People's Democratic Republic of Algeria

Ministry of Higher Education and scientific Research

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# Course

# DISCRETE DYNAMICAL SYSTEM .

# Master 1 (first year) fundamental and applied mathematics

# The first semester



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# CONTENTS

# CHAPTER 1

# BIFURCATION AND CHAOS IN TWO DIMENSIONS

## 1.1 Bifurcation

### 1.1.1 Eigenvalues of $1$ or $-1$

In this section we focus our attention on the bifurcation of two dimensional maps. The extension to higher dimensions should be apparent to the reader after comprehending the two dimensional case. Let us consider the one-parameter family of maps

$$F(\mu, u) : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R} \tag{1.1}$$

with $u = (x, y) \in \mathbb{R}^2$, $\mu \in \mathbb{R}$ and $F \in C^r$, $r \geq 5$. If $(\mu^*, u^*)$ is a fixed point, then we make a change of variables, so that our fixed point is $(0, 0)$. Let $J = D_u F(0, 0)$. Then using the center manifold theorem, we find a onedimensional map $f_\mu(u)$ defined on the center manifold $M_c$. We deduce the following statements:
1. Suppose that $J$ has an eigenvalue equal to 1 . Then we have

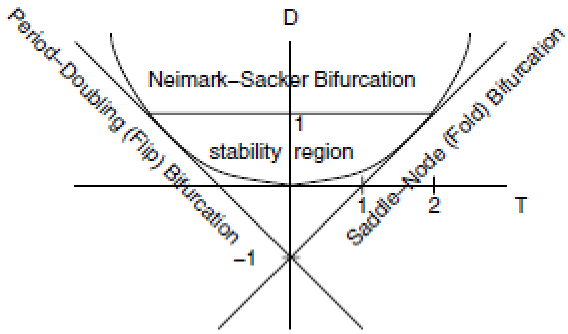


**FIGURE 5.3**
The occurrence of the three main types of bifurcation.

(a) a saddle-node bifurcation (fold bifurcation), if $\frac{\partial F}{\partial \mu}(0,0) \neq 0$ and $\frac{\partial^2 P}{\partial^2 u}(0,0) \neq 0$

(b) a pitchfork bifurcation, if $\frac{\partial f}{\partial \mu}(0,0) = 0$ and $\frac{\partial^2 f}{\partial^2 \mu}(0,0) = 0$

(c) a transcritical bifurcation, if $\frac{\partial f}{\partial \mu}(0,0) = 0$ and $\frac{\partial^2 f}{\partial^2 \mu}(0,0) \neq 0$.

2. If $J$ has an eigenvalue equal to -1 , then we have a period-doubling (flip) bifurcation.

3. If $J$ has a pair of complex conjugate eigenvalues of modulus 1 , a new type of bifurcation, called the Neimark-Sacker bifurcation appears. The Neimark-Sacker bifurcation will be discussed in more details in the sequel. Let $T = \operatorname{tr} J, D = \det J$. Then the following trace-determinant diagram (Fig. 5.3) illustrate the main bifurcation phenomena.

### 1.1.2 A Pair of Eigenvalues of Modulus 1 - The NeimarkSacker Bifurcation

We now turn our attention to the case when the Jacobian matrix $J = D_u F(u_0, \mu_0)$ has two complex conjugate eigenvalues. Let us start with an illustrative example.

**Example 1.1.1** *Consider the family of maps*

$$F_\mu \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (1 + \mu - x_1^2 - x_2^2) \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{1.2}$$

*where $\beta = \beta(\mu)$ is a smooth function of the parameter $\mu$ and $0 < \beta(0) < \pi$. Observe that the origin is a fixed point of the map $F_\mu$ for all $\mu$ with the Jacobian matrix*

$$J = (1 + \mu) \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}.$$

*The matrix $J$ has eigenvalues $\lambda_{1,2} = (1 + \mu)e^{\pm i\beta}$ with $|\lambda_{1,2}| = |1 + \mu|$. Hence, at $\mu = 0$, the eigenvalues cross the unit circle, a clear sign of the appearance of a Neimark-Sacker bifurcation. Clearly, the origin is asymptotically* *stable for* $-2 < \mu < 0$*. To analyze the bifurcation when* $\mu = 0$*, it is more convenient to write the map* $F_\mu$ *in polar coordinates* $(r, \theta)$*. To facilitate this change of coordinates, write Equation (4.2) as a two-dimensional system of difference equations:*

$$\begin{pmatrix} x_1(n+1) \\ x_2(n+1) \end{pmatrix} = \left(1 + \mu - x_1^2(n) - x_2^2(n)\right)$$

$$\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} x_1(n) \\ x_2(n) \end{pmatrix}. \tag{1.3}$$

