Contents

Chapter 1

The Numerical Sequences

1.1 The general concept of a sequence

1.1.1 *Definition*

Definition 1.1.1. *A sequence of real numbers is a real-valued function whose domain is the set of natural numbers* $\mathbb{N}$ *to the real numbers* $\mathbb{R}$ *i.e:*

$$u: \quad \mathbb{N} \longrightarrow \mathbb{R},$$
$$n \longmapsto u_n.$$

The elements of a sequence are called **the terms**. The $n-th$ term u_n or $u(n)$ is called **the general term** of the sequence.

Example 1.1.2. *1.* $(\sqrt{n})_{n\geq 0}$ *is the sequence of terms:* $0, 1, \sqrt{2}, \sqrt{3}, \cdots$.

2. $((-1)^n)_{n\geq 0}$ *is the sequence of terms that are alternated:* $+1, -1, +1, -1, \cdots$.

1.1.2 *Explicit definition*

By an explicit definition of the general term of the sequence (u_n) i.e.: Express u_n in terms of n. For example: $u_n = 3n + 1, \quad v_n = \sin(n\pi/6), \quad w_n = (1/2)^n$.

1.1.3 *Definition by recurrence*

By a recurrence formula, i.e. a relationship that links any term in the sequence to the one that precedes it. In this case, to calculate u_n, you need to calculate all the terms that precede it. For example

$$\begin{cases} u_0 &= 1, \\ u_{n+1} &= 2u_n + 3, \quad n \in \mathbb{N}. \end{cases}$$

1.2 Qualitative features of sequences

1.2.1 *Monotonicity*

Definition 1.2.1. *A sequence u_n is called **increasing** (or **strictly increasing**) if $u_n \leq u_{n+1}$ (or $u_n < u_{n+1}$), for all $n \in \mathbb{N}$.*
*Similarly a sequence u_n is **decreasing** (or **strictly decreasing**) if $u_n \geq u_{n+1}$ (or $u_n > u_{n+1}$), for all $n \in \mathbb{N}$.*
*If a sequence is increasing (or strictly increasing), decreasing (or strictly decreasing), it is said to be **monotonic** (or **strictly monotonic**).*

Example 1.2.2. *The sequence $u_n = \frac{2^n-1}{2^n}$ which starts*

$$\tfrac{1}{2}, \tfrac{3}{4}, \tfrac{7}{8}, \tfrac{15}{16}, \dots$$

is increasing. On the other hand, the sequence $v_n = \frac{n+1}{n}$ which starts

$$\tfrac{2}{1}, \tfrac{3}{2}, \tfrac{4}{3}, \tfrac{5}{4}, \dots$$

is decreasing.

1.2.2 *Boundedness*

Definition 1.2.3. *Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence.*

- *A sequence* $(u_n)_{n\in\mathbb{N}}$ *is* ***bounded from above*** *if* $\exists M \in \mathbb{R},\ \forall\ n,\ u_n \le M$.

- *A sequence* $(u_n)_{n\in\mathbb{N}}$ *is* ***bounded from below*** *if* $\exists m \in \mathbb{R},\ \forall\ n,\ u_n \ge m$.

- *A sequence* $(u_n)_{n\in\mathbb{N}}$ *is bounded iff: it is bounded from above and bounded from below which means:* $\exists M \in \mathbb{R}_+,\ \forall\ n,\ |u_n| \le M$

Remark 1.2.4. *If a sequence* $\{u_n\}_{n=0}^{\infty}$ *is increasing, then it is bounded from below by* u_0*, and if it is decreasing, then it is bounded from above by* u_0.

