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Serie of exercise

DISCRETE DYNAMICAL SYSTEM .

Master 1 (first year) fundamental and applied mathematics

The first semester

By: ROUIBAH KHAOULA

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CONTENTS

CHAPTER 1

STABILITY OF ONE-DIMENSIONAL MAPS AND TWO-DIMENSIONAL MAPS

EXERCISE 1

Let $F(x) = x^2$. Compute the first five points on the orbit of 1/2.

SOLUTION

Let $F(z) = x^2$. Compute the first five points on the orbit of 1/2. (Nole how these fractions visually trace out a parabolal) It appears that

$$F^n\left(\frac{1}{2}\right) = \frac{1}{2^2}$$

in general, and this can be proved by induction (see Exercise 5).

EXERCISE 2

Let $F(x) = x^2 + 1$. Coupute the first five polints on the orbit of 0 .

SOLUTION

Let $F(x) = x^2 + 1$. Compute the first five points on the orbit of 0 .

$$F^1(0) = F(0) = 1$$
$$F^2(0) = F(1) = 2$$
$$F^3(0) = F(2) = 5$$
$$F^{-1}(0) = F(5) = 26$$
$$F^5(0) = F(26) = 677$$

EXERCISE 3

Let $F(x) = x^2 - 2$ Courpute $F^2(x)$ and $F^3(x)$.

SOLUTION

Let $F(x) = x^2 - 2$. Compute $F^2(x)$ and $F^3(x)$.

The second iterate of F is

$$F^2(x) = F(F(x)) = \left(x^2 - 2\right)^2 - 2 = x^4 - 4x^2 + 2$$

while the third iterate is given by

$$\begin{aligned} F^3(x) = F\left(F^2(z)\right) &= F\left(x^4 - 4x^2 + 2\right) \\ &= \left(x^4 - 4x^2 + 2\right)^2 - 2 \\ &= x^4 - 8x^4 + 20x^4 - 16x^2 + 2. \end{aligned}$$

F^4 is a lengthy 16 h-degree polynomial with nine terms. It appears that F^n is a 2^n th degree polynomial with $\left(2^{n-1} + 1\right)$ terms. What patterns do you see in the iterates of F ?

EXERCISE 4

Let $S(x) = \sin(2x)$. Coupute $S^2(x), S^a(x)$, and $S^4(x)$.

EXERCISE 5

Let $F(x) = x^2$. Coumpute $F^2(x), F^3(x)$, and $F^4(x)$. What is the formula for $F^n(x)$?

SOLUTION

5. Let $F(x) = x^2$, Compute $F^2(x), F^3(x)$, and $F^4(x)$. What is the formula for $F^n(x)$?

We have that

$$F^2(x) = F(F(x)) = F\left(x^2\right) = \left(x^2\right)^2 = x^4$$

and

$$F^3(x) = F\left(F^2(x)\right) = F\left(x^4\right) = \left(z^4\right)^2 = x^8.$$

We also have

$$F^4(x) = F\left(F^3(x)\right) = F\left(x^8\right) = \left(x^8\right)^2 = x^{16}.$$

In general, it appears that

$$F''(x) = x^{2^n} \tag{1.1}$$

which we will now show by induction (see Exercise 1 for a special case). We've already werified that 1.1 holds for $n = 1, 2, 3$, and 4 , and so the bsse case of the inductive argument has been established. Now, suppose that the equation is true for $n := k$. Then

$$\begin{aligned} F^{k+1}(x) &= F\left(F^k(x)\right) \\ &= F\left(x^{2^k}\right) \quad \text{by the inductive hypothesis} \\ &= \left(x^{2^k}\right)^2 \\ &= x^{2.2^k} \\ &= x^{2^{k+1}} \end{aligned}$$

which proves that (3.1) holds for all n. Note how this argument mirrors the above computations of $F^2(x), F^3(x)$, and $F^4(x)$, by the way.

EXERCISE 6

Let $A(x) = |x|$. Counpute $A^2(x)$ and $A^3(x)$.

solution

Let $A(x) = |x|$. Compute $A^2(x)$ and $A^3(x)$.

By definition,

$$A(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases},$$

and so $A(x) \geq 0$ for all x (see Figure 1.1). We also have that

$$A^2(x) = A(A(x)) = |A(x)| = A(x)$$

since $A(x) \geq 0$. Similarly,

$$A^3(x) = A\left(A^2\right) = A(A(x)) = |A(x)| = A(x),$$

and in fact,

$$A^{\mathrm{n}}(x) = A(x)$$

for $n \geq 1$. (Can you prove this by induction?) What does this mean? It implies that A has no periodic points of prime period $n > 1$.

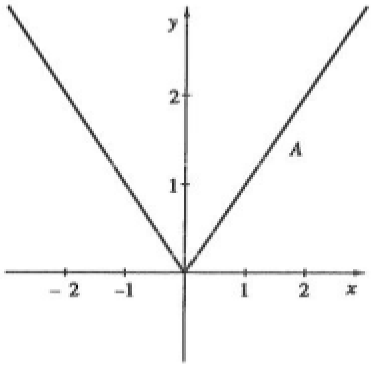


Figure 1.1: The absolute value function $A(x) = |x|$.

EXERCISE 7

Find all real fixed points (if any) for each of the following functions a. $F(x) = 3x + 2$

b. $F(x) = x^2 - 2$

c. $F(x) = x^2 + 1$

d. $F(x) = x^3 - 3x$

e. $F(x) = |x|$

f. $F(x) = x^5$

g. $F(x) = x^6$

h. $F(x) = x \sin x$

solution

Find all real fixed points (if any) for each of the following functions:

7a

$F(x) = 3x + 2 \qquad 3x + 2 = x \Rightarrow 2x = -2 \Rightarrow x = -1$, therefore, $fixF = -1$

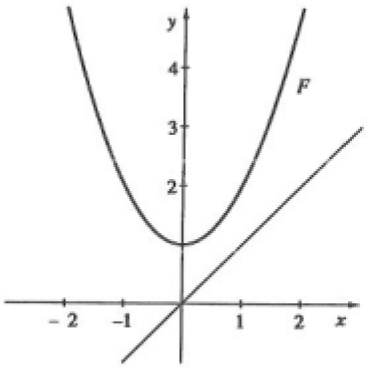


Figure 1.2: The graph of $F(x) = x^2 + 1$ is a parabola with no fixed points.

7b

$F(x) = x^2 - 2$

$x^2 - 2 = x \Rightarrow x^2 - x - 2 = 0$. Applying the quadratic formula to this second degree equation, we get

$$x = \frac{1 \pm \sqrt{1 - 4(1)(-2)}}{2} = \frac{1 \pm \sqrt{9}}{2},$$

and therefore, $fixF = \{-1, 2\}$.

7c

$F(x) = x^2 + 1$ (see Figure 1.2) $x^2 + 1 = x \Rightarrow x^2 - x + 1 = 0$. Again applying the quadratic formula we get

$$x = \frac{1 \pm \sqrt{1 - 4(1)(1)}}{2} = \frac{1 \pm \sqrt{3}i}{2}.$$

In this case, the fixed points are complex and the reader is referred to Chapter 15 rebort for details concerning complex functions.

7d

$F(x) = x^3 - 3x$ (see Figure 1.3)

$x^3 - 3x = x \Rightarrow x^3 - 4x = 0 \Rightarrow x\left(x^2 - 4\right) = 0 \Rightarrow x = 0$ or $x = \pm 2$. Consequently, $fixF = \{0, +2\}$. Consequently, $fixF = \{0, \pm 2\}$

7e

$F(x) = |x|$ (see Figure 1.2)

Since $|x| = x$ for nonnegative x, we have that $fixF = \{x \mid x \geq 0\}$.

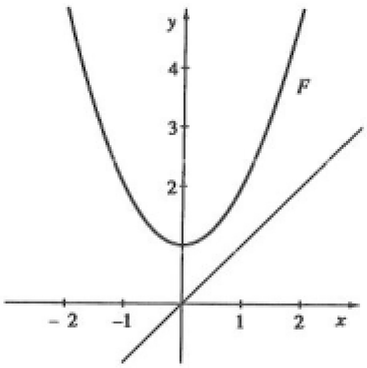


Figure 1.3: The graph of the cubic equation $P(x) = x^3 - 3x$.

7f

$F(x) = x^5$

$x^5 = x \Rightarrow x^5 - x = 0 \Rightarrow x\left(x^4 - 1\right) = 0$ which has the fairly obvious solutions $x = 0$ and $x = \pm 1$. But there are also two other (complex) solutions. Can you find them?

7g

$F(x) = x^6$

$x^6 = x \Rightarrow x^6 - x = 0 \Rightarrow x\left(x^5 - 1\right) = 0$ which has solutions $x = 0$ and $x = 1$, and four others which are not so easy to find (see Chapter 15) Note: The solutions to $x^n - 1 = 0$ are called the nth roots of unity.

7h

$F(x) = x \sin x$

Observe that $x \sin x = x \Rightarrow 0 = x - x \sin x = x(1 - \sin x)$. Thus, $x = 0$ or $1 - \sin x = 0$. Now, $\sin x = 1$ if $x = \pi/2$ or any 2π-multiple of $\pi/2$. Thus,

$$\begin{aligned} fix \quad F &= \left\{\ldots, \frac{\pi}{2} - 4\pi, \frac{\pi}{2} - 2\pi, \frac{\pi}{2}, \frac{\pi}{2} + 2\pi, \frac{\pi}{2} + 4\pi, \ldots\right\} \cup \{0\} \\ &= \left\{\ldots, \frac{-7\pi}{2}, \frac{-3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots\right\} \cup \{0\} \\ &= \{(4k+1)\pi/2 \mid k \in \mathbb{Z}\} \cup \{0\}. \end{aligned}$$

Note: F is an even function, that is, $F(x) = F(-x)$ for all x. See Figure 1.4.

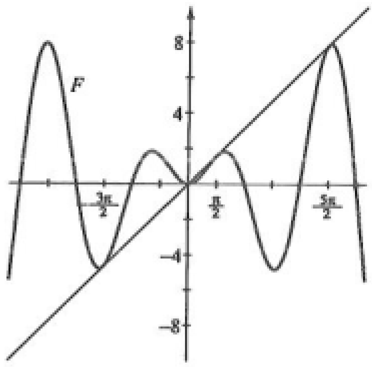


Figure 1.4: The graph of the cubic equation $F(x) = x \sin x$.

EXERCISE 8

8. What are the eventually fiwed points for $A(x) = |x|$?

solution

What are the eventually fixed points for $A(x) = |x|$?

As shown above in Exercise **7e**, $fixA = \{x \mid x \geq 0\}$. But fixed point are eventually fixed with preperiod 0 .

The negative real numbers are also eventually fixed since each becomes positive after just one iteration. Thus, all real numbers are eventually fixed under iteration of A. We write

$$\overline{\text{fix}\, A} = \mathbb{R}$$

Lo denote this fact.

EXERCISE 9

Let $F(x) = 1 = x^2$. Show that 0 lies ou a $2 - cycle$ for this function.

solution

Let $F(x) = 1 - x^2$. Show that 0 lies on a $2 - cycle$ for this function.

This is most certainly true since $F(0) = 1$ and $F(1) = 0$.

EXERCISE 10

Consider the function $F(x) = |x - 2|$.

a. What are the fixed points for F?

b. If m is an odd integer, what can you say about the orbit of m?

c. What happens to the orbit if m is even?

solution

Consider the function

$$F(x) = |x - 2| = \begin{cases} x - 2 \text{ if } x \geq 2 \\ 2 - x \text{ if } x < 2 \end{cases}.$$

See Figure 1.5 for the graph of F

10a

What are the fixed points for F ?

If $x \geq 2$, then $|x - 2| = x \Rightarrow x - 2 = x$, which has no solution.