*Now, putting* $x_1(n) = r(n)\cos\theta(n), x_2(n) = r(n)\sin\theta(n)$ *in Equation (4.3), we obtain*

$$r(n+1) = (1+\mu)r(n) - r^3(n)$$

$$\theta(n+1) = \theta(n) + \beta \tag{1.4}$$

The form of Equation (4.4) enables us to detect easily the presence of an invariant circle by letting $(1+\mu)r - r^3 = r$. Hence, the invariant circle is of radius $r^* = \sqrt{\mu}$. Thus, this invariant circle appears when $\mu$ crosses the value 0 as shown in Fig. 5.4. When $\mu = 0$, the map $r \longmapsto r - r^3$ is one-dimensional and its stability can be determined by using the techniques of Chapter 1.

Note also that the cobweb diagram indicates that the origin is (slowly) asymptotically stable (see Fig. 5.5).

Thus, for $\mu = 0$, the origin is asymptotically stable. When $\mu$ becomes positive, the origin loses its stability and gives rise to an attracting (asymptotically stable) circle with radius $r = \sqrt{\mu}$. The dynamics on this circle are determined by the map $\theta \longmapsto \theta + \beta$, which is a rotation by an angle $\beta$ in the counterclockwise direction. This phenomenon is called a Neimark-Sacker bifurcation (or a Hopf bifurcation).

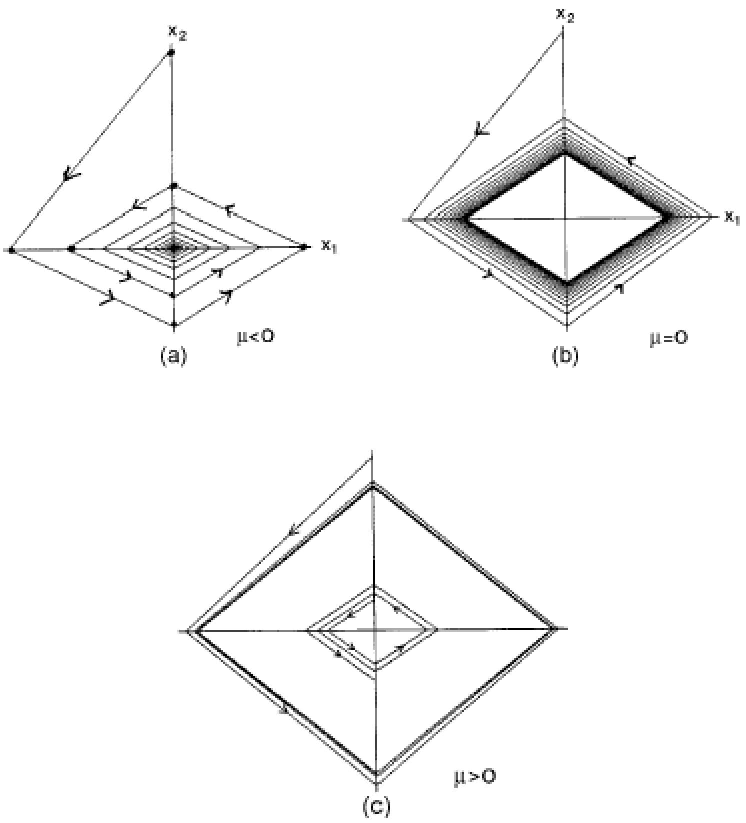


**FIGURE 5.4**
Supercritical Neimark-Sacker bifurcation of the map in (5.18), (a) $\mu < 0$, (b) $\mu = 0$, (c) $\mu > 0$.

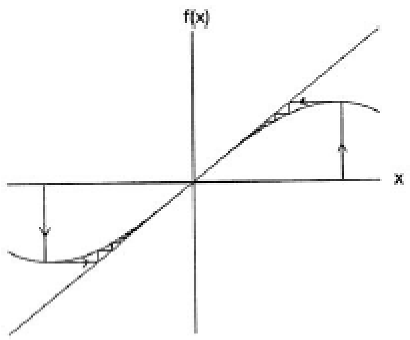


**FIGURE 5.5**
For $\mu = 0$, the Cobweb diagram for the one dimension map $r \to r - r^3$ shows that 0 is asymptotically stable.

