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DISCRETE DYNAMICAL SYSTEM .

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By: ROUIBAH KHAOULA

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CONTENTS

CHAPTER 1

STABILITY OF TWO-DIMENSIONAL MAPS

1.1 Linear Maps vs. Linear Systems

Recall from linear algebra that a map $L : \mathbb{R}^2 \to \mathbb{R}^2$ is called a linear transformation if

1. $L(U_1 + U_2) = L(U_1) + L(U_2)$ for $U_1, U_2 \in \mathbb{R}^2$

2. $L(\alpha U) = \alpha L(U)$ for $U \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$. Moreover, it is always possible to represent f (with a given basis for $\mathbb{R}^2$) by a matrix A. A typical example is

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

which may be written in the form

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

or

$$L(U) = AU, \tag{1.1}$$

where $U = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

By iterating L, we conclude that $L^n(U) = A^n U$. Hence, the orbit of U under f is given by

$$\left\{U, AU, A^2U, \ldots, A^nU, \ldots\right\} \tag{1.2}$$

Thus, to compute the orbit of U, it suffices to compute $A^n U$ for $n \in \mathbb{Z}^+$.
Another way of looking at the same problem is by considering the following two-dimensional system of difference equations

$$x(n+1) = ax(n) + by(n)y(n+1) = cx(n) + dy(n) \tag{1.3}$$

or

$$U(n+1) = AU(n). \tag{1.4}$$

By iteration, one may show that the solution of Equation (**??**) is given by

$$U(n) = A^n U(0). \tag{1.5}$$

So, if we let $U_0 = U(0)$, then $L^n(U_0) = U(n)$. The form of Equation (**??**) is more convenient when we are considering applications in biology, engineering, economics, and so forth. For example, $x(n)$ and $y(n)$ may represent the population sizes at time period n of two competitive cooperative species, or preys and predators.

In the next section, we will develop the necessary machinery to compute A^n for any matrix of order two.

1.2 Computing A^n

Consider a matrix $A = \left(a_{ij}\right)$ of order 2×2. Then, $p(\lambda) = \det(A - \lambda I)$ is called the characteristic polynomial of A and its zeros are called the eigenvalues of A. Associated with each eigenvalue λ of A a nonzero eigenvector $V \in \mathbb{R}^2$ with $AV = \lambda V$.

Example 1.2.1 *Find the eigenvalues and the eigenvectors of the matrix*

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$$

SOLUTION First we find the eigenvalues of A by solving the characteristic equation $\det(A - \lambda I) = 0$ or

$$\begin{vmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{vmatrix} = 0$$

which is

$$\lambda^2 - 6\lambda + 5 = 0.$$

Hence, $\lambda_1 = 1$ and $\lambda_2 = 5$. To find the corresponding eigenvector V_1, we solve the vector equation $AV_1 = \lambda V_1$ or $(A - \lambda_1 I)\, V_1 = 0$. For $\lambda_1 = 1$, we have

$$\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence, $v_{11} + 3v_{21} = 0$. Thus, $v_{11} = -3v_{21}$. So, if we let $v_{21} = 1$, then $v_{11} = -3$. It follows that the eigenvector V_1 corresponding to λ_1 is given by $V_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$.

For $\lambda_2 = 5$, the corresponding eigenvector may be found by solving the equation $(A - \lambda_2 I)\, V_2 = 0$. This yields

$$\begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus, $-3v_{12} + 3v_{22} = 0$ or $v_{12} = v_{22}$. It is then appropriate to let $v_{12} = v_{22} = 1$ and hence $V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

To find the general form for A^n for a general matrix A is a formidable task even for a 2×2 matrix such as in Example 3.2.1. Fortunately, however, we may be able to transform a matrix A to another simpler matrix B whose nth power B^n can easily be computed. The essence of this process is captured in the following definition.

Definition 1.2.1 *The matrices A and B are said to be similar if there exists a nonsingular matrix P such that*

$$P^{-1}AP = B.$$

We note here that the relation "similarity" between matrices is an equivalence relation, i.e.,

1. A is similar to A.

2. If A is similar to B then B is similar to A.

3. If A is similar to B and B is similar to C, then A is similar to C.

The most important feature of similar matrices, however, is that they possess the same eigenvalues. that det $P \neq 0$, *where det denotes determinant.*

Theorem 1.2.1 *Let A and B be two similar matrices. Then A and B have the same eigenvalues.*

Proof. Suppose that $P^{-1}AP = B$ or $A = PBP^{-1}$. Let λ be an eigenvalue of A and V be the corresponding eigenvector. Then, $\lambda V = AV = PBP^{-1}V$. Hence, $B\left(P^{-1}V\right) = \lambda\left(P^{-1}V\right)$. Consequently, λ is an eigenvalue of B with $P^{-1}V$ as the corresponding eigenvector. ■

The notion of similarity between matrices corresponds to linear conjugacy. In other words, two linear maps are conjugate if their corresponding matrix representations are similar. Thus, the linear maps L_1, L_2

on $\mathbb{R}^2$ are linearly conjugate if there exists an invertible map h such that

$$L_1 \circ h = h \circ L_2$$

or

$$h^{-1} \circ L_1 \circ h = L_2.$$

The next theorem tells us that there are three simple "canonical" forms for 2×2 matrices.