On the other hand, if $x < 2$, we have that $|x - 2| = x \Rightarrow 2 - x = x \Rightarrow 2 = 2x \Rightarrow x = 1$. Therefore, $fixF = \{1\}$

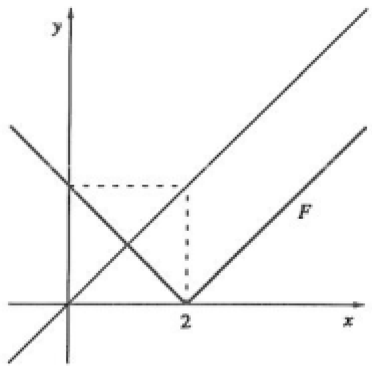


Figure 1.5: The graph of the $F(x) = |x - 2|$.

10b

If x is an odd integer, what can you say about the orbit of x ?

Observe that if x is an odd integer, then so is $x - 2$.

Now, suppose x is an odd positive integer greater than 2

. Then $x - 2$ is an odd positive integer smaller than x, and repeated subtraction eventually produces a value of 1 which is fixed by F. In other words, all odd positive integers are eventually fixed.

But what about odd negative x ? Well, since $F(x) \geq 0$ for all z, all we need to do is apply this very same argument to $F(x)$.

10c

What happens to the orbit if x is even?

Suppose x is an even positive integer greater than or equal to 2 . Then $x - 2$ is an even positive integer smaller than z.

In this case, repeated subtractions eventually vanish, but $F(0) = |0 - 2| = 2$.

And since $F(2) = 0$, we see that the orbit period 2. Similar arguments hold for even negative x

The following four exercises deal with the doubling function D

EXERCISE 12

Does the function $F(x) = -x^3$ have a cycle of prime porbod 2?

solution

Give an explicit formula for $D^2(x)$ and $D^3(x)$. Can you write down a general formula for $D^n(x)$? Recall that the doubling function D is given by the equations

$$\begin{aligned} D(x) &= 2x \bmod 1 \\ &= \begin{cases} 2x & \text{if } 0 \leq x < 1/2 \\ 2x - 1 & \text{if } 1/2 \leq x < 1 \end{cases}. \end{aligned}$$

See Figure 1.6 for the graph of D. To derive a formula for $D^2(x)$, we partition the interval $[0, 1)$ into four

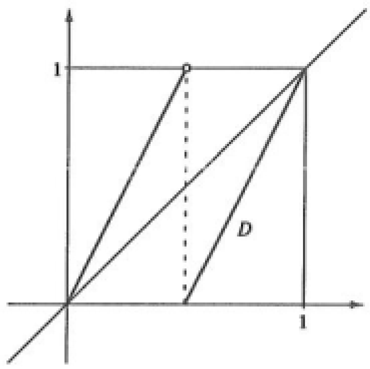


Figure 1.6: The doubling map $D(x) = 2x \bmod 1$.

parts and compute the image of each of these subintervals under D^2.

$$\begin{aligned} 0 \leq x < 1/4 \quad \Rightarrow \quad 0 \leq D(x) < 1/2 \quad \Rightarrow \quad D^2(x) &= D(D(x)) \\ &= D(2x) \\ &= 2(2x) \\ &= 2(2x) \\ &= 4x \end{aligned}$$

$$
\begin{aligned}
1/4 \le x < 1/2 \quad \Rightarrow \quad 1/2 \le D(x) < 1 \quad \Rightarrow \quad D^2(x) &= D(D(x)) \\
&= D(2x) \\
&= D(2x) \\
&= 2(2x) - 1 \\
&= 2(2x) - 1 \\
&= 4x - 1
\end{aligned}
$$

$$
\begin{aligned}
1/2 \le x < 3/4 \quad \Rightarrow \quad 0 \le D(x) < 1/2 \quad \Rightarrow \quad D^2(x) &= D(D(x)) \\
&= D(2x - 1) \\
&= 2(2x - 1) \\
&= 2(2x - 1) \\
&= 4x - 2
\end{aligned}
$$

$$
\begin{aligned}
3/4 \le x < 1 \quad \Rightarrow \quad 1/2 \le D(x) < 1 \quad \Rightarrow \quad D^2(x) &= D(D(x)) \\
&= D(2x - 1) \\
&= 2(2x - 1) - 1 \\
&= 4x - 3
\end{aligned}
$$

Thus, we have shown that

$$
D^2(x) = \begin{cases} 4x & \text{if } 0 \le x < 1/4 \\ 4x - 1 & \text{if } 1/4 \le x < 1/2 \\ 4x - 2 & \text{if } 1/2 \le x < 3/4 \\ 4x - 3 & \text{if } 3/4 \le x < 1 \end{cases} .
$$

(See Figure 1.7 **7a** for the graph of D^2.) An expression for $D^3(z)$ is derived similarly, but this time we partition $[0, 1)$ into eight subintervals. The first couple of steps in the computation of $D^3(x)$ are given below.

$$0 \le x < 1/8 \quad \Rightarrow \quad 0 \le D(x) < 1/4 \text{ and } 0 \le D^2(x) < 1/2$$
$$\Rightarrow D^3(x) = D\left(D^2(x)\right.$$
$$= D(4x)$$
$$= 2(4x)$$
$$= 8x$$

$$1/8 \le x < 1/4 \Rightarrow 1/4 \le D(x) < 1/2 \text{ and } 1/2 \le D^2(x) < 1$$

$$\Rightarrow \quad D^3(x) = D\left(D^2(x)\right)$$
$$= D(4x)$$
$$= 2(4x) - 1$$
$$= 8x - 1$$

Can you predict the outcome? The reader is encouraged to complete the remaining six cases. When the dust clears, you should get

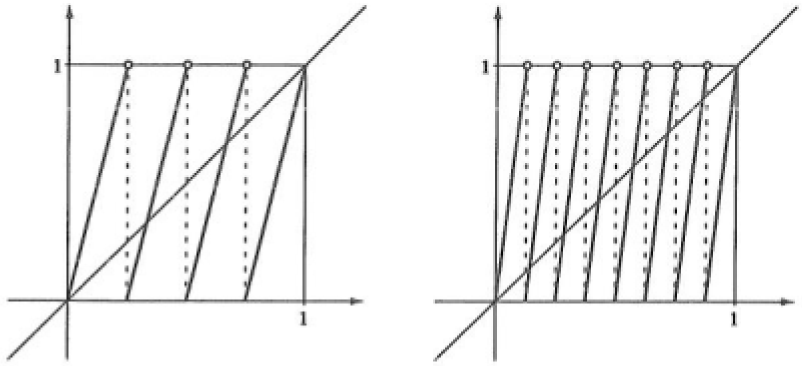
$$D^3(x) = \begin{cases} 8x & \text{if } 0 \le x < 1/8 \\ 8x - 1 & \text{if } 1/8 \le x < 1/4 \\ 8x - 2 & \text{if } 1/4 \le x < 3/8 \\ 8x - 3 & \text{if } 3/8 \le x < 1/2 \\ 8x - 4 & \text{if } 1/2 \le x < 5/8 \\ 8x - 5 & \text{if } 5/8 \le x < 3/4 \\ 8x - 6 & \text{if } 3/4 \le x < 7/8 \\ 8x - 7 & \text{if } 7/8 \le x < 1 \end{cases}.$$

(See Figure 1.7 7 b for the graph of D^3.) Observe the pattern in the expressions for $D^2(x)$ and $D^3(x)$, and then try to write down a comparabl expression for $D^n(x)$. You'll find that

$$D^n(x) = \begin{cases} 2^n x & \text{if } 0 \le x < 1/2^n \\ 2^n x - 1 & \text{if } 1/2^n \le x < 2/2^n \\ 2^n x - 2 & \text{if } 2/2^n \le x < 3/2^n \\ \vdots & \vdots \\ 2^n x - (2^n - 1) & \text{if } \left(2^n - 1\right)/2^n \le x < 1 \end{cases}$$

Putting it more succinctly,

$$D^n(x) = 2^n x - k \text{ if } k/2^n \le x < (k+1)/2^n$$



(a) $D^2(x) = 4x \bmod 1$ (b) $D^3(x) = 8x \bmod 1$

Figure 1.7: The second and third iterates of the doubling map.

for $k = 0, 1, \ldots, 2^n - 1$. This is precisely the meaning of the abbreviated notation

$$D^n(x) = 2^n x \bmod 1,$$

by the way

EXERCISE 13

Consider the map $f(x) = sinx, x \in R$. Discuss the stability character of the fixed point $x = 0$ of the map.

SOLUTION

The derivative of$f(x)$ is$f'(x) = cos(x)$ cos x. Since $|f'(0)| = |cos(0)| = 1$ we cannot apply linear stability analysis to determine the stability of the fixed point origin. We construct the cobweb diagram as shown in 1.8.

1.8 Cobweb diagram of $f(x) = \sin(x)$ depicting the stability character of the fixed point origin
The figure shows that the iterated points move toward the fixed point origin. So the origin is stable.

EXERCISE 14

Consider the map $f(x) = 1 - \lambda x^2$, where $-1 \leq x \leq 1$ and $0 \leq \lambda \leq 2$. Find all fixed points of the map. Also determine their stability characters.

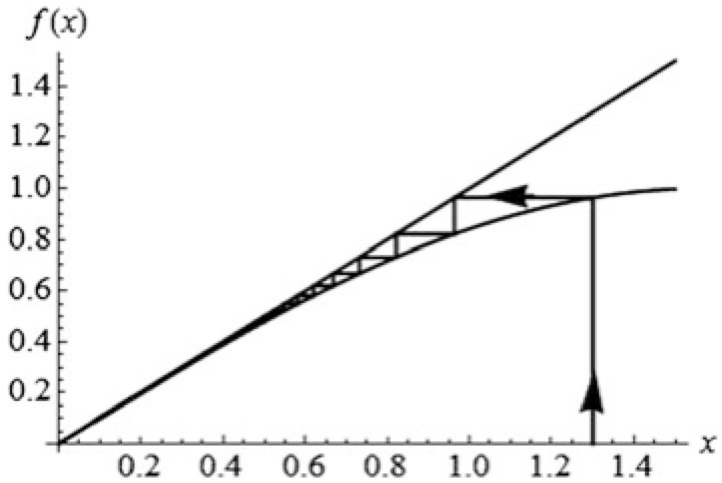


Figure 1.8: Cobweb diagram of $f(x) = \sin(x)$ depicting the stability character of the fixed point origin

SOLUTION

The fixed points of $f(x)$ are the solutions of the equation $f(x) = x$. This gives

$$1 - \lambda x^2 = x \Rightarrow \lambda x^2 + x - 1 = 0 \Rightarrow x = \frac{-1 \pm \sqrt{1+4\lambda}}{2\lambda}.$$

Therefore,

$$x_1^* = \frac{-1 + \sqrt{1+4\lambda}}{2\lambda} \text{ and } x_2^* = \frac{-1 - \sqrt{1+4\lambda}}{2\lambda}$$

are two fixed points of f. Now, $f'(x) = -2\lambda$. Since $\left|f'\left(x_2^*\right)\right| = (1 + \sqrt{1+4\lambda}) > 1 \forall \lambda \in [0, 2]$, the fixed point x_2^* is unstable for all $\lambda \in [0, 2]$. Again, $\left|f'\left(x_1^*\right)\right| = (\sqrt{1+4\lambda} - 1)$. Therefore, $\left|f'\left(x_1^*\right)\right| < 1$ if $\sqrt{1+4\lambda} - 1 < 1$, that is, if $\lambda < 3/4$. So, the fixed point x_1 is stable if $0 < \lambda < 3/4$. And $\left|f'\left(x_1^*\right)\right| > 1$ if $\sqrt{1+4\lambda} - 1 > 1$, that is, if $\lambda > 3/4$. Hence x_1^* is unstable if $\lambda > 3/4$.

EXERCISE 15

Find the fixed points of the one-dimensional map $f(x) = x + \sin x, x \in \mathbb{R}$. Also find the basins of attraction.