Theorem 1.2.5. *If the sequence* (u_n) *is bounded and monotonic, then* $\lim_{n\to\infty} u_n$ *exists.*

Proof 1.2.6. *Suppose that* (u_n) *is increasing sequence, and* $\sup\limits_{n\in\mathbb{N}} u_n = M$. *then for given* $\varepsilon > 0$, *there exists* n_0 *such that* $M - \varepsilon \le u_{n_0}$. *Since* (u_n) *is increasing, we have* $u_{n_0} \le u_n$ *for all* $n \ge n_0$. *This implies that*

$$M - \varepsilon \le u_n \le M \le M + \varepsilon, \qquad \forall n \ge n_0.$$

That is $u_n \longrightarrow M$. *For decreasing sequences we have* $u_n \longrightarrow m$ *such that* $m = \inf\limits_{n\in\mathbb{N}} u_n$ *and its proof is similar.*

1.3 Convergent Sequences

Definition 1.3.1. *We say that the sequence* u_n *converges to the scalar* l *iff*

$$\forall \varepsilon > 0,\ \exists\ n_0 \in \mathbb{N} :\ \forall n \ge n_0 :\ |u_n - l| < \varepsilon.$$

In this case we write $\lim_{n\to\infty} u_n = l,\ (l\ finite)$. *If there is no finite value* l *so that* $\lim_{n\to\infty} u_n = l$, *then we say that the limit does not exist, or equivalently that the sequence diverges.*

Remark 1.3.2. *Any open interval with center* l *contains all the terms of the sequence from a certain rank.*

Example 1.3.3. *1.* $u_n = \left(\frac{3}{4}\right)^n$.

$\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} \left(\frac{3}{4}\right)^n = \lim_{n\to+\infty} e^{n\ln\left(\frac{3}{4}\right)} = 0$. *So* (u_n) *converges to* 0.

2. $v_n = (-1)^n$. v_n *is a divergent sequence.*

3. $w_n = \sin(n)$. *The limit of* w_n *does not exist, so* w_n *is divergent.*

Example 1.3.4. *Consider:*

- The sequence $u_n = \dfrac{n}{n+1}$ converges to 1

Using the definition of convergence, we show that $\lim\limits_{n\to+\infty} u_n = 1$
Let $\varepsilon > 0$ we have:

$$\begin{aligned} &|u_n - 1| \le \varepsilon \\ \Leftrightarrow\ &|\frac{n}{n+1} - 1| \le \varepsilon \\ \Leftrightarrow\ &|\frac{n}{n+1} - 1| \le \varepsilon \\ \Leftrightarrow\ &|1 - \frac{1}{n+1} - 1| \le \varepsilon \\ \Leftrightarrow\ &\frac{1}{n+1} \le \varepsilon \\ \Leftrightarrow\ &\frac{1}{\varepsilon} - 1 \le n \end{aligned}$$

By setting $n_0 = \lfloor\frac{1}{\varepsilon}\rfloor > \frac{1}{\varepsilon} - 1$, we obtain :
$\forall\varepsilon > 0, \exists n_0 \in \mathbb{N}\,(n_0 = \lfloor\frac{1}{\varepsilon}\rfloor), \forall n \in \mathbb{N}; n \ge n_0 \implies |u_n - 1| \le \varepsilon$
$\implies (u_n)_{n\in\mathbb{N}}$ converges to $l = 1$

Proposition 1.3.5. *If the sequence* a_n *is convergent then it has a unique limit.*

Proof 1.3.6. *Assume that* $\lim_{n\to+\infty} u_n = l$, *and* $\lim_{n\to+\infty} u_n = l'$, *we need to show that* $l = l'$.

- $\lim_{n\to+\infty} u_n = l \iff \forall\varepsilon > 0,\ \exists\ n_0 \in \mathbb{N} : \forall\ n \ge n_0 :\ |u_n - l| < \frac{\varepsilon}{2}$.

  and

- $\lim_{n\to+\infty} u_n = l' \Longleftrightarrow \forall\varepsilon > 0,\ \exists\ n_1 \in \mathbb{N} : \forall\ n \geq n_1 :\ |u_n - l'| < \frac{\varepsilon}{2}$.

We have $|l - l'| = |l - u_n + u_n - l'| \leq |l - u_n| + |u_n - l'| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. *So* $\forall\varepsilon > 0 :\ |l - l'| < \varepsilon$, *then* $l = l'$.

Proposition 1.3.7. *If the sequence* u_n *converges to* l*, then* $|u_n|$ *converges to* $|l|$.