Theorem 1.2.2 *Let A be a 2×2 real matrix. Then A is similar to one of the following matrices:*

1. $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$
2. $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$
3. $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$

Proof. Suppose that the eigenvalues λ_1 and λ_2 are real. Then, we have two cases to consider. The first case is where $\lambda_1 \neq \lambda_2$. In this case, we may easily show that the corresponding eigenvectors V_1 and V_2 are linearly independent (Problem 10). Hence, the matrix $P = (V_1, V_2)$, i.e., the matrix P whose columns are these eigenvectors, is nonsingular. Let $P^{-1}AP = J = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$. Then,

$$AP = PJ. \tag{1.6}$$

Comparing both sides of Equation (**??**), we obtain

$$AV_1 = eV_1 + gV_2.$$

Hence,

$$\lambda_1 V_1 = eV_1 + gV_2.$$

Thus, $e = \lambda_1$ and $g = 0$.
Similarly, one may show that $f = 0$ and $h = \lambda_2$. Consequently, J is a diagonal matrix of the form (a).

The second case is where $\lambda_1 = \lambda_2 = \lambda$. There are two subcases to consider here. The first subcase occurs if we are able to find two linearly independent eigenvectors V_1 and V_2 corresponding to the eigenvalue λ. This subcase is then reduced to the preceding case. We note here that this scenario happens when $(A - \lambda I)V = 0$ for all $V \in \mathbb{R}^2$. In particular, one may let $V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, which are clearly linearly independent.

The second subcase occurs when there exists a nonzero vector $V_2 \in \mathbb{R}^2$ such that $(A - \lambda I)V_2 \neq 0$. Equivalently, we are able to find only one eigenvector (not counting multiples) V_1 with $(A - \lambda I)V_1 = 0$. In practice, we find V_2 by solving the equation

$$(A - \lambda I)V_2 = V_1.$$

The vector V_2 is called a generalized eigenvector of A. Note that $AV_1 = \lambda V_1$ and $AV_2 = \lambda V_2 + V_1$. Now, we let $P = (V_1, V_2)$ and $P^{-1}AP = J$. Then,

$$AP = PJ. \tag{1.7}$$

Comparing both sides of Equation (**??**) yields

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}. \tag{1.8}$$

The matrix J is in a Jordan form. Next, we assume that A has a complex eigenvalue $\lambda_1 = \alpha + i\beta$. Since A is assumed to be real, it follows that the second eigenvalue λ_2 is a conjugate of λ_1, that is, $\lambda_2 = \alpha - i\beta$. Let $V = V_1 + iV_2$ be the eigenvector corresponding to λ_1. Then,

$$\begin{aligned} AV &= \lambda_1 V \\ A(V_1 + iV_2) &= (\alpha + i\beta)(V_1 + iV_2). \end{aligned}$$

Hence,

$$\begin{aligned} AV_1 &= \alpha V_1 - \beta V_2 \\ AV_2 &= \beta V_1 + \alpha V_2, \end{aligned}$$

letting $P = (V_1, V_2)$ we get $P^{-1}AP = J$. Comparison of both sides of Equation (**??**) yields

$$J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}. \tag{1.9}$$

Theorem 3.2.2 gives us a simple method of computing the general form of A^n for any 2×2 real matrix. In the first case, when $P^{-1}AP = D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, we have

$$\begin{aligned} A^n &= \left(PDP^{-1}\right)^n \\ &= PD^nP^{-1} \\ &= P\begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}P^{-1}. \end{aligned} \tag{1.10}$$

In the second case, when $P^{-1}AP = J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, then

$$\begin{aligned} A^n &= PJ^nP^{-1} \\ &= P\begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}P^{-1}. \end{aligned} \tag{1.11}$$

Equation (**??**) may be easily proved by mathematical induction. In the third case, we have $P^{-1}AP = J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$. Let $\omega = \arctan(\beta/\alpha)$. Then $\cos\omega = \alpha/|\lambda_1|, \sin\omega = \beta/|\lambda_1|$. Now, we write the matrix J in the form

$$J = |\lambda_1|\begin{pmatrix} \alpha/|\lambda_1| & \beta/|\lambda_1| \\ -\beta/|\lambda_1| & \alpha/|\lambda_1| \end{pmatrix} = |\lambda_1|\begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix}.$$

By mathematical induction one may show that

$$J^n = |\lambda_1|^n\begin{pmatrix} \cos n\omega & \sin n\omega \\ -\sin n\omega & \cos n\omega \end{pmatrix}. \tag{1.12}$$

and thus

$$A^n = |\lambda_1|^n P\begin{pmatrix} \cos n\omega & \sin n\omega \\ -\sin n\omega & \cos n\omega \end{pmatrix}P^{-1} \tag{1.13}$$

■

Example 1.2.2 *Solve the system of difference equations*

$$X(n+1) = AX(n) \tag{1.14}$$

where

$$A = \begin{pmatrix} -4 & 9 \\ -4 & 8 \end{pmatrix}, X(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