SOLUTION

The fixed points of f satisfy

$$f(x) = x \Rightarrow x + \sin x = x \Rightarrow \sin x = 0 \Rightarrow x = n\pi, n = 0, \pm 1, \pm 2, \ldots$$

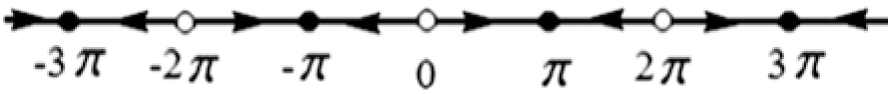


Figure 1.9: Baxins of attraction of $f(x) = x + \sin x$

So, the fixed points of the given map are $x^* = n\pi, n = 0, \pm 1, \pm 2, \ldots$ Now, the derivative $f'(x) = 1 + \cos x$. So,

$$f'(n\pi) = 1 + \cos(n\pi) = \begin{cases} 2, \text{ for } n = 2m \\ 0, \text{ for } n = 2m + 1 \end{cases}, m = 0, \pm 1, \pm 2, \ldots$$

Therefore, $f'(0) = 2, f'(\pi) = 0, f'(2\pi) = 2$. This shows that $x^* = \pi$ is an attracting fixed point, while $x^* = 0, 2\pi$ are repelling fixed points. So the basin of attraction of π is $W^x(\pi) = (0, 2\pi)$. Similarly, the basin of attraction of 3π is $W^s(3\pi) = (2\pi, 4\pi)$. In general, we get $\left|f'(2m\pi)\right| = 2 > 1$ and $|\ f'((2m+1)\ \pi)\ |= 0 < 1, 2m\pi, m \in \mathbb{Z}$ are repelling fixed points, while $(2m+1)\pi$ are attracting fixed points. The basins of attraction of the fixed points $(2m+1)\pi$ are 1.9

$$W^x((2m+1)\pi) = (2m\pi, (2m+2)\pi), m \in \mathbb{Z}.$$

EXERCISE 16

Consider the map $x_{x+1} = f(x_*)$, where $f(x) = 1 - x^2, x \in [-1, 1]$. Find all the periodic 2-cycles of f.

SOLUTION

The fixed points of f are given by

$$x = f(x) = 1 - x^2 \Rightarrow x^2 + x - 1 = 0 \Rightarrow x = (-1 \pm \sqrt{5})/2.$$

We denote $x_1^* = (-1 + \sqrt{5})/2$ and $x_2 = (-1 - \sqrt{5})/2$. The point x_2 lies outside the domain of f. Now, for periodic 2 cycles we see $f^2(x) = f(f(x)) = f\left(1 - x^2\right) = 1 - \left(1 - x^2\right)^2 = 2x^2 - x^4$. For periodic 2-cycles, we have $f^2(x) = x$. This gives

$$2x^2 - x^4 = x \Rightarrow x^4 - 2x^2 + x = 0 \Rightarrow x(x-1)\left(x^2 + x - 1\right) = 0$$
$$\Rightarrow x = 0, 1, (-1 \pm \sqrt{5})/2.$$

So the fixed points of $f^2(x)$ are $x = 0, 1, (-1 \pm \sqrt{5})/2$. But $(-1 \pm \sqrt{5})/2$ are the fixed points of f. Again, $f(0) = 1, f(1) = 0$ and $f^2(0) = 0, f^2(1) = 1$. This shows that the set $\{0, 1\}$ forms the period 2-cycle of the map f.

EXERCISE 17

Find the period of the point $\frac{1}{8}(5 + \sqrt{5})$ for the map $f(x) = 4x(1 - x), x \in [0, 1]$. Also determine its stability.

SOLUTION

Given map is $f(x) = 4x(1 - x), x \in [0, 1]$. This is a quadratic map. Now,

$$f\left(\frac{1}{8}(5 + \sqrt{5})\right) = 4 \cdot \frac{1}{8}(5 + \sqrt{5}) \cdot \left(1 - \frac{1}{8}(5 + \sqrt{5})\right) = \frac{1}{16}(5 + \sqrt{5})(3 - \sqrt{5}) = \frac{1}{8}(5 - \sqrt{5}),$$
$$f\left(\frac{1}{8}(5 - \sqrt{5})\right) = 4 \cdot \frac{1}{8}(5 - \sqrt{5}) \cdot \left(1 - \frac{1}{8}(5 - \sqrt{5})\right) = \frac{1}{16}(5 - \sqrt{5})(3 + \sqrt{5}) = \frac{1}{8}(5 + \sqrt{5})$$
$$\text{Again, } f^2\left(\frac{1}{8}(5 + \sqrt{5})\right) = f\left(f\left(\frac{1}{8}(5 + \sqrt{5})\right)\right) = f\left(\frac{1}{8}(5 - \sqrt{5})\right) = \frac{1}{8}(5 + \sqrt{5}).$$

Again, $f^2\left(\frac{1}{8}(5 + \sqrt{5})\right) = f\left(f\left(\frac{1}{8}(5 + \sqrt{5})\right)\right) = f\left(\frac{1}{8}(5 - \sqrt{5})\right) = \frac{1}{8}(5 + \sqrt{5})$. This shows that the point $(5 + \sqrt{5})/8$ is a fixed point of the map f^2 and hence it is a periodic point of period-2 of the given map. We shall now examine the stability of this periodic-2 point. We have

$$f^2(x) = f(f(x)) = f(4x(1 - x)) = 4 \cdot 4x(1 - x)\{1 - 4x(1 - x)\}$$
$$= 16x - 80x^2 + 128x^3 - 64x^4.$$

We shall use the derivative test for finding the stability character of the periodic point of the map. We see that $\left(f^2\right)'(x) = 16 - 160x + 384x^2 - 256x^3$. Since

$$\left|\left(f^2\right)'\left(\frac{1}{8}(5 + \sqrt{5})\right)\right| = 16(244 + 105\sqrt{5}) > 1,$$

the periodic-2 point $(5 + \sqrt{5})/8$ of f is unstable. Stability of Periodic Cycles Stability of periodic cycles is a collective property. Let $\{x_1, x_2, \ldots, x_4\}$ be a periodic n-cycle of a map $f : \mathbb{R} \to \mathbb{R}$. As per definition of periodic cycle each $x_i (i = 1, 2, \ldots, n)$ is a fixed point of the map f^n. The cycle is stable (respectively unstable) if and only if the points $x_i (i = 1, 2, \ldots, n)$ are stable (respectively unstable) fixed points of the map $f^n(x)$. Using the chain rule of differentiation of the composition map $f''(x_i)$, we get

$$\begin{aligned}
(f^n)'(x_i) &= \left(f^{n-1}\right)'(f(x_i))\, f'(x_i) = \left(f^{n-1}\right)'(x_{i+1})\, f'(x_i) \\
&= \left(f^{n-2}\right)'(f(x_{i+1}))\, f'(x_{i+1})\, f'(x_i) \\
&= \left(f^{n-2}\right)'(x_{i+2})\, f'(x_{i+1})\, f'(x_i) \\
&= \cdots = f'(x_{i+\infty-1})\, f'(x_{i+n-2}) \cdots f'(x_{i+1})\, f'(x_i) \\
&= f'(x_1)\, f'(x_2) \cdots f'(x_{x-1})\, f'(x_n) \quad [\text{ Since } \{x_1, x_2, \ldots, x_{-}\} \text{ is a cycle of } f]
\end{aligned}$$

Now if x_i is a stable fixed point of $f^n(x)$, then from linear stability analysis $\left|(f^n)'(x_1)\right| < 1$. This implies that $\left|f'(x_1)\, f'(x_2) \cdots f'(x_4)\right| < 1$. Similarly, if x_1 is an unstable fixed point of f^n, then $\left|f'(x_1)\, f'(x_2) \cdots f'(x_s)\right| > 1$. Hence we have the following definition:

The cycle is said to be stable (sink or atrracting) if $f'(x_1)\, f'(x_2) \cdots f'(x_a)\ | < 1$ and it is unstable (source or repelling) if $\left|f'(x_1)\, f'(x_2) \cdots f'(x_n)\right| > 1$. But these criteria are weak in nature.

EXERCISE 18

Find all fixed points of $f(x) = x^2 - 1, x \in \mathbb{R}$. Determine their stabilities. Show that $\{0, -1\}$ is a periodic orbit of period-2. Are the periodic cycle attracting?

SOLUTION

The fixed points of f are given by

$$x = f(x) = x^2 - 1 \Rightarrow x^2 - x - 1 = 0 \Rightarrow x = \frac{1 \pm \sqrt{5}}{2}.$$

So, the fixed points of the map are $\left(\frac{1+\sqrt{5}}{2}\right)$ and $\left(\frac{1-\sqrt{5}}{2}\right)$. Now, $f'(x) = 2x$. Since $f'\left(\frac{1+\sqrt{5}}{2}\right) = 2\left(\frac{1+\sqrt{5}}{2}\right) = (1+\sqrt{5}) > 1$ and $f'\left(\frac{1-\sqrt{5}}{2}\right) = 2\left(\frac{1-\sqrt{5}}{2}\right) = (1-\sqrt{5}) < 1$, both the fixed points are unstable. For periodic orbit, we find $f(0) = -1, \quad f(-1) = 0, \quad f^2(0) = f(f(0)) = f(-1) = 0$ and $f^2(-1) = f(f(-1)) = f(0) = -1$.

This shows that $\{0, -1\}$ is a periodic orbit of period-2 of the map f. Since $\left|f'(0)f'(-1)\right| = |0.(-2)| = 0 < 1$, the cycle is stable.

EXERCISE 19

Consider the map $Q(x) = x^2 - 0.85$ defined on the interval $[-2, 2]$. Find the 2-cycles and determine their stability.

SOLUTION

Observe that $Q^2(x) = \left(x^2 - 0.85\right)^2 - 0.85$. The 2-periodic points are obtained by solving the equation

$$Q^2(x) = x, \text{ or } x^4 - 1.7x^2 - x - 0.1275 = 0.$$

This equation has four roots, two of which are fixed points of the map $Q(x)$. These two fixed points are the roots of the equation

$$x^2 - x - 0.85 = 0.$$

To eliminate these fixed points of $Q(x)$ from (1.6.2) we divide the left-hand side of (1.6.2) by the left-hand side of (1.6.3) to obtain the second-degree equation

$$x^2 + x + 0.15 = 0.$$

The 2-periodic points are now obtained by solving (1.6.4). They are given by

$$a = \frac{-1 + \sqrt{0.4}}{2}, \quad b = \frac{-1 - \sqrt{0.4}}{2}.$$

To check the stability of the cycle $\{a, b\}$ we apply Theorem 1.21. Now,

$$|Q'(a)Q'(b)| = |(-1 + \sqrt{0.4})(-1 - \sqrt{0.4})| = 0.6 < 1.$$

Hence by Theorem 1.21, part (i), the 2-cycle is asymptotically stable.

EXERCISE 20

Show that $\{-1, 1\}$ is an attracting 2 -cycle of the map $f(x) = -x^{1/3}, x \in \mathbb{R}$. Find the stability character of the fixed points of f.

SOLUTION

Here $f(x) = -x^{1/3}, x \in \mathbb{R}$. We see that $f(-1) = -(-1)^{1/3} = -(-1) = 1, f(1) = -(1)^{1/3} = -1, f^2(-1) = f(f(-1)) =$

$$f(1) = -1, f^2(1) = f(f(1)) = f(-1) = 1$$

The points $\{-1, 1\}$ form a cycle. Again we see that.

$$f^3(-1) = f\left(f^2(-1)\right) = f(-1) = 1, f^3(1) = f\left(f^2(1)\right) = f(1) = -1.$$

This shows that $\{-1,1\}$ is a 2-cycle of the map f. We shall use the derivative test for stability character of the cycle. The derivative of f gives

$$f'(x) = -\frac{1}{3}x^{-(2/3)} = -\frac{1}{3x^{(2/3)}}$$
$$\therefore f'(-1) = -\frac{1}{3} \text{ and } f'(1) = -\frac{1}{3}$$

The cycle $\{-1,1\}$ will be linearly stable if $\left|f'(-1)f'(1)\right| < 1$. Since $\left|f'(-1)f'(1)\right| = \left|\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)\right| = \frac{1}{9} < 1$, so the 2 -cycle $\{-1,1\}$ is stable. We now find the fixed points of the map. The fixed points are obtained by solving the equation

$$f(x) = x \Rightarrow -x^{1/3} = x \Rightarrow -x = x^3 \Rightarrow x^3 + x = 0 \Rightarrow x\left(x^2+1\right) = 0$$
$$\Rightarrow x = 0(\because x \in \mathbb{R})$$

So, $x^* = 0$ is the only fixed point of the map f. Since $\left|f'(x)\right| > 1$ in the neighborhood of the fixed point 0 , the fixed point origin is repelling.