**Remark 1.1.1** *An analogous but different scenario may occur if we consider the map*

$$\tilde{F}_\mu \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \left(1 + \mu + x_1^2 + x_2^2\right) \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{1.5}$$

*This map undergoes a Neimark-Sacker bifurcation at $\mu = 0$ but in a manner different from that of Equation (4.2) (see Fig. 5.6). The reader is asked to provide details in Problem 4.*

To this end, we have described in detail the occurrences of the NeimarkSacker bifurcation for the map in Example 4.1.1. In the sequel, we will show that the dynamics of this map are typical for a certain class of two-dimensional maps with one parameter. So, let us consider the family of $C^r$ maps ($r \geq 5$) $F_\mu : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ such that the following conditions hold:

1. $F_\mu(0) = 0$, i.e., the origin is a fixed point of $F_\mu$.

2. $\underline{D}F_\mu(0)$ has two complex conjugate eigenvalues $\lambda(\mu) = r(\mu)e^{i\theta(\mu)}$ and $\bar{\lambda}(\mu)$, where $r(0) = 1, r'(0) \neq 0, \theta(0) = \theta_0$. Thus, $|\lambda(0)| = 1$.

3. $e^{ik\theta_0} \neq 0$ for $k = 1, 2, 3, 4, 5$, i.e., $\lambda(0)$ is not a low root of unity.

Based on the above assumptions, we make the following claims.

1. By a change of basis in $R^2$, we may assume, without loss of generality, that

$$J = DF_\mu(0,0) = (1+\mu)\begin{pmatrix} \cos\beta(\mu) & -\sin\beta(\mu) \\ \sin\beta(\mu) & \cos\beta(\mu) \end{pmatrix}.$$

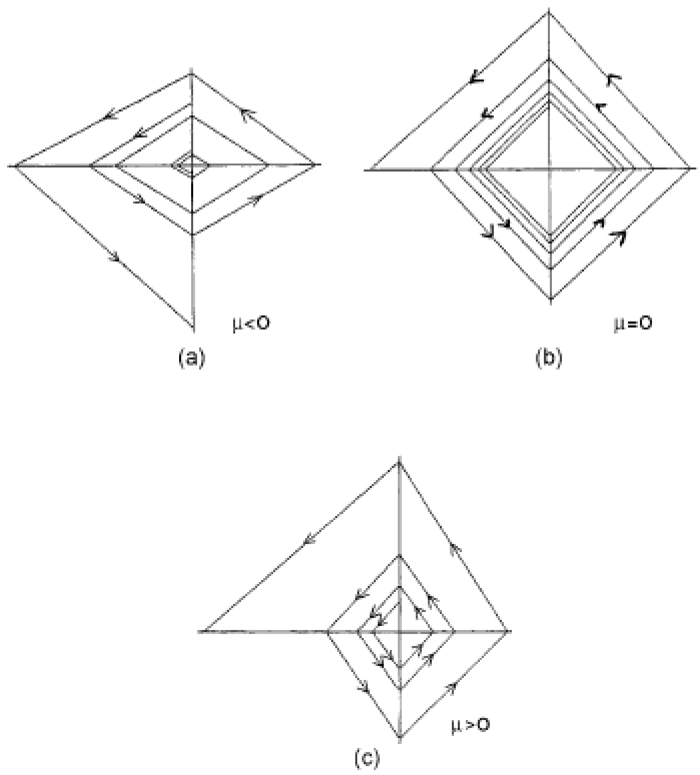


**FIGURE 5.6**

Supercritical Neimark-Sacker bifurcation of the map in (4.5), (a) $\mu < 0$, (b) $\mu = 0$, (c) $\mu > 0$.

2. From assumption 3 , by a change of coordinates, we may assume that our map $F_\mu$ takes the form

$$F_\mu\begin{pmatrix}x_1\\x_2\end{pmatrix} = N_\mu\begin{pmatrix}x_1\\x_2\end{pmatrix} + O\left(\left|\begin{pmatrix}x_1\\x_2\end{pmatrix}\right|^5\right) \tag{1.6}$$

where

$$N_\mu\begin{pmatrix}r\\\theta\end{pmatrix} = \begin{pmatrix}(1+\mu)r - f(\mu)r^3\\\theta + \beta(\mu) + g(\mu)r^2\end{pmatrix}, \tag{1.7}$$

with $f(0) \neq 0$. Notice that the radius of the invariant circle is given by $r = \sqrt{\mu/f(\mu)}$.