SOLUTION The eigenvalues of A are repeated: $\lambda_1 = \lambda_2 = 2$. The only eigenvector that we are able to find is $V_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$. To construct P we need to find a generalized eigenvector V_2. This is accomplished by solving the equation $(A - 2I)V_2 = V_1$. Then, V_2 may be taken as any vector $\begin{pmatrix} x \\ y \end{pmatrix}$, with $3y - 2x = 1$. We take $V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Now if we put $P = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$, then $P^{-1}AP = J = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. Thus, the solution of Equation (**??**) is given by

$$\begin{aligned} X(n) &= PJ^nP^{-1}x(0) \\ &= \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2^n & n2^{n-1} \\ 0 & 2^n \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= 2^n\begin{pmatrix} 1-3n \\ -2n \end{pmatrix}. \end{aligned}$$

Remark 1.2.1 *If a map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $f(X_0) = AX_0$, then $f^n(X_0) = A^nX_0 = PJ^nP^{-1}X_0$. In particular, if $X_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then $f^n(X_0) = 2^n\begin{pmatrix} 1-3n \\ -2 \end{pmatrix}$ for all $n \in \mathbb{Z}^+$.*

Exercises

In Problems 1-5, find the eigenvalues and eigenvectors of the matrix A and compute A^n. 1. $A = \begin{pmatrix} -4.5 & 5 \\ -7.5 & 8 \end{pmatrix}$

2. $A = \begin{pmatrix} 4.5 & -1 \\ 2.25 & 1.5 \end{pmatrix}$

3. $A = \begin{pmatrix} 8/3 & 1/3 \\ -4/3 & 4/3 \end{pmatrix}$

4. $A = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}$

5. $A = \begin{pmatrix} -2 & -3 \\ 1 & 1 \end{pmatrix}$

6. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $L(X) = AX$, where A is as in Problem 1 . Find $L^n\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

7. Solve the difference equation $X(n+1) = AX(n)$, where A is as in Problem 3, and $X(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

8. Solve the difference equation $X(n+1) = AX(n)$, where A is as in Problem 4, and $X(0) = X_0$.

9. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(X) = AX$, with A as in Problem 5 . Find $f^n\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

10. Let A be a 2×2 matrix with distinct real eigenvalues. Show that the corresponding eigenvectors of A are linearly independent.

11. (a) If $J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, show that $J^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$.

(b) If $J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$, show that $J^n = |\lambda|^n \begin{pmatrix} \cos n\omega \sin n\omega \\ -\sin n\omega \cos n\omega \end{pmatrix}$, where $|\lambda| = \sqrt{\alpha^2+\beta^2}, \omega = \arctan\left(\frac{\beta}{\alpha}\right)$.

12. Let a matrix A be in the form

$$A = \begin{pmatrix} 0 & 1 \\ -p_2 & -p_1 \end{pmatrix}.$$

(a) Show that if A has distinct eigenvalues λ_1 and λ_2, then

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

where $P = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}$.

(b) Show that if A has a repeated eigenvalue λ, then

$$P^{-1}AP = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

where $P = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$.

(c) Show that if A has complex eigenvalues $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$, then

$$P^{-1}AP = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix},$$

where $P = \begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix}$.

1.3 Fundamental Set of Solutions

Consider the linear system

$$X(n+1) = AX(n), \tag{1.15}$$

where A is a 2×2 matrix. Then, two solutions $X_1(n)$ and $X_2(n)$ of Equation (**??**) are said to be linearly independent if $X_2(n)$ is not a scaler multiple of $X_1(n)$ for all $n \in \mathbb{Z}^+$. In other words, if $c_1X_1(n)+c_2X_2(n) = 0$ for all $n \in \mathbb{Z}^+$, then $c_1 = c_2 = 0$. A set of two linearly independent solutions $\{X_1(n), X_2(n)\}$ is called a fundamental set of solutions of Equation (**??**).

Definition 1.3.1 *Let $\{X_1(n), X_2(n)\}$ be a fundamental set of solutions of Equation (**??**). Then*

$$X(n) = k_1 X_1(n) + k_2 X_2(n), k_1, k_2 \in \mathbb{R} \tag{1.16}$$

*is called a general solution of Equation (**??**). Finding $X_1(n)$ and $X_2(n)$ is generally an easy task. We now give an explicit derivation.*

In the sequel λ_1, λ_2 denote the eigenvalues of A; V_1, V_2 are the corresponding eigenvectors of A.

We have three cases to consider.

Case (i)

Suppose that $P^{-1}AP = J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. Then a general solution may be given by

$$\begin{aligned} X(n) &= A^n X(0) = PJ^n P^{-1} X(0) \\ &= (V_1, V_2) \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \end{aligned}$$

where $\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = P^{-1}X(0)$. Then,

$$X(n) = k_1 \lambda_1^n V_1 + k_2 \lambda_2^n V_2. \tag{1.17}$$

Here, $X_1(n) = \lambda_1^n V_1$ and $X_2(n) = \lambda_2^n V_2$ constitute a fundamental set of solutions since in this case V_1 and V_2 are linearly independent eigenvectors. Note that one may check directly that $\lambda_1^n V_1$ and $\lambda_2^n V_2$ are indeed solutions of Equation (**??**).