EXERCISE 21

Consider the map $f(x) = -x^3, x \in \mathbb{R}$. Show that the origin is an attracting fixed point and $\{-1,1\}$ is a repelling 2 -cycle of the map.

SOLUTION

Solution The fixed points of the map f are given by

$$\begin{aligned} x &= f(x) = -x^3 \\ &\Rightarrow \quad x^3 + x = 0 \\ &\Rightarrow x\left(x^2+1\right) = 0 \\ &\Rightarrow x = 0. \end{aligned}$$

So the origin is the only fixed point of f. Now $f'(x) = -3x^2$. Since $f'(0) = 0 < 1$, the fixed point origin is stable. Again, $f(1) = -1, f(-1) = 1$. Now $f^2(x) = f(f(x)) = -\left(-x^3\right)^3 = x^9$, $f^3(x) = -x^{27}$ and $f^4(x) = x^{81}$. Therefore, $f^2(1) = 1$, $f^2(-1) = -1$, $f^3(1) = -1$ and $f^3(-1) = 1$, $f^4(1) = 1$ and $f^4(-1) = -1$. This shows that $\{-1,1\}$ is a periodic 2 -cycle. The stability condition of the cycle gives that $\left|f'(-1)f'(1)\right| = |(-3)(-3)| = 9 > 1$. Hence $\{-1,1\}$ is a repelling 2 -cycle.

EXERCISE 22

Show that the map $f(x) = -\frac{3}{2}x^2 + \frac{5}{2}x + 1, x \in \mathbb{R}$ has a 3-cycle. Comment about the stability of the cycle.

SOLUTION

Take three points 0,1 , and 2 . We see that

$$f(0) = 1, \quad f(1) = -\frac{3}{2} + \frac{5}{2} + 1 = 2,$$
$$f(2) = -\frac{3}{2}(4) + \frac{5}{2}(2) + 1 = -6 + 5 + 1 = 0$$

So $f(0) = 1, f(1) = 2, f(2) = 0$. And $f^2(0) = f(f(0)) = f(1) = 2, f^2(1) = f(f(1)) = f \neq 29, f^2(2) = f(f(2)) = f(0) = 1 f^3(0) = f\left(f^2(0)\right) = f(2) = 0, f^3(1) = f\left(f^2(1)\right) = f(0) = 1, f^3(2) = f\left(f^2(2)\right) = f(1) = 2$. The three points 0,1 , and 2 are fixed points of f^3. Also, $f^4(0) = f\left(f^3(0)\right) = f(0) = 1, f^4(1) = f\left(f^3(1)\right) = f(1) = 2, f^4(2) = f\left(f^3(2)\right) = f(2) = 0$.

This shows that $\{0, 1, 2\}$ is a periodic 3 cycle of the map f. We shall test the stability of the 3-cycle using derivative test. This gives the condition $\left|f'(0)f'(1)f'(2)\right| < 1$. Now, $f'(x) = -3x + \frac{5}{2}$. So, $f'(0) = \frac{5}{2}, f'(1) = -3 + \frac{5}{2} = -\frac{1}{2}, f'(2) = -6 + \frac{5}{2} = -\frac{7}{2}$. Since $\left|f'(0)f'(1)f'(2)\right| = \left|\frac{5}{2}\left(-\frac{1}{2}\right)\left(-\frac{7}{2}\right)\right| = \frac{35}{8} > 1$, the cycle $\{0, 1, 2\}$ is unstable.

EXERCISE 23

Find all periodic two orbits of the quadratic map $f(x) = 4.x(1-x), x \in [0, 1]$. Show that they are unstable.

SOLUTION

It can be easily shown that the fixed points of $f^2(x)$, where $f(x) = r(1-x), x \in [0, 1]$ are

$$x^* = 0, \left(1 - \frac{1}{r}\right), p, q$$

where $p = \frac{r+1+\sqrt{(r+1)(r-3)}}{2r}$ and $q = \frac{r+1-\sqrt{(r+1)(r-3)}}{2r}$. But $x^* = 0, \left(1 - \frac{1}{r}\right)$ are the fixed points of $f(x) = r(1-x)$. Here $r = 4$. So

$$p = \frac{4+1+\sqrt{(4+1)(4-3)}}{24} = \frac{5+\sqrt{5}}{8} \text{ and } q = \frac{5-\sqrt{5}}{8}. \text{ Now}$$
$$f(p) = 4p(1-p) = 4\left(\frac{5+\sqrt{5}}{8}\right)\left(1 - \frac{5+\sqrt{5}}{8}\right) = \frac{5+\sqrt{5}}{2} \cdot \frac{3-\sqrt{5}}{8} = \frac{5-\sqrt{5}}{8}$$
$$= q.$$

That is, $f(p) = q$. Similarly, $f(q) = p$. Thus we get $f(p) = q, f(q) = p$ and $f^2(p) = p$ and $f^2(q) = q$. Therefore the periodic-2 cycle of f is $\{p, q\}$, i.e., $\left\{\frac{5+\sqrt{5}}{8}, \frac{5-\sqrt{5}}{8}\right\}$. Here $f'(x) = 4 - 8x$. So,

$$f'(p) = 4 - 8p = 4 - 8\left(\frac{5+\sqrt{5}}{8}\right) = -(1+\sqrt{5}) \text{ and}$$
$$f'(q) = 4 - 8q = 4 - 8\left(\frac{5-\sqrt{5}}{8}\right) = -(1-\sqrt{5}).$$

The derivative test of the cycle $\{p, q\}$ gives $\left|f'(p)f'(q)\right| =| (1+\sqrt{5})(1-\sqrt{5}) |= 4 > 1$. Hence the cycle $\{(5+\sqrt{5})/8, (5-\sqrt{5})/8\}$ is unstable.

EXERCISE 24

Find the source and sink of the map $f(x) = x^2$.

SOLUTION

consider the map $f(x) = x^2$. The fixed points of the map are $x^* = 0, 1$. Since $\left|f'(0)\right| = 0 \neq 1$ and $\left|f'(1)\right| = 2 \neq 1$, the fixed points $x^* = 0, 1$ are hyperbolic. Hyperbolic Periodic Point Let p be a periodic point of period n of a map $f : \mathbb{R} \to \mathbb{R}$. Then p is said to be hyperbolic periodic point if $\left|(f^n)'(p)\right| \neq 1$. Let $f(x) = -\left(x + x^3\right)/2, x \in \mathbb{R}$. Clearly, the points ± 1 are periodic-2 points of f. Now, we see that $\left|\left(f^2\right)'(\pm 1)\right| = 4 \neq 1$. Hence the periodic points ± 1 are hyperbolic.

EXERCISE 25

Find the source and sink of the map $f(x) = -\left(x^2 + x\right)/2, x \in \mathbb{R}$. Show that they are hyperbolic in nature.

SOLUTION

The fixed points of the map f are given by $f(x) = x$. Therefore, the fixed points are given by $x^* = 0, -3$. Now, $f'(x) = -(2x+1)/2$ and the derivatives of f (x) at the points 0 and -3 are $f'(0) = -1/2$ and $f'(-3) = 5/2$, respectively. Since $\left|f'(0)\right| = 1/2 < 1$, the fixed point $x = 0$ is stable (sink). Again, $\left|f'(-3)\right| = 5/2 > 1$, the fixed point $x = -3$ is unstable (source). We see that $\left|f'(x)\right| \neq 1$ for both the fixed points. This implies that the source at $x = -3$ and the sink at $x = 0$ are hyperbolic in nature. These fixed points are known as hyperbolic source and hyperbolic sink under the flow f.

EXERCISE 26

determine the stability of the fixed point origin of the maps (i) $f(x) = \sin x, x \in \mathbb{R}$, (ii) $f(x) = x^3 - x^2 + x, x \in \mathbb{R}$.

SOLUTION

(i) Here $f(x) = \sin x, x \in \mathbb{R}$. Clearly, the origin is a fixed point of the map. Now, $f'(x) = \cos x, f''(x) = -\sin x, f''(x) = -\cos x$. Therefore, $f'(0) = \cos 0 = 1$. Also, the third derivative is continuous and $f'''(0) = -\cos 0 = -1 \neq 0$. Since $f''(0) = -\sin 0 = 0$ and $f'''(0) = -1 < 0$, by Theorem 9.6 , the origin is asymptotically stable. (ii) Here $f(x) = x^3 - x^2 + x$. Clearly, the origin is a fixed point of f. Now, $f'(x) = 3x^2 - 2x, f''(x) = 6x$ and $f'''(x) = 6$. Here $f''(x)$ is continuous and $f'''(0) \neq 0$. Since $f''(0) = 0$ and $f''(0) = 6 > 0$, by Theorem 9.6 , the origin is unstable.

EXERCISE 27

determine the stability of the fixed point origin of the maps $f(x) = \left(3x - x^3\right)/2, x \in \mathbb{R}$.

SOLUTION

Consider the map $f(x) = \left(3x - x^3\right)/2, x \in \mathbb{R}$. The fixed points of f are given by

$$f(x) = x \Rightarrow \left(3x - x^3\right)/2 = x \Rightarrow x = -1, 0, 1.$$

The derivative of $f(x)$ is $f'(x) = 3\left(1 - x^2\right)/2$. Since $f'(\pm 1) = 0$, the fixed points $x = \pm 1$ are superstable.

EXERCISE 28

determine the stability of the fixed point origin of the maps $f(x) = rx(1 - x), x \in [0, 1], r \geq 0$.

SOLUTION

Here $f'(x) = r(1 - 2x)$. In this context we define an important map, the unimodal map. It is a very simple nonlinear map with a single point of extremum. The map $f(x)$ is unimodal. For superstable 1-cycle, we must have $f'(x) = 0$. This gives $x = 1/2$. However, 1 -cycles are the fixed points of the map. So $f(x) = x$ at $x = 1/2$. This yields the parameter value $r = 2$. Thus a superstable 1 -cycle of $f(x) = rx(1 - x)$ exists when $r = 2$ and the cycle contains only the point $x = 1/2$. For superstable 2-cycle $\{p, q\}$ we have the condition

$f'(p)f'(q) = 0$. This shows that $x = 1/2$ must be an element of the 2 -cycle. Since every element of the 2 -cycle of $f(x)$ is a fixed point of $f^2(x)$, $x = 1/2$ is a fixed point of $f^2(x)$. By solving the equation $f^2(x) = x$ for $x = 1/2$ we get three values of r, namely $r = 2, (1-\sqrt{5}), (1+\sqrt{5})$. But $r = 2$ corresponds to superstable 1-cycle of f and $r = (1-\sqrt{5})$ is negative. So for superstable 2-cycle we must have $r = (1+\sqrt{5})$. With this value of r the superstable 2 -cycle is given by $\left\{\frac{1}{2}, \frac{1+\sqrt{5}}{4}\right\}$.

EXERCISE 29

determine the nature of the fixed point origin of the maps $f(x) = \sin x, x \in \mathbb{R}$

SOLUTION

A fixed point x^* of a map $f : \mathbb{R} \to \mathbb{R}$ is said to be non-hyperbolic if $\left|f'(x^*)\right| = 1$. Thus, for a non-hyperbolic fixed point x^* either $f'(x^*) = 1$ or $f'(x^*) = -1$. . Here $x^* = 0$ is a fixed point of f. Since $f'(x^*) = \cos(x^*) = \cos 0 = 1$, the fixed point $x^* = 0$ is non-hyperbolic. Similarly, $x = 0$ is a non-hyperbolic fixed point of the map $g(x) = \tan x$.

It is very difficult to say whether a non-hyperbolic fixed point is attracting or repelling. In this situation we use cobweb diagram to analyze the nature of the fixed point. We can also determine the stability of a non-hyperbolic fixed point

EXERCISE 30

Determine the stability behavior of the non-hyperbolic fixed points of the quadratic map $Q(x) = ax^2 + bx + c, a \neq 0$.

