{R}$, we say that:*

- *f is continuous on the right of $x_0 \in I$ if:* $\lim_{\substack{> \\ x \longrightarrow x_0}} f(x) = f(x_0)$.

- f *is continuous on the left of* $x_0 \in I$ *if:* $\lim_{x \overset{<}{\longrightarrow} x_0} f(x) = f(x_0)$.

- f *is continuous on* $x_0 \in I$ *if:* $\lim_{x \overset{>}{\longrightarrow} x_0} f(x) = \lim_{x \overset{<}{\longrightarrow} x_0} f(x) = f(x_0)$.

Example 1.3.3. *Let*

$$\begin{aligned} f : \mathbb{R}_+^* &\longrightarrow \mathbb{R}_+ \\ x &\longrightarrow f(x) = \sqrt{x} \end{aligned}$$

We show that f *is continuous at every point* $x_0 \in \mathbb{R}_+^*$*, i.e.*

$$\forall \varepsilon > 0,\ \exists\ \delta > 0,\ \forall x \in \mathbb{R}_+^* :\ |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon,$$

then, $\forall \varepsilon > 0$ *we have:*

$$\begin{aligned} |f(x) - f(x_0)| < \varepsilon \ &\Rightarrow\ \left|\sqrt{x} - \sqrt{x_0}\right| < \varepsilon \\ &\Rightarrow\ \left|0.9\tfrac{x - x_0}{\sqrt{x} + \sqrt{x_0}}\right| < \varepsilon \\ &\Rightarrow\ 0.9\tfrac{|x - x_0|}{\sqrt{x} - \sqrt{x_0}} < \varepsilon \Rightarrow |x - x_0| < \varepsilon\left(\sqrt{x} - \sqrt{x_0}\right). \end{aligned}$$

So $\exists\ \delta = \varepsilon\left(\sqrt{x} - \sqrt{x_0}\right)$ *such that:* $|f(x) - f(x_0)| < \varepsilon$*, then* f *is continous at* x_0.

1.3.3 *Properties of continuous functions*

Theorem 1.3.4. *If* $f, g : I \longrightarrow \mathbb{R}$ *are continuous function at* $x_0 \in I$ *and* $k \in \mathbb{R}$*, then* $k.f, f + g$*, and* $f.g$ *are continuous at* x_0*. Moreover, if* $g(x_0) \neq 0$ *then* f/g *is continuous at* x_0*.*

Theorem 1.3.5. *Let* $f : I \longrightarrow \mathbb{R}$ *and* $g : J \longrightarrow \mathbb{R}$ *where* $f(I) \subset J$*. If* f *is continuous at* $x_0 \in I$ *and* g *is continuous at* $f(x_0) \in J$*, then* $g \circ f : I \longrightarrow \mathbb{R}$ *is continuous at* x_0*.*

Proof 1.3.6. *Fix* $\varepsilon > 0$*. Since* g *is continuous at* $b = f(x_0)$*,*

$$\exists\ \delta > 0,\ \forall\ y \in J :\ |y - b| < \delta \Longrightarrow |g(y) - g(b)| < \varepsilon.$$

Fix this $\delta > 0$*. From the continuity of* f *at* x_0*,*

$$\exists\, \gamma > 0,\ \forall\, x \in I :\ |x - x_0| < \gamma \Longrightarrow |f(x) - f(x_0)| < \delta.$$

From the above, it follows that

$$\forall\, \varepsilon > 0,\ \exists \gamma > 0,\ \ \forall\, x \in I :\ \ |x - x_0| < \gamma \Longrightarrow |g(f(x)) - g(f(x_0))| < \varepsilon.$$

This proves continuity of $g \circ f$ *at* x_0.

Proposition 1.3.7. *Let* $f : I \longrightarrow \mathbb{R}$ *and* $x_0 \in I$*, then:*

f *is continuous at* $x_0 \Longrightarrow$ *for any sequence* (u_n) *that converges to* x_0*, the sequence* $(f(x_0))$ *converges to* $f(x_0)$.

1.3.4 *Continuous extension to a point*

Definition 1.3.8. *Let* f *be a function defined in the neighborhood of* x_0 *except at* x_0 $(x_0 \notin D_f)$*, and* $\lim_{x\to x_0} f(x) = l$*. Then the function which is defined by*

$$\widetilde{f} = \begin{cases} f(x) & : \ x \neq x_0, \\ l & : \ x = x_0. \end{cases}$$

is continuous at x_0. $\widetilde{f}$ *is the continuous extension of* f *at* x_0.

Example 1.3.9. *Show that:*

$$f(x) = 0.9\frac{x^2+x-6}{x^2-4}, \quad x \neq 2.$$

has a continuous extension to $x = 2$*, and find that extension.*

Solution:

$\lim_{x\to 2} f(x) = 0.9\lim_{x\to 2} \frac{x^2+x-6}{x^2-4} = \lim_{x\to 2} \frac{(x-2)(x+3)}{(x-2)(x+2)} = \frac{5}{4}$*, exists. So* f *has a continuous extension at* $x = 2$ *defined by*

$$\widetilde{f} = \begin{cases} 0.9\frac{x^2+x-6}{x^2-4} & : \ x \neq 2, \\ \\ 0.9\frac{5}{4} & : \ x = 2. \end{cases}$$

1.3.5 *Discontinuous functions*

When f is not continuous at x_0, we say f is discontinuous at x_0, or that it has a discontinuity at x_0.