Case (ii)

Suppose that $P^{-1}AP = J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$. Then, a general solution may be given by

$$\begin{aligned} X(n) &= PJ^n P^{-1} X(0) \\ &= (V_1, V_2) \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \\ &= k_1 \lambda^n V_1 + k_2 \left(n\lambda^{n-1} V_1 + \lambda^n V_2 \right) \end{aligned} \tag{1.18}$$

Hence, $X_1(n) = \lambda^n V_1$ and $X_2(n) = \lambda^n V_2 + n\lambda^{n-1} V_1$ constitute a fundamental set of solutions of Equation (**??**). **Case (iii)**

Suppose that $P^{-1}AP = J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$. If $\omega = \arctan(\beta/\alpha)$, then the general solution may be given by

$$\begin{aligned} X(n) =& PJ^nP^{-1}X(0) \\ =& (V_1V_2)\,|\lambda_1|^n \begin{pmatrix} \cos n\omega & \sin n\omega \\ -\sin n\omega & \cos n\omega \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \\ =& |\lambda_1|^n\,[k_1 \cos n\omega + k_2 \sin n\omega)\,V_1 \\ &+ (-k_1 \sin n\omega + k_2 \cos n\omega)\,V_2]\,. \end{aligned} \tag{1.19}$$

Hence, $X_1(n) = |\lambda_1|^n\,[(k_1 \cos n\omega)\,V_1 - (k_1 \sin(n\omega))\,V_2]$ and $X_2(n) = (|\lambda_1|^n\,[(k_2 \sin(n\omega))\,V_1 + (k_2 \cos(n\omega)]\,V_2$ constitute a fundamental set of solutions.

Example 1.3.1 *Solve the system of difference equations*

$$X(n+1) = AX(n), X(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

where

$$A = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix}$$

SOLUTION The eigenvalues of A are $\lambda_1 = -2 + 3i$ and $\lambda_2 = -2 - 3i$. The corresponding eigenvectors are $V = \begin{pmatrix} -1 \\ i \end{pmatrix}$ and $\bar{V} = \begin{pmatrix} -1 \\ -i \end{pmatrix}$, respectively.

This time, we take a short cut and use Equation (**??**). The vectors V_1 and V_2 referred to in this formula are the real part of $V, V_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, and the imaginary part of $V, V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Now, $|\lambda_1| = \sqrt{13}, \omega = \arctan\left(\frac{-3}{2}\right) \approx 123.69°$. Thus,

$$\begin{aligned} X(n) =& (13)^{n/2}\left[(k_1 \cos n\omega + k_2 \sin n\omega)\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right. \\ &\left. + (-k_1 \sin n\omega + k_2 \cos n\omega)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] \\ X(0) =& \begin{pmatrix} 1 \\ 2 \end{pmatrix} = k_1\begin{pmatrix} -1 \\ 0 \end{pmatrix} + k_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned}$$

Hence, $k_1 = 1, k_2 = 2$. Thus,

$$\begin{aligned} X(n) &= (13)^{n/2}\left[(\cos n\omega + 2\sin n\omega)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + (-\sin n\omega + 2\cos n\omega)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] \\ &= (13)^{n/2}\begin{pmatrix} -\cos n\omega - 2\sin n\omega \\ -\sin n\omega + 2\cos n\omega \end{pmatrix}. \end{aligned}$$

1.4 Second-Order Difference Equations

A second-order difference equation with constant coefficients is a scalar equation of the form

$$u(n+2) + p_1 u(n+1) + p_2 u(n) = 0 \tag{1.20}$$

Although one may solve this equation directly, it is sometimes beneficial to convert it to a two-dimensional system. The trick is to let $u(n) = x_1(n)$ and $u(n+1) = x_2(n)$.

Then we have

$$x_1(n+1) = x_2(n)$$

$$x_2(n+1) = -p_2 x_1(n) - p_1 x_2(n)$$

which is of the form

$$X(n+1) = AX(n) \tag{1.21}$$

where

$$X(n) = \begin{pmatrix} x_1(n) \\ x_2(n) \end{pmatrix}, \text{ and } A = \begin{pmatrix} 0 & 1 \\ -p_2 & -p_1 \end{pmatrix}.$$

The characteristic equation of A is given by

$$\lambda^2 + p_1\lambda + p_2 = 0. \tag{1.22}$$

Observe that we may obtain the characteristic Equation (**??**) by letting $u(n) = \lambda^n$ in Equation (**??**). Thus, if λ_1 and λ_2 are the roots of Equation (**??**), then $u_1(n) = \lambda_1^n$ and $u_2(n) = \lambda_2^n$ are solutions of Equation (**??**).

Using Eqs. (**??**), (**??**), and (**??**), we can make the following conclusions:

1. If $\lambda_1 \neq \lambda_2$ and both are real, then the general solution of Equation (**??**) is given by

$$u(n) = c_1\lambda_1^n + c_2\lambda_2^n, \tag{1.23}$$

2. If $\lambda_1 = \lambda_2 = \lambda$, then the general solution of Equation (**??**) is given by

$$u(n) = c_1\lambda^n + c_2 n\lambda^n, \tag{1.24}$$

3. If $\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta$, then the general solution of Equation (**??**) is given by

$$u(n) = |\lambda_1|^n \left(c_1 \cos n\omega + c_2 \sin n\omega\right), \tag{1.25}$$

where $\omega = \arctan(\beta/\alpha)$.

Example 1.4.1 *Solve the second-order difference equation*

$$x(n+2) + 6x(n+1) + 9x(n) = 0, x(0) = 1, x(1) = 0.$$

SOLUTION The characteristic equation associated with the equation is given by $\lambda^2 + 6\lambda + 9 = 0$.