SOLUTION

The fixed points x^* of $Q(x)$ satisfy $Q(x^*) = x^*$. This yields two fixed points $x^*_{\pm} = \frac{-(b-1)\pm\sqrt{(b-1)^2-4ax}}{2a}$. If x^* is non-hyperbolic, then either $Q'(x^*) = 1$ or $Q'(x^*) = -1$ (i) Let $Q'(x^*) = 1$. Then $2ax^* + b = 1$. This gives the fixed point $x^* = (1-b)/2a$ and it exists when $(b-1)^2 - 4ac = 0$, that is, $(b-1)^2 = 4ac$ Since $Q''(x^*) = 2a > 0$, from Theorem 9.6 it follows that the fixed point $x^* = (1-b)/2a$ is semi-stable. (ii) Let $Q'(x^*) = -1$. Then $2ax^* + b = -1$. This gives the fixed point $x^* = -(b+1)/2a$ and it exists when $-(b-1)\pm\sqrt{(b-1)^2 - 4ac} = -(b+1)$, that is, when $(b-1)^2 = 4(ac+1)$ Calculate $Q''(x^*) = 2a$ and $Q'''(x^*) = 0$. Since $SQ(x^*) = -6a^2 < 0$,so the non-hyperbolic fixed point $x^* = -(b+1)/2a$ is asymptotically stable.

EXERCISE 31

Determine the stability of the fixed point origin of the function $f(x) = -\sin x$.

SOLUTION

Clearly, $x = 0$ is a fixed point f. We calculate $f(x) = -\cos x$, $f''(x) = \sin x$ and $f'''(x) = \cos x$. Obviously, $f'(x)$ is continuous and $f'(0) = -1$. The Schwarzian derivative of f at the origin is given by

$$\begin{aligned} Sf(0) &= \frac{f'''(0)}{f'(0)} - \frac{3}{2}\left(\frac{f''(0)}{f'(0)}\right)^2 \\ &= \frac{1}{(-1)} - \frac{3}{2}\left(\frac{0}{(-1)}\right)^2 = -1 < 0. \end{aligned}$$

This shows that the origin is asymptotically stable.

EXERCISE 32

Study the tent function in a comprehensive and complete study

SOLUTION

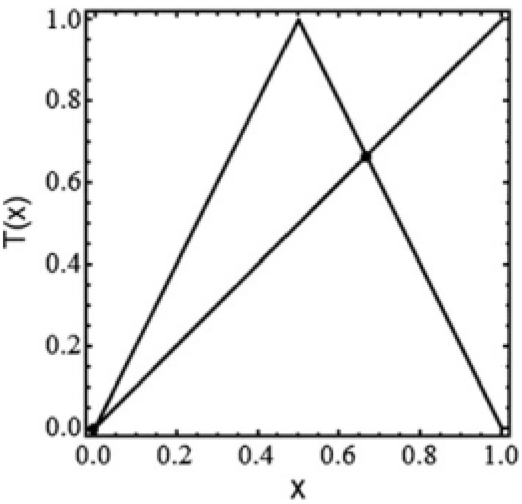


Figure 1.10: Graphical representation of $T(x)$ depicting the fixed points $x^* = 0$ and $x^* = 2/3$

segments of different slopes meet. The tent map has some interesting properties that we explore below. Generally, the tent map $T : [0,1] \to [0,1]$ is defined as

$$T(x) = \begin{cases} 2\lambda x, & 0 \le x \le 1/2 \\ 2\lambda(1-x), & 1/2 \le x \le 1 \end{cases} \tag{20.1}$$

where $\lambda(0 \le \lambda \le 1)$ is a control parameter. We can also write the tent map by an iterative sequence as follows:

$$x_{n+1} = f(x_n) = \lambda\left(1 - 2\left|x_n - \frac{1}{2}\right|\right)$$

As we know that a fixed point is an invariant solution of a map, the nature of the fixed points plays an important role in analyzing the dynamical behavior of the map.we shall analyze the fixed points of the tent map for the parameter value $\lambda = 1$. The fixed points satisfy the relation $x = T(x)$. So, we get $x = 2x \Rightarrow x = 0 \in [0, 1/2]$ and $x = 2(1-x) \Rightarrow x = 2/3 \in [1/2, 1]$. Thus, the only two fixed points of the tent map are $x^* = 0$ and $x^* = 2/3$. The graphical representation of $T(x)$ is shown in 1.10.

The map $T^2(x)$:

Using the definition of $T(x)$, the twofold composition of the tent map can be obtained as follows:

$$T^2(x) = T(T(x)) = \begin{cases} 2(2x), & 0 \le 2x \le 1/2 \\ 2(1-2x), & 1/2 \le 2x \le 1 \\ 2 \cdot 2(1-x), & 0 \le 2(1-x) \le 1/2 \\ 2(1-2(1-x)), & 1/2 \le 2(1-x) \le 1 \end{cases}$$

$$= \begin{cases} 4x, & 0 \le x \le 1/4 \\ 2-4x, & 1/4 \le x \le 1/2 \\ -2+4x, & 1/2 \le x \le 3/4 \\ 4-4x, & 3/4 \le x \le 1 \end{cases}$$

For determining the fixed points of $T^2(x)$, we have to solve the equation $x = T^2(x)$. Now,

$$\text{for } 0 \le x \le \frac{1}{4}, x = T^2(x) \Rightarrow x = 4x \Rightarrow x = 0$$
$$\text{for } \frac{1}{4} \le x \le \frac{1}{2}, x = T^2(x) \Rightarrow x = 2-4x \Rightarrow x = \frac{2}{5}$$
$$\text{for } \frac{1}{2} \le x \le \frac{3}{4}, x = T^2(x) \Rightarrow x = -2+4x \Rightarrow x = \frac{2}{3}$$
$$\text{for } \frac{3}{4} \le x \le 1, x = T^2(x) \Rightarrow x = 4-4x \Rightarrow x = \frac{4}{5}$$

Therefore, the fixed points of $T^2(x)$ are given by $x^* = 0, \frac{2}{5}, \frac{2}{3}, \frac{4}{5}$. The diagrammatic representation of

$T^2(x)$ displaying the fixed points is presented in figure 1.10.

figure 1.11 The twofold composition $T^2(x)$ depicting four fixed points

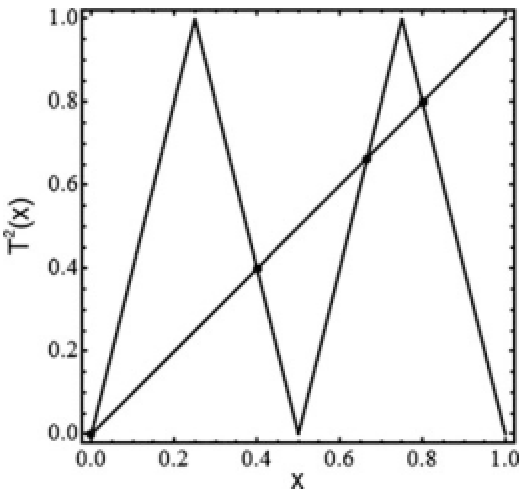


Figure 1.11: the twofold composition $T^2(x)$ depicting four fixed points

The Map $T^3(x)$

Using $T(x)$ and the twofold composition $T^2(x)$, we can show the representation of $T^3(x)$ as follows:

$$T^3(x) = \begin{cases} 8x, & 0 \le x \le \frac{1}{8} \\ 2-8x, & \frac{1}{8} \le x \le \frac{1}{4} \\ -2+8x, & \frac{1}{4} \le x \le \frac{3}{8} \\ 4-8x, & \frac{3}{8} \le x \le \frac{1}{2} \\ -4+8x, & \frac{1}{2} \le x \le \frac{5}{8} \\ 6-8x, & \frac{5}{8} \le x \le \frac{3}{4} \\ -6+8x, & \frac{3}{4} \le x \le \frac{7}{8} \\ 8-8x, & \frac{7}{8} \le x \le 1 \end{cases}$$

and its fixed point calculated as $x^* = 0, \frac{2}{3}, \frac{2}{7}, \frac{4}{7}, \frac{6}{7}, \frac{2}{9}, \frac{4}{9}, \frac{8}{9}$. The graphical representation of $T^3(x)$ displaying the fixed points is presented in Figure 1.12.

Periodic Orbits of the Tent Map

Q1: The set of points $\left\{\frac{2}{5}, \frac{4}{5}\right\}$ forms a periodic-2 cycle of the tent map.

Solution: We show that one orbit of the tent map is $\frac{2}{5}, \frac{4}{5}, \frac{2}{5}, \frac{4}{5}, \ldots$, that is, an orbit which repeats itself exactly every second iteration. Now,

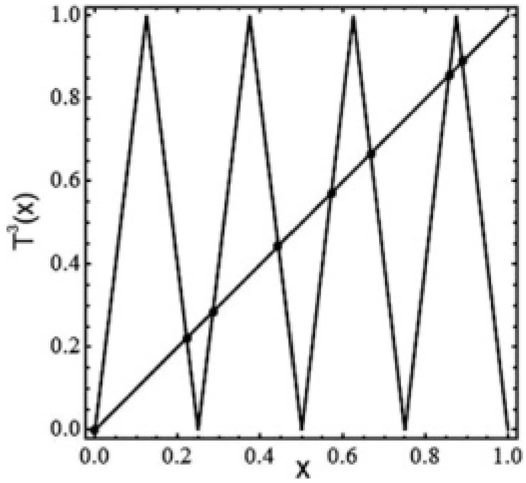


Figure 1.12: Graphical representation of $T^3(x)$

$$T(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ 2(1-x) & \frac{1}{2} \le x \le 1 \end{cases}$$

Since $\frac{2}{5} \in \left[0, \frac{1}{2}\right], T\left(\frac{2}{5}\right) = 2\left(\frac{2}{5}\right) = \frac{4}{5}$. Similarly, $T\left(\frac{4}{5}\right) = 2\left(1 - \frac{4}{5}\right) = \frac{2}{5}$.

Also, $T^2\left(\frac{2}{5}\right) = T\left(T\left(\frac{2}{5}\right)\right) = T\left(\frac{4}{5}\right) = \frac{2}{5}$ and $T^2\left(\frac{4}{5}\right) = T\left(T\left(\frac{4}{5}\right)\right) = T\left(\frac{2}{5}\right) = \frac{4}{5}$.

Therefore, $T\left(\frac{2}{5}\right) = \frac{4}{5}, T\left(\frac{4}{5}\right) = \frac{2}{5}, T^2\left(\frac{2}{5}\right) = \frac{2}{5}$ and $T^2\left(\frac{4}{5}\right) = \frac{4}{5}$.

This shows that $\left\{\frac{2}{5}, \frac{4}{5}\right\}$ forms a periodic-2 cycle of the tent map T. The derivative test, gives the stability behavior of the cycle. Now, $\left|T'\left(\frac{2}{5}\right)T'\left(\frac{4}{5}\right)\right| = 4 > 1$. Hence the periodic-2 cycle is unstable.

Q2 The cycles $\left\{\frac{2}{7}, \frac{4}{7}, \frac{6}{7}\right\}$ and $\left\{\frac{2}{9}, \frac{4}{9}, \frac{8}{9}\right\}$ form two periodic- 3 cycles of the tent map.