**Theorem 1.1.1** ***(Neimark-Sacker).***
*Suppose that $F_\mu$ satisfies assumptions 1-3. Then, for sufficiently small $\mu$, $F_\mu$ has an invariant closed curve enclosing the origin if $\mu/f(\mu) > 0$. Moreover, if $f(0) > 0$, this curve is attracting, and if $f(0) < 0$, it is repelling.*

## 1.2 The Hénon Map

**Example 1.2.1** ***(The Hénon Map).***
*Consider the Hénon map*

$$H_{ab}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1 - ax^2 + y\\bx\end{pmatrix}$$

*As we have seen in Chapter 4, the Hénon map has two fixed points if $a > -\frac{1}{4}(1-b)^2$. These fixed points are*

$$u_1^* = \left(\frac{1}{2a}\left(b - 1 + \sqrt{(1-b)^2 + 4a}\right), \frac{b}{2a}\left(b - 1 + \sqrt{(1-b)^2 + 4a}\right)\right)^T$$
$$u_2^* = \left(\frac{1}{2a}\left(b - 1 - \sqrt{(1-b)^2 + 4a}\right), \frac{b}{2a}\left(b - 1 - \sqrt{(1-b)^2 + 4a}\right)\right)^T$$

*Recall that for $|b| < 1$ and $a \in \left(-\frac{1}{4}(1-b)^2, \frac{3}{4}(1-b)^2\right) = I$, the fixed point $u_1^*$ is asymptotically stable while $u_2^*$ is a saddle. Now at the left end point $a_1 = -\frac{1}{4}(1-b)^2$ of the parameter interval $I$, we have $u_2^* = u_1^* = \left(\frac{-(1-b)}{2a}, \frac{-b(1-b)}{2a}\right)$. Moreover, the Jacobian matrix $J = D_uH\left(u_1^*, a_1\right) = \begin{pmatrix}1-b & 1\\ b & 0\end{pmatrix}$ has an eigenvalue equal to +1 . Hence, by the center manifold theorem, there is a saddle node bifurcation at $u_1^*$.*

*On the other hand, at the right end point $a_2 = \frac{3}{4}(1-b)^2$ of $I$, we have $u_1^* = \left(\frac{(1-b)}{2a}, \frac{b(1-b)}{2a}\right)^T$. Furthermore, the Jacobian matrix $J = D_uH\left(u_1^*, a_2\right) = \begin{pmatrix}-(1-b) & 1\\ b & 0\end{pmatrix}$ has an eigenvalue equal to -1. Again, by the center manifold theorem, we have a period-doubling bifurcation. For a fixed b, we may plot a bifurcation diagram showing the x components of an orbit and the parameter a on the x axis. Figure 5.16 shows the bifurcation diagram of the Hénon map for $a \in [0, 1.4]$, and $b = 0.3$. We see a 4-cycle going to a chaotic region and then when $a = 2$ we have two pieces of a chaotic attractor .*

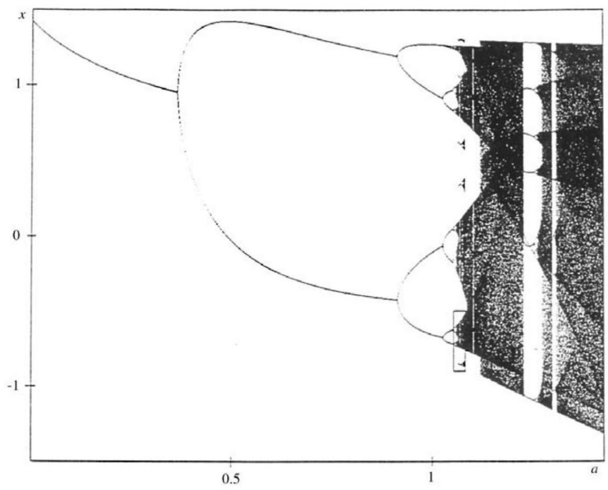


***FIGURE 5.16*** *The bifurcation diagram of the Henon map for a fixed $b = 0.3$.*

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