Hence, the characteristic roots are $\lambda_1 = \lambda_2 = -3$. The general solution is given by

$$\begin{aligned} x(n) &= 9(-3)^n + c_2 n(-3)^n \\ x(0) &= 1 = c_1 \\ x(1) &= 0 = -3c_1 - 3c_2. \end{aligned}$$

Thus, $c_2 = -1$ and, consequently,

$$\begin{aligned} x(n) &= (-3)^n - n(-3)^n \\ &= (-3)^n(1-n) \end{aligned}$$

1.5 Phase Space Diagrams

One of the best graphical methods to illustrate the various notions of stability is the phase portrait or the phase space diagram. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a given map. Then, starting from an initial point $X_0 = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$, we plot the sequence of point $X_0, f(X_0), f^2(X_0), f^3(X_0), \ldots$ and then connect the points by straight lines. An arrow is placed on these connecting lines to indicate the direction of the motion on the orbit. In many instances, we need to be prudent in choosing our initial points in order to get a better phase portrait. In this section, we consider linear systems for which $f(X) = AX$, where A is a 2×2 matrix. Observe that if $A - I$ is nonsingular, i.e., $\det(A - I) \neq 0$, then the origin $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the only fixed point of the map f. Equivently, $X^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the only fixed point of the system

$$X(n+1) = AX(n) \tag{1.26}$$

As stipulated in Theorem 3.2.2, there exists a nonsingular matrix P such that $P^{-1}AP = J$ where J is one of the forms (1), (2), or (3) in Theorem 3.2.2. If we let

$$X(n) = PY(n) \tag{1.27}$$

in Equation (**??**), we obtain

$$Y(n+1) = JY(n). \tag{1.28}$$

Our plan here is to draw the phase portrait of Equation (**??**), then use the transformation (**??**) to obtain the phase portrait of the original system (**??**).

(I). We begin our discussion by assuming that J is in the diagonal form $J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, where λ_1 and λ_2 are not necessarily distinct. Here we have two linearly independent solutions:

$$Y_1(n) = \lambda_1^n V_1, \text{ and } Y_2(n) = \lambda_2^n V_2, \text{ where } V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

are the eigenvectors of A corresponding to the eigenvalues λ_1 and λ_2, respectively.

Observe that $Y_1(n)$ is a multiple of V_1, and thus must stay on the line emanating from the origin in the direction of V_1; in this case, the x axis. Similarly, $Y_2(n)$ must stay on the line passing through the origin and in the direction of V_2; in this case, the y axis. These two solutions are called straight line solutions. The general solution is given by

$$Y(n) = k_1\lambda_1^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2\lambda_2^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}, Y(0) = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}. \tag{1.29}$$

We have the following cases to consider:

1. If $|\lambda_1| < 1$ and $|\lambda_2| < 1$, then all solutions tend to the origin as $n \to \infty$. Observe that if $|\lambda_1| < |\lambda_2| < 1$, then, $\left|\lambda_1^n\right|$ goes to zero faster than $\left|\lambda_2^n\right|$.

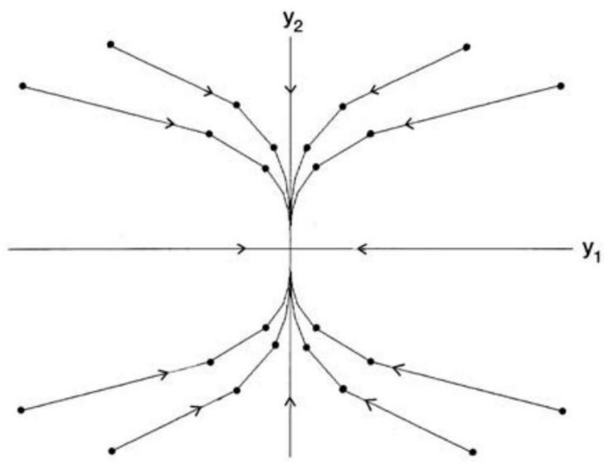


FIGURE 4.1a

(a) A sink: $0 < \lambda_1 < \lambda_2$.

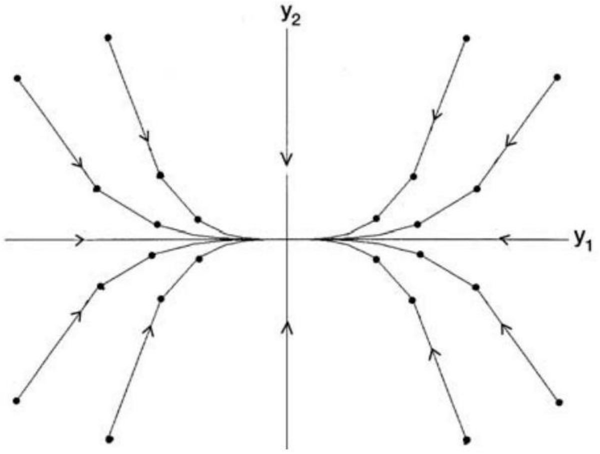


FIGURE 4.1b
(b) A sink: $0 < \lambda_2 < \lambda_1 < 1$.