Solution We shall show that the points $\left\{\frac{2}{7}, \frac{4}{7}, \frac{6}{7}\right\}$ and $\left\{\frac{2}{9}, \frac{4}{9}, \frac{8}{9}\right\}$ repeat itself every third iterations. Now,

$$T\left(\frac{2}{7}\right) = 2 \cdot \frac{2}{7} = \frac{4}{7}, T\left(\frac{4}{7}\right) = 2\left(1 - \frac{4}{7}\right) = \frac{6}{7}, \text{ and } T\left(\frac{6}{7}\right) = 2\left(1 - \frac{6}{7}\right) = \frac{2}{7}$$

$$T^2\left(\frac{2}{7}\right) = T\left(T\left(\frac{2}{7}\right)\right) = T\left(\frac{4}{7}\right) = \frac{6}{7}$$

$$T^2\left(\frac{4}{7}\right) = T\left(T\left(\frac{4}{7}\right)\right) = T\left(\frac{6}{7}\right) = \frac{2}{7} \text{ and } T^2\left(\frac{6}{7}\right) = T\left(T\left(\frac{6}{7}\right)\right) = T\left(\frac{2}{7}\right) = \frac{4}{7}$$

Again, $T^3\left(\frac{2}{7}\right) = T\left(T^2\left(\frac{2}{7}\right)\right) = T\left(\frac{6}{7}\right) = \frac{2}{7}$. Similarly, $T^3\left(\frac{4}{7}\right) = \frac{4}{7}$ and $T^3\left(\frac{6}{7}\right) = \frac{6}{7}$

This shows that $\left\{\frac{2}{7}, \frac{4}{7}, \frac{6}{7}\right\}$ forms a periodic- 3 cycle of T. In the similar manner we can prove that $\left\{\frac{2}{9}, \frac{4}{9}, \frac{8}{9}\right\}$ also forms a periodic-3 cycle of the tent map. The derivative test for 3-cycle gives $\left|T'\left(\frac{2}{7}\right)T'\left(\frac{4}{7}\right)T'\left(\frac{6}{7}\right)\right| = 2^3 > 1$. Similarly, the derivative test for the other cycle gives that value 2^3. So, both the cycles are unstable.

Periodic cycles of $T^n(x)$

From the above analysis, it can be shown easily that the n-fold composition, $T^n(x)$, of $T(x)$ has 2^n fixed points. For $n = 1$, there are two period-1 orbits, $x_0^* = 0$ and $x_1^* = 2/3$. In each fold of compositions there is a fixed point $x_0^* = 0.T^2(x)$ has four fixed points, of which two are new, $x^* = 2/5, 4/5$. But the new pair of fixed points $\left\{\frac{2}{5}, \frac{4}{5}\right\}$ forms a period-2 orbit of the map. $T^3(x)$ has eight fixed points, among which two fixed points $x_0^* = 0$ and $x_1^* = 2/3$ belong to $n = 1$ and the other fixed points, viz., $\left\{\frac{2}{9}, \frac{4}{9}, \frac{8}{9}\right\}$ and $\left\{\frac{2}{7}, \frac{4}{7}, \frac{6}{7}\right\}$ form two period-3 orbits of the tent map. $T^4(x)$ has 2^4 fixed points, among which two belong to $n = 1$ and another two belong to $n = 2$. So, the other 12 fixed points of $T^4(x)$ must form three distinct period-4 orbits. In general, $T^n(x)$ has 2^n fixed points and we see from above analyses that there is at least one period-n orbit with the points $\left\{\frac{2}{2^n+1}, \frac{2^2}{2^n+1}, \frac{2^3}{2^n+1}, \ldots, \frac{2^n}{2^n+1}\right\}$ (Davies [1]). So, there are periodic orbits of every period. Note that none of these periodic cycles contain the points $0, 1/2$ and 1 , where the derivatives of the map are not defined. The derivative test can be applied to these cycles for their stability analyses. These give $T'(x^*) = \pm 2, (T^n)'(x^*) = \pm 2^n \Rightarrow \left|(T^n)'(x^*)\right| = 2^n > 1$. This implies that all cycles are unstable in nature. In other words, we can say that infinitely many unstable periodic orbits can exist for the tent map.

EXERCISE 33

Study The Quadratic Map $Q_c(x) = x^2 + c, x \in \mathbb{R}$ in a comprehensive and complete study

SOLUTION

The family of quadratic map is often denoted by Q_c and is defined by $Q_c(x) = x^2 + c, x \in \mathbb{R}$.
where $c \in \mathbb{R}$ is a parameter. This map is called the Myrbrg family of maps on $\mathbb{R}$ as the domain. Myrbrg was one of the first to study this map extensively, see [5, 6]. The study of dynamics of Q_c with varying c is interesting and we shall discuss it below. First we calculate the fixed points of the quadratic map.

Fixed points: The fixed points of Q_c are simply the roots of the quadratic equation $Q_c(x) - x \equiv x^2 + c - x = 0$.
This yields two fixed points

$$x_+^* = \frac{1}{2}(1 + \sqrt{1 - 4c})$$

and

$$x_-^* = \frac{1}{2}(1 - \sqrt{1 - 4c})$$

. Note that the points $x_\pm^*$ depend on the parameter c. Furthermore, they are real if and only if $c \leq 1/4$. So, the fixed points of Q_c exist only when $c \leq 1/4$. At $c = 1/4, x_+^* = x_-^* = 1/2$. No fixed points will appear when $c > 1/4$. Figure **??** depicts the three cases.

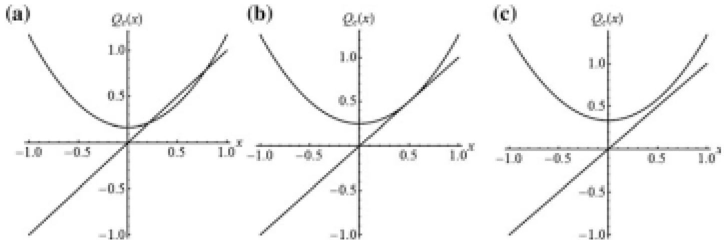


Figure 1.13: Quadratic map for a $c < 1/4$, **b**$c = 1/4$, and $cc > 1/4$

Stabilities of the fixed points are determined by the derivative condition $Q_c'(x_\pm^*) = 1 \pm \sqrt{1-4c}$. Note that $Q_c'(x_+^*) > 1$ when $c < 1/4$ and $Q_c(x_+^*) = 1$ at $c = 1/4$. Hence the fixed point x_+^* is repelling when $c < 1/4$ and it is neutral when $c = 1/4$. At $x_-^*, Q_c'(x_-^*) = 1$ when $c = 1/4$ and $Q_c'(x_-^*) < 1$ for c slightly below 1/4. We now see that $\left|Q_c'(x_-^*)\right| < 1$ if and only if $-3/4 < c < 1/4$. At $c = -3/4, Q_c'(x_-^*) = -1$ and $Q_c'(x_-^*) < -1$ when $c < -3/4$. Therefore, the fixed point x_-^* is attracting when $-3/4 < c < 1/4$ and repelling when $c < -3/4$. At $c = 1/4, -3/4$, the stability test fails. Using the cobweb diagram, the fixed point $x_+^* = x_-^* = 1/2$ at $c = 1/4$ is semi-stable. Similarly, at $c = -3/4$, the fixed point x_-^* is stable while the fixed point x_+^* is unstable. Figure **??** displays the stability characters of the fixed points for $c = 1/4, -3/4$.

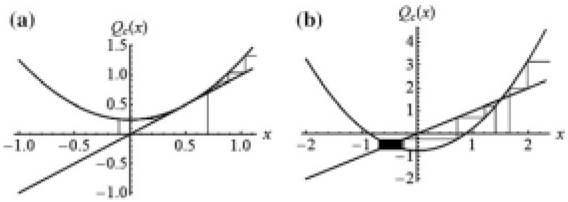


Figure 1.14: Stability characters of the fixed points at a $c = 1/4$ and $bc = -3/4$

Periodic points: The period-1 points of Q_c are the fixed points of the map. The period-2 points of Q_c are the fixed points of the map Q_c^2 and are the roots of the equation

$$f(x) \equiv Q_c^2(x) - x = x^4 + 2cx^2 - x + c^2 + c = 0.$$

Since the fixed points of Q_c are also the fixed points of $Q_c^2, \left(x^2 - x + c\right)$ is a factor of the polynomial $f(x)$. Dividing $f(x)$ by this factor, we get the other factor as $\left(x^2 + x + c + 1\right)$. The solutions $p, q = \frac{-1 \pm \sqrt{-(4c+3)}}{2}$ of the equation $x^2 + x + c + 1 = 0$ yield the other two fixed points of Q_c^2. Note that the points p, q exist only when $c \leq -3/4$, they collide at $(-1/2)$ when $c = -3/4$. Now, calculate

$$\begin{aligned} Q_c(p) = p^2 + c &= \frac{1}{4}(1 - 2\sqrt{-4c-3} - 3 - 4c) + c \\ &= \frac{-1 - \sqrt{-(4c+3)}}{2} = q, \end{aligned}$$

and

$$Q_c(q) = q^2 + c = \left(\frac{-1 - \sqrt{-(4c+3)}}{2}\right)^2 + c$$
$$= \frac{-1 + \sqrt{-(4c+3)}}{2} = p.$$

This shows that $\{p, q\}$ forms a period-2 cycle of $Q_c(x)$. Stability of this cycle is determined by the derivative condition of the cycle $Q_c'(p)Q_c'(q) = 4pq = 4(c+1)$. Therefore, $\left|Q_c'(p)Q_c'(q)\right| < 1$ when $-1 < 4(c+1) < 1$, that is, when $-5/4 < c < -3/4$. Similarly, $\left|Q_c'(p)Q_c'(q)\right| > 1$ when $c < -5/4$ and $\left|Q_c'(p)Q_c'(q)\right| = 1$ when $c = -3/4, -5/4$. Thus, the period-2 cycle $\{p, q\}$ of the quadratic map Q_c is stable when $-5/4 < c < -3/4$ and is unstable (repelling) when $c < -5/4$ figure 1.15.

It can be shown that the quadratic map has two period-3 cycles and they occur at $c = -1.75$. Figure 1.16 shows two attracting 3 -cycles of the quadratic map at $c = -1.77$.

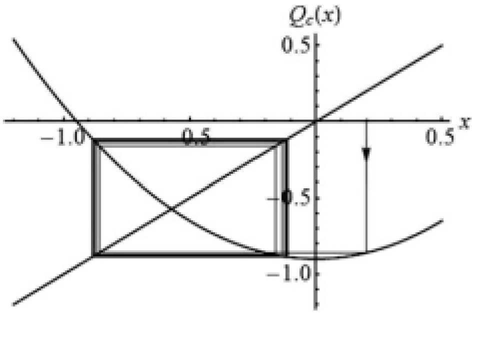


Figure 1.15: Period-2 cycle of the quadratic map for $c = -0.9$

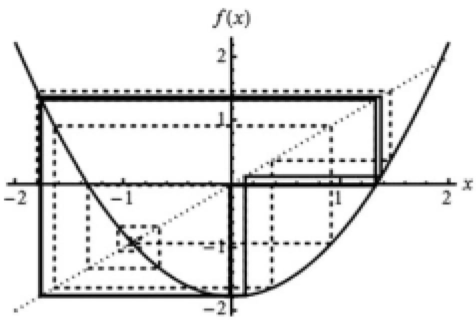


Figure 1.16: Two period-3 cycles of the quadratic map

EXERCISE 34

Prove that the logistic map L_4 is conjugate to $G : [-1,1] \to [-1,1]$,

$$G(x) = 2x^2 - 1$$

SOLUTION

We know that If $F(x) = ax^2 + bx + c$, and $G(x) = rx^2 + sx + t$ where $a \neq 0, r \neq 0$ and $c = \frac{b^2-s^2+2s-2b+4rt}{4a}$, then F and G are linearly conjugate via $h(x) = \frac{a}{r}x + \frac{b-s}{2r}$.

Now,

$$L_4 = 4x(1-x) = -4x^2 + 4x \text{ so, } a = -4, b = 4, c = 0, \text{ and}$$

$$G(x) = 2x^2 - 1 \text{ so, } r = 2, s = 0, t = -1, \text{ and } h(x) = -2x + 1$$

So it is sufficient to prove that $c = \frac{b^2-s^2+2s-2b+4rt}{4a} = 0$. Which is clear $16 - 0 + 0 - 2(4) + 4(2)(-1) = 0$. Thus $L_4 \approx G$ via $h(x) = -2x + 1$.

EXERCISE 35

Consider the map $T_a(x); a \in R$ and $a > 1$ that defined as

$$T_a(x) = \begin{cases} a(x+1) \text{ for } x \leq 0 \\ a(1-x) \text{ for } x > 0 \end{cases}$$

Verify that a period -2 orbit of $T_a(x)$ is given by $x_1 = \frac{a(1-a)}{1+a^2}; x_2 = \frac{a(1+a)}{1+a^2}$ and determine the stability of this orbit.