Proposition 1.3.8. *any convergent sequence is bounded.*

Proof 1.3.9. *Suppose a sequence* (u_n) *converges to* u. *Then, for* $\varepsilon = 1$*, there exist* N *such that*

$$|u_n - u| \leq 1, \qquad \forall n \geq N.$$

This implies $|u_n| \leq |u| + 1$ *for all* $n \geq N$. *If we let*

$$M = \max\{|u_1|,\ |u_2|, \cdots, |u_{N-1}|\},$$

then $|u_n| \leq M + |u| + 1$ *for all* n. *Hence* (u_n) *is a bounded sequence.*

Remark 1.3.10.

- *If* $(u_n)_{n\in\mathbb{N}}$ *is increasing and bounded from above, then* $(u_n)_{n\in\mathbb{N}}$ *converges to* $l = \sup u_n$.
- *If* $(u_n)_{n\in\mathbb{N}}$ *is decreasing and bounded from below, then* $(u_n)_{n\in\mathbb{N}}$ *is converges to* $l = \inf u_n$.

1.4 The usual rules of limits

If (u_n) and (v_n) are convergent sequences to l and l' respectively, and α is any real constant then:

1) $\lim_{n\to+\infty}(u_n + v_n) = l + l'$,
2) $\lim_{n\to+\infty}(u_n \times v_n) = l \times l'$,
3) $\lim_{n\to+\infty}(\alpha u_n) = \alpha l$,
5) $\lim_{n\to+\infty} \frac{1}{u_n} = \frac{1}{l},\quad l \neq 0$,
6) $if\ u_n \leq v_n,\quad then\ \ l \leq l'$,
7) $if l = l',\quad and\ \ u_n \leq w_n \leq v_n,\quad then\ \ \lim_{n\to+\infty} w_n = l$.

1.5 Adjacent sequences

Definition 1.5.1. *We say that two real sequences (u_n), and (v_n) are adjacent if they satisfy the following properties:*

1. (u_n) *is increasing, and* (v_n) *is decreasing,*

2. $\lim_{n\to\infty}(u_n - v_n) = 0.$

Theorem 1.5.2. *If (u_n) and (v_n) are adjacent sequences, then they converge to the same limit.*

Proof. We assume that (u_n) is increasing and (v_n) is decreasing. Let $w_n = u_n - v_n$, then

$$\begin{aligned} w_{n+1} - w_n &= u_{n+1} - v_{n+1} - u_n + v_n, \\ &= (u_{n+1} - u_n) - (v_{n+1} - v_1), \\ &\geq 0. \end{aligned}$$

and $\lim_{n\to\infty} w_n = \lim_{n\to\infty}(u_n - v_n) = 0$. Since (w_n) is an increasing sequence and $\lim_{n\to\infty} w_n = 0$, then $\forall n \in \mathbb{N}: \; w_n \leq 0 \Rightarrow u_n \leq v_n$.

Therefore, $\forall n \in \mathbb{N}: \; u_0 \leq u_n \leq v_n \leq v_0$. the sequence (u_n) is convergent since it is increasing and bounded from above by v_0, also the sequence (v_n) is convergent, and since $\lim_{n\to\infty}(u_n - v_n) = 0$ we deduce that $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n$. □

Exercise 1.5.3. *Show that the two sequences (u_n) and (v_n) are adjacent:*

- $u_n = 1 + \frac{1}{n!}$, *and* $v_n = \frac{n}{n+1}$.

- $u_n = \sum_{k=1}^{n} \frac{1}{k^2}$ *and* $v_n = u_n + \frac{2}{n+1}$.

1.6 Subsequences

It is useful to sometimes consider only some terms of a sequence. A subsequence of $\{u_n\}_{n=1}^{\infty}$ is a sequence that contains only some of the numbers from $\{u_n\}_{n=1}^{\infty}$ in the same order.