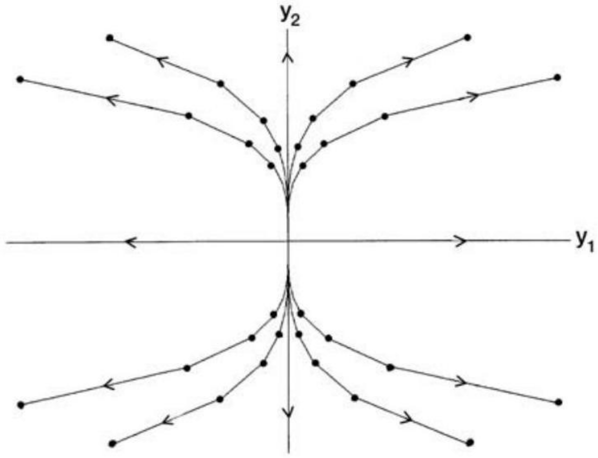


FIGURE 4.2a (a) A source: $\lambda_1 > \lambda_2 > 1$.

And consequently, any solution $Y(n)$ in the form (**??**) is asymptotic to the straight line solution $Y_2(n) = \lambda_2^n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (see Fig. 4.1a).

On the other hand, if $|\lambda_1| > |\lambda_2|$, then $Y(n)$ is asymptotic to $Y_1(n) = \lambda_1^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (see Fig. 4.1b).

Phase portraits 4.1a and 4.1 b are called sinks.

2. If $|\lambda_1| > 1$, and $|\lambda_2| > 1$, then we obtain an source as illustrated in Figs. 4.2a and 4.2b.

Note that if $|\lambda_1| > |\lambda_2| > 1$, then $Y(n)$ is asymptotic to $Y_2(n) = \lambda_2^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (the y axis) when $n \to -\infty$ and is dominated by $Y_1(n) = \lambda_1^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ when $n \to \infty$.

3. If $|\lambda_1| < 1$ and $|\lambda_2| > 1$, then we obtain a saddle (Fig. 4.3). In this case, when $n \to \infty Y(n)$, is asymptotic to $Y_2(n) = \lambda_2^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as $n \to \infty$ and is asymptotic to $Y_1(n) = \lambda_1^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as $n \to -\infty$. Similar analysis is readily available for the case $|\lambda_1| > 1$ and $|\lambda_2| < 1$.

4. If $\lambda_1 = \lambda_2$, then

$$Y(n) = k_1 \lambda^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 \lambda^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \lambda^n \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

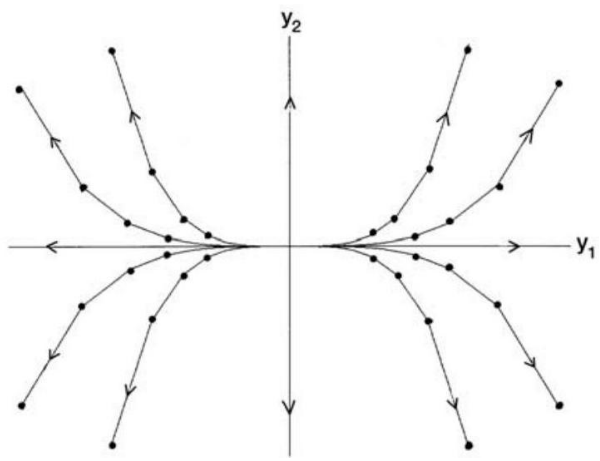


FIGURE 4.2b (b) A source: $\lambda_2 > \lambda_1 > 1$.

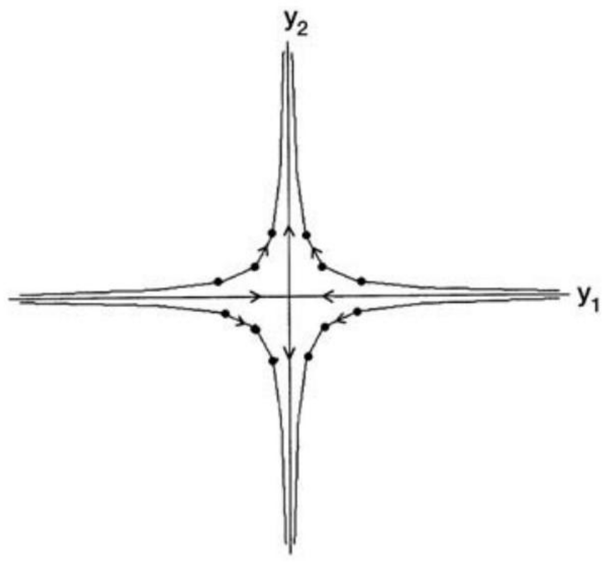


FIGURE 4.3 A saddle: $0 < \lambda_1 < 1, \lambda_2 > 1$.