SOLUTION

Since $a > 1$ we have that $x_1 < 0, x_2 > 0$

$$T_a(x_1) = a\left[\frac{a(1-a)}{1+a^2} + 1\right] = \frac{a^2 - a^3 + a + a^3}{1+a^2} = \frac{a(a+1)}{1+a^2}$$

$$= x_2$$

$$= \frac{a-a^2}{1+a^2} = \frac{a(1-a)}{1+a^2} = x_1$$

Thus $\{x_1, x_2\}$ is z-cycle orbit. Now $T_a'(x) = \begin{cases} a & \text{for} \quad x \leqslant 0 \\ -a & \text{for} \quad x > 0, \end{cases}$

$$\left|T_a^1\,(x_1) \cdot \left| x_a'\right)\right| = |a(-a)\,| = a^2 > 1 \quad (\text{ since} \quad a > 1)$$

Thus, the orbit is unstable .

CHAPTER 2

BIFURCATIONS IN ONE-DIMENSIONAL DISCRETE SYSTEMS

EXERCISE 36

Study the Logistic Map $r(x) = r(1-x)$ in a comprehensive and complete study

SOLUTION

This map is associated with the logistic pattern of population growth (linear growth model, that is, $r(x) = r(1-x)$, a linear decrease of $r(x)$ with increasing population x) and may be represented by

$$x_{n+1} = f(x_n) = rx_n(1-x_n) \tag{2.1}$$

which is basically a discrete-time analog of the logistic equation for the population growth model $\dot{x} = rx(1-x), x \in [0,1]$. Here $x_n \geq 0$ is a dimensionless measure of the population in the nth generation and $r \geq 0$ is the intrinsic growth rate (population growth parameter). The graph of 2.1 represents a parabola with a maximum value of $(r/4)$ at $x = 1/2$. Figure **??** depicts a sketch of $f(x)$ at the parameter value $r = 4$.

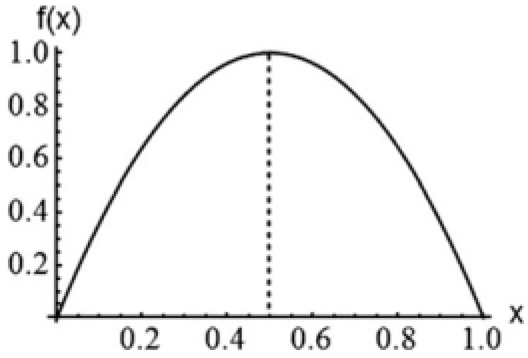


Figure 2.1: Sketch of $f(x)$ at the parameter value $r = 4$

Some Properties of the Logistic Map

Q1 Find all fixed points of the logistic map $x_{n+1} = rx_n(1 - x_n)$ for $0 \leq x_n \leq 1$ and $0 \leq r \leq 4$. Also determine their stability behaviors.

Solution The logistic map function is given by $f(x) = rx(1 - x), 0 \leq x \leq 1$ and $0 \leq r \leq 4$. For the fixed points of $f(x)$, we have a relation

$$f(x) = x \Rightarrow rx(1 - x) = x \Rightarrow x\{(r - 1) - rx\} = 0 \Rightarrow x = 0, \left(1 - \frac{1}{r}\right)$$

So, the map has two fixed points, namely $x_0^* = 0$ and $x_1^* = \left(1 - \frac{1}{r}\right)$. Clearly, $x_0^* = 0$ is a fixed point of the map f for all values of the parameter r. But $x_1^* = \left(1 - \frac{1}{r}\right)$ is a fixed point of f if $r \geq 1$ (since $0 \leq x \leq 1$). Therefore, the fixed points of the logistic map are $x_0^* = 0 \forall r \in [0, 4]$ and $x_1^* = \left(1 - \frac{1}{r}\right)$ for $1 \leq r \leq 4$. The stability of the fixed points depends on the absolute value of $f'(x) = r - 2rx$. Since $\left|f'(0)\right| = |r| = r$, the fixed point $x_0^* = 0$ is stable when $r < 1$ and unstable when $r > 1$. Again, since $\left|f'\left(1 - \frac{1}{r}\right)\right| = |2 - r|$, the fixed point x_1^* is stable if $|2 - r| < 1$, that is, if $1 < r < 3$ and unstable if $|2 - r| > 1$, that is, in the range $3 < r \leq 4$.

Q2 Prove that the logistic map $f(x) = rx(1 - x)$ has a 2-cycle for all $r > 3$.

Solution

The growth rate parameter r is the deciding factor for the evolution of the logistic map. We now find periodic-2 orbit for logistic map which depends upon the parameter r. Now, $f^2(x) = rf(x)\{1 - f(x)\} = r^2x(1 - x)\{1 - rx(1 - x)\}$. For finding fixed points of twofold composition f^2, we get a relation

$$
\begin{aligned}
& f^2(x) = x \\
\text{or,} \quad & r^2x(1-x)\{1-rx(1-x)\} = x \\
\text{or,} \quad & x\left[r^2(1-x)\{1-rx(1-x)\}-1\right] = 0 \\
\text{or,} \quad & x\left[r^2(1-x)-r^3x(1-x)^2-1\right] = 0 \\
\text{or,} \quad & x\left[r^2(1-x)+r^3(1-x-1)(1-x)^2-1\right] = 0 \\
\text{or,} \quad & x\left[r^3(1-x)^3-r^3(1-x)^2+r^2(1-x)-1\right] = 0 \\
\text{or,} \quad & x\left\{x-\left(1-\frac{1}{r}\right)\right\}\left[r^3(1-x)^2+r^2(1-r)(1-x)+r\right] = 0 \\
\text{or,} \quad & x = 0, 1-\frac{1}{r} \text{ and } 1-x = \frac{-r^2(1-r) \pm \sqrt{r^4(1-r)^2-4r^4}}{2r^3} \\
& = \frac{r-1 \pm \sqrt{(r+1)(r-3)}}{2r}
\end{aligned}
$$

or,

$$
\begin{aligned}
x &= 0, 1-\frac{1}{r} \text{ and } 1-\frac{r-1 \pm \sqrt{(r+1)(r-3)}}{2r} = \frac{r+1 \pm \sqrt{(r+1)(r-3)}}{2r} \\
&= p, q(\text{ say })
\end{aligned}
$$

Therefore, there are four fixed points of f^2, given by $x^* = 0, \frac{1}{r}, p, q$. But the two, $x^* = 0, \frac{1}{r}$, of them are the fixed points of $f(x)$ and the other two are real only for $r \geq 3$. We examine the cycle property of f.

Now,

$$
\begin{aligned}
f(p) = rp(1-p) &= r \cdot \frac{r+1+\sqrt{(r+1)(r-3)}}{2r}\left\{1-\frac{r+1+\sqrt{(r+1)(r-3)}}{2r}\right\} \\
&= \frac{r+1+\sqrt{(r+1)(r-3)}}{2}\left\{\frac{r-1-\sqrt{(r+1)(r-3)}}{2r}\right\} \\
&= \frac{r^2-\{1+\sqrt{(r+1)(r-3)}\}^2}{4r} \\
&= \frac{r^2-1-2\sqrt{(r+1)(r-3)}-\left(r^2-2r-3\right)}{4r} \\
&= \frac{2r+2-2\sqrt{(r+1)(r-3)}}{4r} \\
&= \frac{r+1-\sqrt{(r+1)(r-3)}}{2r} = q \\
&\Rightarrow f(p) = q
\end{aligned}
$$

Similarly, we can show that $f(q) = p$. Again, $f^2(p) = f(f(p)) = f(q) = p$ and $f^2(q) = f(f(q)) = f(p) = q$. According to the definition of 2-cycle it is clear that the logistic map $f(x)$ has a periodic-2 orbit or cycle $\{p, q\}$ when $r > 3$. Note that for $r < 3$ the roots are complex, which means that a cycle does not exist. Hence a 2-cycle of the logistic map appears when $r > 3$. The graph of $f^2(x)$ for $r > 3$ is shown in

Figure 2.2.

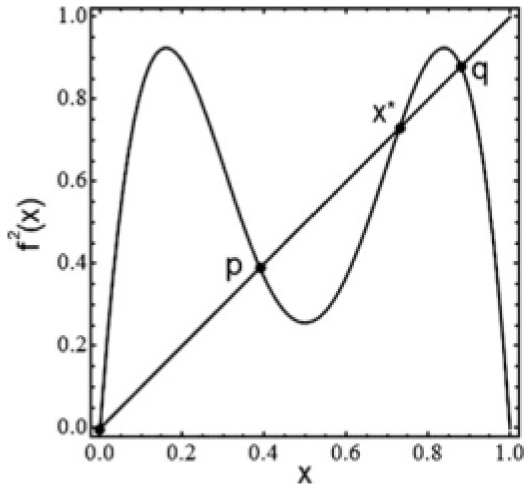


Figure 2.2: 10.5 Graphical representation of the logistic map at the second iteration for $r = 3.7$

Q3

Explain bifurcation of a system. Show that the logistic map $f_r(x) = rx(1-x), x \in [0,1]$ undergoes a transcritical bifurcation at $r = 1$ and a period-doubling bifurcation at $r = 3$.

Solution

Bifurcation is basically a change in the structure of the orbit as a system parameter (known as control parameter) varies continuously through critical values. Bifurcation theory is concerned with equilibrium solutions of system. The characters of the fixed points and the period orbits are altered figure 2.2).

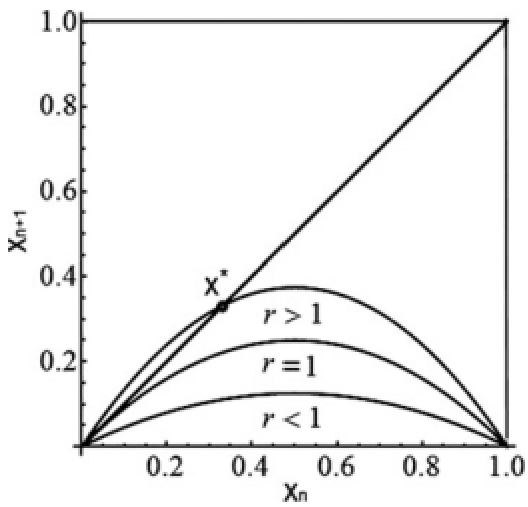


Figure 2.3: Graphical representation of the logistic map for different values of r

When the parameter is increased, the two fixed points collide, whereupon they exchange their stabilities. This type of bifurcation is called transcritical bifurcation and this is not a common type of bifurcation occurred in one-dimensional discrete system. Now, the fixed points of the logistic map f are given by

$$x = f(x) = rx(1-x) \Rightarrow x = 0, \left(1 - \frac{1}{r}\right)$$

The fixed $x = 0$ exists for all values of r, whereas the other fixed point $x = \left(1 - \frac{1}{r}\right)$ exists when $r \geq 1$. So, the fixed points of the logistic map are given by

$$x_0^* = 0 \quad \forall r \in [0, 4] \text{ and } x_1^* = 1 - \frac{1}{r} \text{ for } 1 \leq r \leq 4$$

As discussed earlier, the fixed point origin is stable when $r < 1$ and unstable when $r > 1$. Similarly, the fixed point x_1^* is stable when $1 < r < 3$ and unstable when $r > 3$. Geometrically, the stability behaviors can be explained in a better way. We draw the graphs of the logistic map and the line $y = x$ in $x - f(x)$ plane.

For the parameter value $r < 1$, the parabola, which represents the logistic map $f(x)$, lies below the diagonal line $y = x$ and the origin is the only fixed point, and it is stable. With increasing values of r, say at $r = 1$, the parabola becomes tangent to the diagonal line $y = x$. For $r > 1$, the parabola intersects the diagonal line at the fixed point x_1^*, while the fixed point origin loses its stability. Thus, we see that at $r = 1$ the map undergoes a bifurcation resulting a transcritical bifurcations by exchanging stabilities of the fixed points.

When r increases beyond 1 , the slope of the function f gets increasingly steep and the critical slope is attained when $r = 3$ (at $r = 3$, fixed points are $p = q = 2/3$). This indicates that the logistic map undergoes another bifurcation leading to period-2 cycle. This bifurcation is known as a period-doubling bifurcation or a flip bifurcation. The iterates flip from side to side of the fixed point. So, flip bifurcation is basically a period-doubling bifurcation and occurs at the critical value $r = 3$ for the logistic map.