Definition 1.6.1. *The sequence $(u_{\phi(n)})_{n\in\mathbb{N}}$ is a subsequence of the sequence $(u_n)_{n\in\mathbb{N}}$ if $\phi:\mathbb{N}\longrightarrow\mathbb{N}$ is a strictly increasing sequence of natural numbers.*

Example 1.6.2. *Consider the sequence*

$$u_n=\left(\tfrac{1}{n}\right)_{n=1}^{\infty}=\left\{1,\tfrac{1}{2},\tfrac{1}{3},\cdots,\tfrac{1}{n},\cdots\right\},$$

then letting $n_k=2k$ yields the subsequence

$$u_{2k}=\left(\tfrac{1}{2k}\right)_{k=1}^{\infty}=\left\{\tfrac{1}{2},\tfrac{1}{4},\cdots,\tfrac{1}{2k},\cdots\right\},$$

and letting $n_k=2k+1$ yields the subsequence

$$u_{2k+1}=\left(\tfrac{1}{2k+1}\right)_{k=1}^{\infty}=\left\{\tfrac{1}{3},\tfrac{1}{5},\cdots,\tfrac{1}{2k+1},\cdots\right\}.$$

Proposition 1.6.3. *If $\{u_n\}_{n=1}^{\infty}$ is a convergent sequence, then every subsequence $\{u_{n_i}\}_{i=1}^{\infty}$ is also convergent, and*

$$\lim_{n\to+\infty}u_n=\lim_{i\to+\infty}u_{n_i}.$$

Proof 1.6.4. *Let u_{n_i} denote a subsequence of u_n. Note that $n_i\geq i$ for all i. This easy to prove by induction: in fact, $n_1\geq 1$ and $n_i\geq i$ implies that $n_{i+1}>n_i\geq i$ and hence $n_{i+1}\geq i+1$. Let $\lim u_n=u$, and let $\varepsilon>0$. There exists N so that $n>N$ implies $|u_n-u|<\varepsilon$. Now*

$$i>N\Longrightarrow n_i>N\Longrightarrow |u_{n_i}-u|<\varepsilon.$$

therefore $\lim\limits_{i\to\infty}u_{n_i}=u$.

Corollary 1.6.5. *Let (u_n) be a sequence, if it admits a divergent subsequence, or if it admits two subsequences converging to distinct limits, then (u_n) is diverges.*

Theorem 1.6.6. ***(Bolzano-Weierstrass)***

Every bounded sequence has a convergent subsequence.

To prove the Bolzano-Weierstrass theorem, we will first need two lemmas.

Lemma 1.6.7. *All bounded monotone sequences converge.*

Proof 1.6.8. *Let (u_n) be a bounded, nondecreasing sequence. Let U denote the set $u_n, \quad n \in \mathbb{N}$. Then let $b = \sup\ U$ (the supremum of U).*
Choose some $\varepsilon > 0$. Then there is a corresponding N such that $u_N > b - \varepsilon$. Since (u_n) is nondecreasing, for all $n > N$, $u_n > b - \varepsilon$. But (u_n) is bounded, so we have $b - \varepsilon < u_n \leq b$. But this implies $|u_n - b| < \varepsilon, \quad so \quad \lim u_n = b$.
(The proof for nonincreasing sequences is analogous.)

Lemma 1.6.9. *Every sequence has a monotonic subsequence.*

Proof 1.6.10. *First a definition: call the nth term of a sequence dominant if it is greater than every term following it. For the proof, note that a sequence (u_n) may have finitely many or infinitely many dominant terms.*
First we suppose that (u_n) has infinitely many dominant terms. Form a subsequence (u_{n_k}) solely of dominant terms of (u_n). Then $u_{n_{k+1}} < u_{n_k}k$ by definition of dominant, hence (u_{n_k}) is a decreasing (monotone) subsequence of (u_n).
For the second case, assume that our sequence (u_n) has only finitely many dominant terms. Select n_1 such that n_1 is beyond the last dominant term. But since n_1 is not dominant, there must be some $m > n_1$ such that $u_m > u_{n_1}$. Select this m and call it n_2. However, n_2 is still not dominant, so there must be an $n_3 > n_2$ with $u_{n_3} > u_{n_2}$, and so on, inductively. The resulting sequence $u_1,\ u_2,\ u_3, \cdots$ is monotonic (nondecreasing).

Proof 1.6.11. ***(of Bolzano-Weierstrass)***
The proof of the Bolzano-Weierstrass theorem is now simple: let (u_n) be a bounded sequence. By Lemma (1.6.9) it has a monotonic subsequence. By Lemma (1.6.7), the subsequence converges.