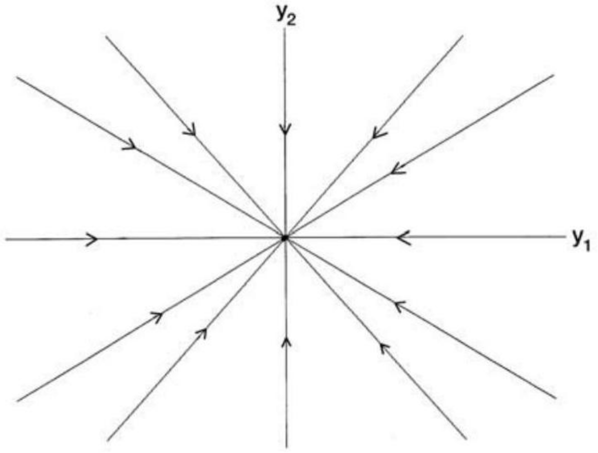


FIGURE 4.4 A sink: $0 < \lambda_1 < \lambda_2 < 1$.

Hence, every solution $Y(n)$ lies on a line passing through the origin with a slope k_2/k_1 (see Fig. 4.4).

Observe that in each of the four subcases, the presence of a negative eigenvalue will cause the solution $Y(n)$ to oscillate around the origin and the phase portrait will not look as nice as in Figs. 4.1a-4.4.

(II). Suppose that J is in the form

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

Then, we only have one straight line solution, $Y_1(n) = \lambda^n V_1 = \lambda^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The general solution is given by

$$\begin{aligned} Y(n) &= k_1 \lambda^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 \left(n\lambda^{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \\ &= (k_1 \lambda + k_2 n)\, \lambda^{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 \lambda^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

Now, if $|\lambda| < 1$, then, $Y(n) \to 0$ as $n \to \infty$, since $\lim_{n\to\infty} n\lambda^{n-1} = 0$ (by L'Hopital Rule). Since the term $k_1 \lambda^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ tends to the origin, as $n \to \infty$, faster than the term $(k_1 \lambda + k_2 n)\, \lambda^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, our solution $Y(n)$ tends to the origin asymptotic to the x axis (see Fig. 4.5a). In this case, the origin is called a

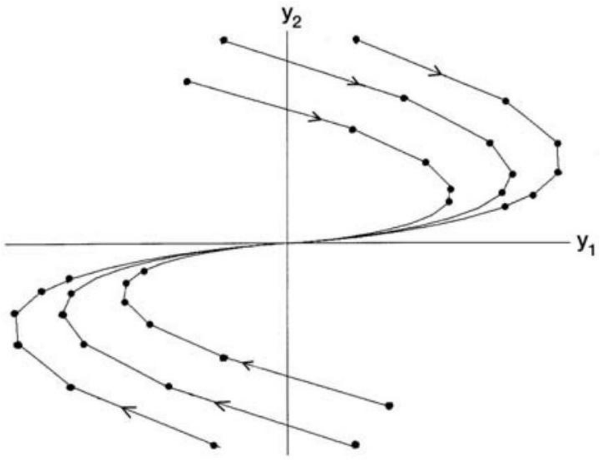


FIGURE 4.5a

(a) A degenerate sink: $\lambda_1 = \lambda_2 = \lambda, \quad 0 < \lambda < 1$.

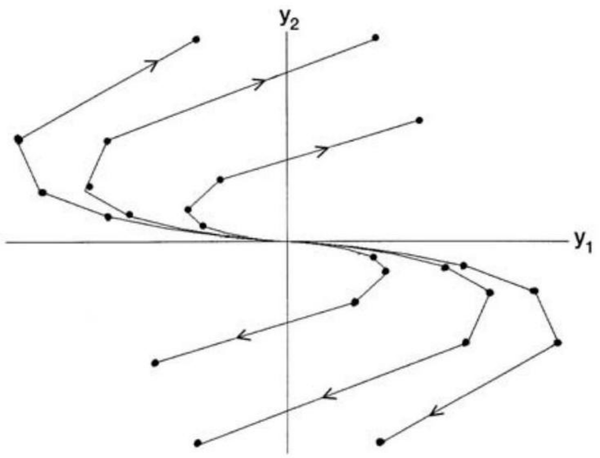


FIGURE 4.5b

(b) A degenerate source: $\lambda_1 = \lambda_2 = \lambda, \lambda > 1$.

degenerate sink. Figure 4.5 b depicts the case when $|\lambda| > 1$ and in this case, the origin is called a degenerate source.

(III). Suppose that J is in the form

$$J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

In this case, we have no straight line solutions due to the presence of $\cos n\beta$ and $\sin n\beta$ in the solutions $Y_1(n) = |\lambda_1|^n (k_1 \cos n\beta + k_2 \sin n\beta) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $Y_2(n) = |\lambda_1|^n (-k_1 \sin n\beta + k_2 \cos n\beta) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The general solution is given by

$$Y(n) = |\lambda_1|^n \begin{pmatrix} k_1 \cos n\beta + k_2 \sin n\beta \\ -k_1 \sin n\beta + k_2 \cos n\beta \end{pmatrix}$$

with $Y(0) = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$. Define an angle γ by setting $\cos\gamma = k_1/r_0$, and $\sin\gamma = k_2/r_0$, where $r_0 = \sqrt{k_1^2 + k_2^2}$. Then

$$y_1(n) = |\lambda_1|^n \, r_0 \cos(n\omega - \gamma)$$
$$y_2(n) = -\, |\lambda_1|^n \, r_0 \sin(n\omega - \gamma).$$

Thus, the solution in polar coordinates is given by

$$\begin{aligned} r(n) &= \sqrt{y_1^2(n) + y_2^2(n)} \\ &= r_0 \, |\lambda_1|^n \end{aligned} \tag{1.30}$$

$$\begin{aligned} \theta(n) &= \arctan\left(\frac{y_2(n)}{y_1(n)}\right) \\ &= -(nw - \gamma) \end{aligned} \tag{1.31}$$

It follows from Eqs. (**??**) and (**??**) that

1. If $|\lambda_1| < 1$, then we have a stable focus where each orbit spirals toward the origin [Fig. 4.6(a)]. On the other hand, if $|\lambda_1| > 1$, then we have an unstable focus, where each orbit spirals away from the origin [Fig. 4.6(b)].
2. If $|\lambda_1| = 1$, then we have a center, where the orbits follow a circular path [Fig. 4.6(c)]. This is due to the fact that $y_1^2(n) + y_2^2(n) = r_0^2$.