Q4

Prove that the period-2 cycle of the logistic map is linearly stable when $3 < r < (1 + \sqrt{6}) = 3.449\ldots$

Solution

We know that the period 2-cycle of the logistic map is $\{p, q\}$, where

$$p, q = \frac{r + 1 \pm \sqrt{(r+1)(r-3)}}{2r}$$

So, $p + q = \frac{r+1}{r}$ and $pq = \frac{(r+1)^2 - (r+1)(r-3)}{4r^2} = \frac{r+1}{r^2}$.

The derivative of $f(x)$ is given by $f'(x) = r - 2rx$. Therefore, $f'(p) = r - 2rp$ and $f'(q) = r - 2rq$. The linear stability of the 2-cycle $\{p, q\}$ gives the following condition as

$$
\begin{aligned}
&\left|f'(p)f'(q)\right| < 1 \\
\Rightarrow \quad & |(r-2rp)(r-2rq)| < 1 \\
\Rightarrow \quad & r^2|(1-2p)(1-2q)| < 1 \\
\Rightarrow \quad & r^2|1-2(p+q)+4pq| < 1 \\
\Rightarrow \quad & r^2\left|1-2\left(\tfrac{r+1}{r}\right)+4\left(\tfrac{r+1}{r^2}\right)\right| < 1 \\
\Rightarrow \quad & \left|r^2-2r(r+1)+4(r+1)\right| < 1 \\
\Rightarrow \quad & \left|-r^2+2r+4\right| < 1 \\
\Rightarrow \quad & \left|r^2-2r-4\right| < 1 \\
\Rightarrow \quad & \left|(r-1)^2-5\right| < 1 \\
\Rightarrow \quad & -1 < (r-1)^2-5 < 1 \\
\Rightarrow \quad & 4 < (r-1)^2 < 6 \\
\Rightarrow \quad & 2 < (r-1) < \sqrt{6} \\
\Rightarrow \quad & 3 < r < 1+\sqrt{6}.
\end{aligned}
$$

Hence, the period 2-cycle of the logistic map is linearly stable when $3 < r < (1+\sqrt{6})$.

EXERCISE 37

Study the Cubic Map $f(x) = rx - x^3$ in a comprehensive and complete study

SOLUTION

The Cubic Map

A family of one-dimensional cubic maps is defined by $f(x) = rx - x^3$, where $r \in \mathbb{R}$ is the control parameter. The dynamics of the cubic maps are more complicated than the quadratic maps. The fixed points of f are given by $x = 0, \pm\sqrt{(r-1)}$. Clearly, $x = 0$ is a fixed point of f for every value of r, but the other two fixed points, $x_{\pm} = \pm\sqrt{(r-1)}$, exist only when $r \geq 1$, and they coincide with $x = 0$ when $r = 1$. Figure 10.16 shows the fixed points of the map at $r = 2, -1$. The fixed points are the intersecting points of the function $f(x)$ with the diagonal line.

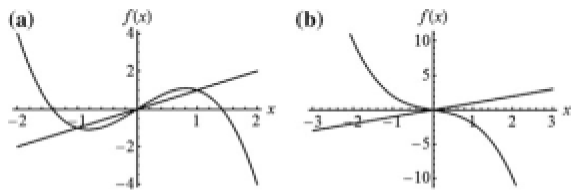


Figure 2.4: Fixed points of the cubic map at at $r = 2$, b$r = -1$

A bifurcation may occur at $r = 2$. We shall now determine the period-2 cycle of f, which are simply the solutions of the equation $g(x) \equiv f^2(x) - x = r\left(rx - x^3\right) - \left(rx - x^3\right)^3 - x = 0$. Since the fixed points of f are also the fixed points of f^2, $x^3 - (\mathrm{r} - 1)x$ must be a factor of the polynomial $\mathrm{g}(x)$. On division, we obtain the other factor as $\left(x^2 - r - 1\right)\left(x^4 - rx^2 + 1\right)$ and this gives six 2 -cycle points, $a_\pm = \pm\sqrt{r+1}$, $b_\pm = \pm\sqrt{\frac{r+\sqrt{r^2-4}}{2}}$ and $c_\pm = \pm\sqrt{\frac{r\sqrt{r^2-4}}{2}}$. The points $a_\pm$exist if $r \geq -1$. At $r = -1$, they coincide with the fixed point $x = 0$. The other four points exist when $r \geq 2$. We see that $f(a_+) = r\sqrt{r+1} - (r+1)\sqrt{r+1} = -\sqrt{r+1} = a_-$, $f(a_-) = -r\sqrt{r+1} + (r+1)\sqrt{r+1} = \sqrt{r+1} = a_+$. Thus, $\{a_+, a_-\}$forms a period-2 cycle of the cubic map. Similarly, it can be shown that the other two period-2 cycles of the cubic map are $\{b_+, c_-\}$and $\{b_-, c_-\}$. To determine the stability of the 2-cycles, we evaluate $\left(f^2\right)'(x) = \left(r - 3x^2\right)\left(r - 3x^2\left(r - x^2\right)^2\right)$. Using a little algebraic manipulation, we see that the 2 -cycle $\{a_+, a_-\}$is stable when $1 < r < 2$ and the other two 2 -cycles are stable when $2 < r < \sqrt{5}$.

Hénon Map

This is a two-dimensional map and was first studied by the French astronomer Michel Hénon (1931-2013) in the year 1976. The Hénon map $H_{ab} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined as

$$(x, y) \to H_{a,b}(x, y) = \left(a - x^2 + by, x\right)$$

$$\text{or,} \begin{bmatrix} x_n \\ y_n \end{bmatrix} \to \begin{bmatrix} x_{n+1} = a - x_n^2 + by_n \\ y_{n+1} = x_n \end{bmatrix}$$

where a, b are two real parameters (Fig. 10.22). The Hénon map has a nonlinear term x^2. The fixed points of this map satisfy the following equations:

$$f(x, y) = (x, y) \Rightarrow a - x^2 + by = x \text{ and } y = x$$

which is equivalent to

$$x^2 - (b-1)x - a = 0.$$

The roots are given by

$$x = \frac{b - 1 \pm \sqrt{(b-1)^2 + 4a}}{2}.$$

Clearly, the roots are real if and only if

$$(b-1)^2 + 4a \geq 0, \text{ that is, } 4a \geq -(b-1)^2.$$

Hence the Hénon map has fixed points if the relation $4a \geq -(b-1)^2$ is satisfied and the fixed points lie along the diagonal line $y = x$. When $4a = -(b-1)^2$, the map has only one fixed point $((b-1)/2, (b-1)/2)$

and when $4a > -(b-1)^2$, it has two distinct fixed points (α, α) and (β, β) where

$$\alpha, \beta = \frac{b - 1 \pm \sqrt{(b-1)^2 + 4a}}{2}.$$

For example, if $a = 0$ and $b = 0.4$, then the Hénon map has two distinct fixed points $(0, 0)$ and $(-0.6, -0.6)$. The Jacobian matrix of $H_{a,b}$ is given by

$$J(x, y) = \begin{pmatrix} -2x & b \\ 1 & 0 \end{pmatrix}$$

This gives the eigenvalues

$$\lambda = x \pm \sqrt{x^2 + (b/2)^2}.$$

The Jacobian determinant of the Hénon map is constant and it is given by $\det(J) = -b$. Note that $\det(J) \neq 0$ if and only if $b \neq 0$. Hence the Hénon map is invertible if and only if $b \neq 0$ and the inverse is given by

$$H_{a,b}^{-1} = \left(y, \left(x - a + y^2\right)/b\right).$$

We also see that $|\det(J)| < 1$ if and only if $-1 < b < 1$. Hence the Hénon map is area-contracting (a map $f(x, y)$ is said to be area-contracting if $|\det(J(x, y))| < 1$ everywhere) for $-1 < b < 1$. That is, it contracts the area of any region in each iteration by a constant factor of $|b|$.

Period-2 points

For finding period-2 points we have a relation

$$f^2(x, y) = (x, y) \Rightarrow a - \left(a - x^2 + by\right)^2 + bx = x \text{ and } a - x^2 + by = y.$$

Solving the second equation for y, $(1 - b)y = a - x^2$ and then substituting in the first equation, we get

$$\left(x^2 - a\right)^2 + (1 - b)^3 x - (1 - b)^2 a = 0.$$

By factorization we see that $\left(x^2 + (1 - b)x - a\right)$ must be a factor of the equation. On division, we get the other factor of the equation as $\left(x^2 - (1 - b)x - a + (1 - b)^2\right)$ and the period-2 points are given by the roots of the equation $x^2 - (1 - b)x - a + (1 - b)^2 = 0$, while $(1 - b)y = a - x^2$. The roots are given by

$$x_1, x_2 = \frac{1}{2}\left[(1 - b) \pm \sqrt{4a - 3(1 - b)^2}\right].$$

The period two points lie on $x + y = 1 - b$. It is evident that the Hénon map has a period-2 orbit if and only if $4a > 3(1 - b)^2$.

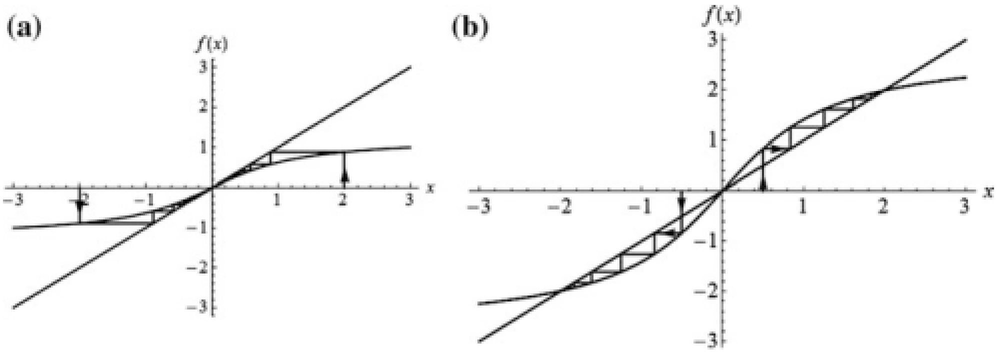


Figure 2.5: The graphical representations of $f_\lambda(x)$ for **a**$\lambda < 1$ and **b**$\lambda > 1$

EXERCISE 38

Discuss bifurcations and draw bifurcation diagrams for the one-dimensional map $f(x, \lambda) = \lambda \tan^{-1}(x); \lambda, x \in \mathbb{R}$

SOLUTION

consider the one-dimensional map $f(x, \lambda) = \lambda \tan^{-1}(x); \lambda, x \in \mathbb{R}$. For $\lambda < 1$, $f_\lambda(x) = \lambda \tan^{-1}(x) = x \Rightarrow x = 0$ is only fixed point since $x/\tan^{-1}(x) > 1$. Since $f'_\lambda(0) = \lambda < 1$, the fixed point origin is stable. For $\lambda > 1, f_\lambda(x) = \lambda \tan^{-1}(x)$ has three fixed points at $x = 0, x = x_+ > 0$ and $x = x_- < 0$. The map $f_\lambda(x)$ has three points of intersection. Now, $f_\lambda(x_+) = \lambda \tan^{-1}(x_+) = x_+ \Rightarrow \lambda = \frac{x_+}{\tan^{-1}(x_+)}$. This implies

$$f'_\lambda(x_+) = \frac{\lambda}{1 + (x_+)^2} = \frac{x_+}{1 + (x_+)^2} \frac{1}{\tan^{-1}(x_+)} < 1.$$

The fixed point at $x_+$is stable for $\lambda > 1$.

Similarly, the fixed point at $x = x_-$is also stable. But the fixed point $x = 0$ is unstable, since $f'_\lambda(0) = \lambda > 1$. Thus the map undergoes a bifurcation when the parameter λ crosses the value $\lambda = 1$, called bifurcating point. The graphical representations of $f_\lambda(x)$ for $\lambda < 1$ and $\lambda > 1$ are shown in Figure 2.5

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