Democratic and Popular Republic of Algeria

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Analysis 1

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Exercises Serie N° 2

Exercise 1

Show by induction that:

1. $1+2+3+\cdots+n=\dfrac{n(n+1)}{2}$.
2. $1^2+2^2+3^2+\cdots+n^2=\dfrac{n(n+1)(2n+1)}{6}$.

Exercise 2

Determine, by justifying your answers, if the following sequences are convergent:

1. $U_n=\dfrac{\cos n-2}{n^4},\quad \forall n\in\mathbb{N}^*$.
2. $V_n=\dfrac{3n+5(-1)^n}{2n+1},\quad \forall n\in\mathbb{N}$.
3. $W_n=(-1)^n(\dfrac{n+1}{n}),\quad \forall n\in\mathbb{N}^*$.
4. $Z_n=\sqrt{2n+1}-\sqrt{2n-1},\quad \forall n\in\mathbb{N}^*$.*(*)*

Exercise 3

Let $(u_n)_{n\in\mathbb{N}}$ *be the sequence of real numbers defined by* $u_0\in]0.1]$, *and by the recurrence relation*

$$u_{n+1}=\frac{u_n}{2}+\frac{(u_n)^2}{4}$$

1. *Show that:* $\forall n\in\mathbb{N},\quad u_n>0$.
2. *Show that:* $\forall n\in\mathbb{N},\quad u_n\leq 1$.
3. *Show that the sequence is monotonic. Deduce that the sequence is convergent.*
4. *Determine the limit of the sequence* $(u_n)_{n\in\mathbb{N}}$.

Exercise 4

Prove that the following two sequences are adjacent

$$\forall n\in\mathbb{N},\qquad u_n=\sum_{k=1}^{n}\frac{1}{k^2},\qquad v_n=u_n+\frac{1}{n}.$$

Exercise 5

1. *Let* $u_n=\dfrac{E(\sqrt{n})}{n}$, *for all* $n\in\mathbb{N}^*$, *show that*

$$\lim_{n\to+\infty}u_n=0.$$

2. *Let* $v_n=\dfrac{E(\sqrt{n})^2}{n}$, *for all* $n\in\mathbb{N}^*$, *show that the sequence* $(v_n)_{n\in\mathbb{N}^*}$ *converges and determine its limit. (*)*

Exercise 6

Calculate the following limits, if they exist, of the following sequences:

1. $u_n = \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{(n+1)(n+2)}$.
2. $v_n = \frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n-1}{n^2}$.
3. $w_n = \frac{\ln(n+1)}{\ln n}$.
4. $z_n = \sqrt{n^2+n+1} - \sqrt{n^2-n+1}$. *(*)*

Exercise 7

We consider the sequence $(u_n)_{n\geq 1}$ *given by:* $u_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$.

1. *Show that* $\frac{1}{n^2} \leq \frac{1}{n-1} - \frac{1}{n}$.
2. *Show that the sequence* $(u_n)_{n\geq 1}$ *is bounded above by 2.*
3. *Show that the sequence* $(u_n)_{n\geq 1}$ *is increasing.*
4. *Deduce that* $(u_n)_{n\geq 1}$ *is converges.*

Exercise 8

We consider the sequence $(u_n)_{n\in\mathbb{N}}$ *defined by* $u_0 = 0$ *and by the recurrence relation*

$$u_{n+1} = \frac{1}{6}u_n^2 + \frac{3}{2}$$

1. *Show that for all* $n \in \mathbb{N}^*, \quad u_n > 0$.
2. *Calculate the limit of the sequence* $(u_n)_{n\in\mathbb{N}}$.
3. *Show that for all* $n \in \mathbb{N}, \quad u_n < 3$.
4. *Show that the sequence is increasing, what can we conclude from this?*

Exercise 9 *(Supplementary)*

We consider the sequence $(u_n)_{n\in\mathbb{N}^*}$ *defined by*

$$u_n = \frac{1}{3+|\sin(1)|\sqrt{1}} + \frac{1}{3+|\sin(2)|\sqrt{2}} + \cdots + \frac{1}{3\,|\sin(n)|\sqrt{n}}$$

Show that $\lim_{n\to+\infty} u_n = +\infty$.

Exercises marked with (*) are left to students.