We say that the function f is not continuous in the following cases:

1. If f is not defined at x_0, then f is discontinuous at x_0.

2. If f defined in the neighborhood of x_0, then f is discontinuous at x_0 if

$$\exists\ \varepsilon > 0,\ \forall \delta > 0,\ \exists\ x \in I:\ |x - x_0| < \delta,\ \ and\ \ |f(x) - f(x_0)| \geq \varepsilon.$$

3. If $\lim_{x \overset{>}{\longrightarrow} x_0} f(x) \neq \lim_{x \overset{<}{\longrightarrow} x_0} f(x)$, then f is discontinuous at x_0, and x_0 is a discontinuous point of the first kind.

4. If one of the two limits $\lim_{x \overset{>}{\longrightarrow} x_0} f(x)$, $\lim_{x \overset{<}{\longrightarrow} x_0} f(x)$ or both limits does not exist or are not finite, then f is discontinuous at x_0, and x_0 is a discontinuous point of the second kind.

5. If $\lim_{x \overset{<}{\longrightarrow} x_0} f(x) = \lim_{x \overset{>}{\longrightarrow} x_0} f(x) \neq f(x_0)$, then f is discontinuous at x_0.

1.3.6 *Uniform continuity*

Definition 1.3.10. *Let $f : I \longrightarrow \mathbb{R}$. Then f is uniformly continuous on I if:*

$$\forall \varepsilon > 0,\ \exists\ \delta > 0,\ \forall x',\ x'' \in I:\ |x' - x''| < \delta \Longrightarrow |f(x') - f(x'')| < \varepsilon.$$

Remark 1.3.11.

1. *Uniform continuity is a property of the interval form, whereas continuity can be defined at a point.*

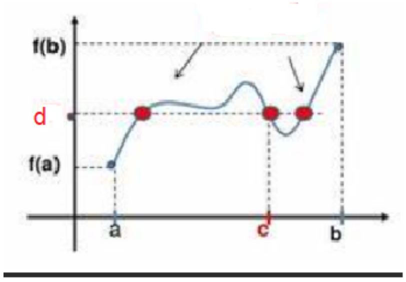
2. *The number δ does not depend on ε whereas for continuity δ depends on ε and x_0.*

3. *Let $f : I \longrightarrow \mathbb{R}$ be a function. If f is uniformly continuous, then f is continuous.*

Example 1.3.12. *$f(x) = x$ and $g(x) = \sin x$ are uniformly continuous on $\mathbb{R}$ (we find $\delta = \varepsilon$).*

1.3.7 *The intermediate value theorem*

Theorem 1.3.13. *Suppose that $f : [a, b] \longrightarrow \mathbb{R}$ is a continuous function on a closed bounded interval. Then for every d strictly between $f(a)$ and $f(b)$ there is a point $a < c < b$ such that $f(c) = d$.*



Corollary 1.3.14. *Suppose that $f : [a, b] \longrightarrow \mathbb{R}$ is a continuous function on a closed bounded interval. If $f(a).f(b) < 0$, then there is a point $a < c < b$ such that $f(c) = 0$.*

Corollary 1.3.15. *Let $f : D \longrightarrow \mathbb{R}$ is a continuous function and $I \subseteq D$ is an interval, then $f(I)$ is an interval.*

Theorem 1.3.16. *Let $I = [a, b]$ be a closed interval, and $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function. Then f is uniformly continuous.*

Theorem 1.3.17. *Any continuous function on a closed interval $[a, b]$ is bounded on $[a, b]$, i.e:* $\sup_{[a,b]} |f(x)| < +\infty$.

Remark 1.3.18.

1. *The image by a continuous function of a closed interval of $\mathbb{R}$ is a closed interval.*
2. *If I is not closed then the interval $f(I)$ is not necessarily of the nature of I. For example: $f(x) = x^2$, then $f(]-1, 1[) = [0, 1[$.*

1.3.8 *Fixed point theorem*

Definition 1.3.19. *Let* $f : I \longrightarrow I$ *and let* $\dot{x} \in I$*, we say that* $\dot{x} \in I$ *is a fixed point of* f *if:* $f(\dot{x}) = \dot{x}$.

Theorem 1.3.20. *Let* $f : [a, b] \longrightarrow [a, b]$ *be a continuous function, then* f *admits at least one fixed point in* $[a, b]$ *i.e:* $\exists \dot{x} \in [a, b]$ *such that* $f(\dot{x}) = \dot{x}$.

Exercise 1.3.21. *Let* f *be a continuous function on* $[a, b]$ *and* $x_1, \ x_2, \ \cdots, x_n \in [a, b]$*. Prove that there exists* $c \in [a, b]$ *with*

$$f(c) = \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}.$$

Solution:

Let $\alpha = \min\{f(x) : \ x \in [a, \ b]\}$, and $\beta = \max\{f(x) : \ x \in [a, \ b]\}$. Then

$$\frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} \leq \frac{n\beta}{n} = \beta.$$

Similarly,

$$\frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} \geq \alpha.$$

Then the conclusion follows from the Intermediate Value Theorem.

Exercise 1.3.22. *Consider* k *distinct points* $x_1, \ x_2, \cdots, x_k \in \mathbb{R}, \ k \geq 1$*. Find a function defined on* $\mathbb{R}$ *that is continuous at each* $x_i, \ \ i = 1, \cdots, k$ *and discontinuous at all other points.*

Solution: Consider

$$f(x) = \begin{cases} (x - a_1)(x - a_2) \cdots (x - a_k), & if \quad x \in \mathbb{Q}, \\ 0, & if \quad x \in \mathbb{Q}^c. \end{cases}$$