1.7 Cauchy Sequences

Definition 1.7.1. *A real sequence (u_n) is called a Cauchy sequence if for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $\forall m, n \in \mathbb{N}$, if $m, n \geq N$ then*

$$|u_n - u_m| \leq \varepsilon.$$

Proposition 1.7.2. *If a sequence is Cauchy, then it is bounded.*

Proof 1.7.3. *we have a Cauchy sequence:*

$$\forall \varepsilon > 0, \quad \exists\ N \quad s.t \quad \forall\ n,\ m > N,\ |u_n - u_m| < \varepsilon.$$

we want to prove: this sequence is bounded: $\forall\ n, \quad |u_n| < C$. Note that $|u_n| = |u_n - u_m + u_m| \leq |u_n - u_m| + |u_m|$ by the Triangle Inequality set $\varepsilon = 1$, because this sequence is Cauchy, $\exists\ N$ such that $\forall\ m,\ n > N, \quad |u_n - u_m| < 1$. Set $m = N + 1$. Combined with our initial note, we can rewrite the following: $|u_n| < 1 + |u_{N+1}|$, and this is true for $\forall\ n > N$.
This bounds all the terms beyond the Nth. Looking at the terms before the Nth term, we can find the maximum of them and note that this bounds that part of the sequence:

$$|u_n| < \max\ (|u_1|,\ |u_2|, \cdots, |u_N|)$$

and this is true for $n \leq N$. By choosing the maximum of either $1 + |u_{N+1}|$ or the maximum of the aforementioned set, we can find our C which bounds all the terms in the sequence. We have shown the sequence is bounded.

Proposition 1.7.4. *A sequence of real numbers is Cauchy if and only if it converges.*

Proof 1.7.5. *Suppose (x_n) is a convergent sequence, and $\lim(x_n) = x$. Let $\varepsilon > 0$. We can find $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - x| < \frac{\varepsilon}{2}$. Therefore, by the triangle inequality, for all $m,\ n \geq N$, $|x_m - x_n| \leq |x_m - x| + |x - x_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. So (x_n) is Cauchy.*
Conversely, suppose (x_n) is Cauchy. Let $\varepsilon > 0$. By a result proved in class, (x_n) is bounded. By Bolzano-Weierstrass, it has a convergent subsequence (x_{n_k}) with $\lim(x_{n_k}) = x$ for some x. We

can find $K \in \mathbb{N}$ *such that for all* $k \geq K$, $|x_{n_k} - x| < \dfrac{\varepsilon}{2}$. *We can also find* M *such that for all* $m,\ n \geq M$, $|x_m - x_n| < \dfrac{\varepsilon}{2}$. *Let* $N = \sup\ K, M$. *Then since* $n_k \geq k$ *for all* k, *if* $k \geq N$, *we have that* $k,\ n_k \geq M$ *and* $n_k \geq K$. *Therefore, for all* $k \geq N$, $|x_n - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x| < \dfrac{\varepsilon}{2} + \dfrac{\varepsilon}{2} = \varepsilon$ *by the Triangle Inequality. Therefore,* (x_n) *is Cauchy.*

1.8 Arithmetic sequences

1.8.1 *Definition*

A simple way to generate a sequence is to start with a number u_0, and add to it a fixed constant r, over and over again. This type of sequence is called an **arithmetic sequence**.

Definition 1.8.1. *the sequence* (u_n) *is an arithmetic sequence with first term* u_0 *and common difference* r *if and only if for any integer* $n \in \mathbb{N}$ *we have*

$$u_{n+1} = u_n + r, \qquad (u_n = u_0 + n.r).$$

More generally: $u_n = u_p + (n - p).r$.

1.8.2 *Sum of* n *terms*

For the arithmetic sequence

$$0.9S_n = u_0 + u_1 + \cdots + u_{n-1} = n.\tfrac{u_0 + u_{n-1}}{2}.$$

1.9 Geometric sequences

1.9.1 *Definition*

Another simple way of generating a sequence is to start with a number v_0 and repeatedly multiply it by a fixed nonzero constant q. This type of sequence is called a geometric sequence.

Definition 1.9.1. *the sequence (v_n) is a geometric sequence with first term v_0 and common ratio $q \in \mathbb{R}^*$ if and only if for any integer $n \in \mathbb{N}$ we have*

$$v_{n+1} = q.v_n, \qquad (v_n = v_0.q^n).$$

More generally: $v_n = v_p.q^{n-p}$.

1.9.2 *Sum of* n *terms*

For a geometric sequence, if $S_n = 1 + q + q^2 + \cdots + q^n$, then

$$S_n = \begin{cases} n+1 & si \quad q = 1, \\ 0.9\frac{1-q^{n+1}}{1-q} & si \quad q \neq 1. \end{cases}$$

Exercise 1.9.2. *Let $(a_n)_n$ be a sequence defined by:*

$$\begin{cases} a_1 = \sqrt{2}, \\ a_{n+1} = \sqrt{a_n + 2}, \quad for \quad n \geq 1. \end{cases}$$

1. *Prove that $a_n < 2$ for all $n \in \mathbb{N}$.*
2. *Prove that $\{a_n\}$ is an increasing sequence.*
3. *Prove that $\lim\limits_{n\to\infty} a_n = 2$.*

Solution:

1. Clearly, $a_1 < 2$. Suppose that $a_k < 2$ for $k \in \mathbb{N}$. Then

$$a_{k+1} = \sqrt{2 + a_k} < \sqrt{2+2} = 2.$$

   By induction, $a_n < 2$ for all $n \in \mathbb{N}$.

2. Clearly, $a_1 = \sqrt{2} < \sqrt{2 + \sqrt{2}} = a_2$. Suppose that $a_k < a_{k+1} fork \in \mathbb{N}$. Then

$$a_k + 2 < a_{k+1} + 2$$

which implies

$$\sqrt{a_k + 2} < \sqrt{a_{k+1} + 2}.$$

Thus, $a_{k+1} < a_{k+2}$. By induction, $a_n < a_{n+1}$ for all $n \in \mathbb{N}$. Therefore, $\{a_n\}$ is an increasing sequence.

3. By the monotone convergence theorem, $\lim\limits_{n\to\infty} a_n$ exists. Let $l = \lim\limits_{n\to\infty} a_n$, since $a_{n+1} = \sqrt{2 + a_n}$ and $\lim\limits_{n\to\infty} a_{n+1} = l$, we have

$$l = \sqrt{2 + l}, \qquad \text{or} \qquad l^2 = 2 + l.$$

Solving this quadratic equation yields $l = -1$ or $l = 2$. Therefore, $\lim\limits_{n\to\infty} a_n = 2$.

Exercise 1.9.3. *Let a and b be two positive real numbers with $a < b$. Define $a_1 = a$, $b_1 = b$, and*

$$a_{n+1} = \sqrt{a_n b_n}, \qquad \textit{and} \qquad b_{n+1} = \frac{a_n + b_n}{2}, \quad \textit{for } n \geq 1.$$

Show that $\{a_n\}$ and $\{b_n\}$ are convergent to the same limit.

Solution:

Observe that

$$b_{n+1} = \frac{a_n + b_n}{2} \geq \sqrt{a_n b_n} = a_{n+1} \quad \text{for all} \quad n \in \mathbb{N}.$$

Thus

$$a_{n+1} = \sqrt{a_n b_n} \geq \sqrt{a_n a_n} = a_n \quad \text{for all} \quad n \in \mathbb{N}.$$

Hence

$$b_{n+1} = \frac{a_n + b_n}{2} \leq \frac{b_n + b_n}{2} = b_n \quad \text{for all} \quad n \in \mathbb{N}.$$

It follows that $\{a_n\}$ is monotone increasing and bounded above by b_1, and $\{b_n\}$ is decreasing and bounded below by a_1. Let $x = \lim\limits_{n\to\infty} a_n$, and $y = \lim\limits_{n\to\infty} b_n$. Then

$$x = \sqrt{xy} \qquad \text{and} \qquad y = \frac{x+y}{2}.$$

Therefore, $x = y$.