To this end, we have obtained the phase portraits of Equation (**??**), which may be called "canonical" phase portraits. To obtain the phase portraits of the original system (**??**), we apply (**??**), i.e., we apply P to the orbits of Equation (**??**). Since $P\binom{1}{0} = (V_1, V_2)\binom{0}{1} = V_1$, and $P\binom{0}{1} =$ $(V_1, V_2)\binom{0}{1} = V_2$, applying P to the orbits $Y(n)$ amounts to rotating the coordinates; the x axis to V_1 and the y axis to V_2. In other words, the straight line solutions are now along the eigenvectors V_1 and V_2. Using this observation, one may opt to sketch the phase portrait of Equation (**??**) directly and without going through the canonical forms.

The set of points on the line emanating from the origin along V_1 is called the stable subspace W^2; the set of points on the line passing through the origin in the direction of V_2 is called the unstable subspace W^4. Hence,

$$W^* = \left\{X \in \mathbb{R}^2 : A^n X \to 0 \text{ as } n \to \infty\right\},$$
$$W^n = \left\{X \in \mathbb{R}^2 : A^{-n} X \to 0 \text{ as } n \to \infty\right\}$$

The following example illustrates the above-described direct method to sketch the phase portrait.

Example 1.5.1 *Sketch the phase portrait of the system $X(n+1) = AX(n)$, where*

$$A = \begin{pmatrix} 1 & 1 \\ 0.25 & 1 \end{pmatrix} \quad \square$$

SOLUTION The eigenvalues of A are $\lambda_1 = \frac{3}{2}$, and $\lambda_2 = \frac{1}{2}$; the corresponding eigenvectors are $V_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $V_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, respectively. Hence, we have two straight line solutions, $X_1(n) = (1.5)^n \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $X_2(n) = (0.5)^n \begin{pmatrix} 2 \\ -1 \end{pmatrix}$. The general solution is given by $X(n) = k_1(1.5)^n \begin{pmatrix} 2 \\ 1 \end{pmatrix} + k_2(0.5)^n \begin{pmatrix} 2 \\ -1 \end{pmatrix}$. Note that $x(n)$ is asymptotic to the line through $V_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ (see Fig. 4.7).

1.6 Stability Notions

Consider the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ and let $X^* = \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix}$ be a fixed point of f; i.e., $f(X^*) = X^*$.

Our main objective in this section is to introduce the main stability notions pertaining to the fixed point x^*. Observe that these notions were previously

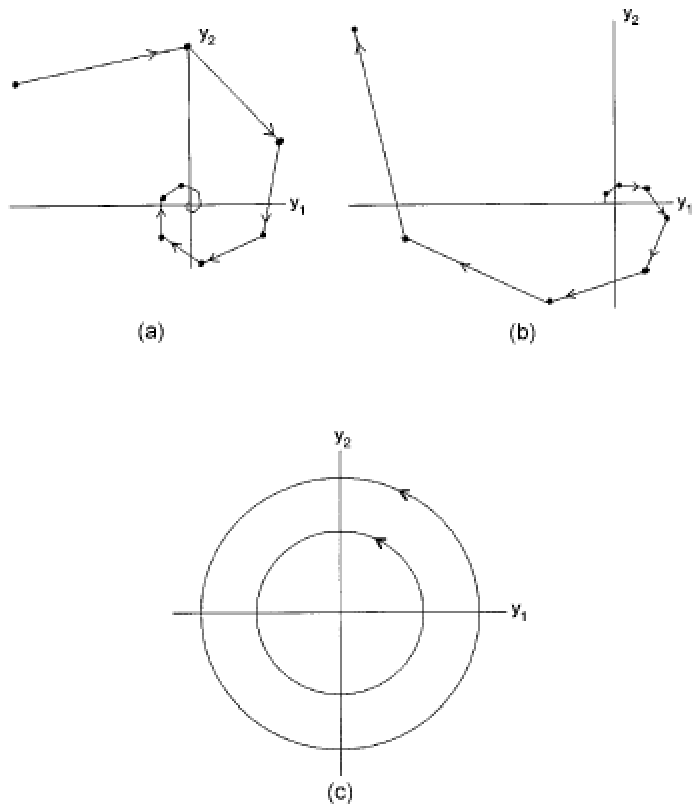


FIGURE 4.6
(a) Stable focus: $|\lambda_1| < 1$. (b) Source: $|\lambda_1| > 1$. (c) Center: $\lambda_{1,2} = \alpha \pm i\beta$, $|\lambda_{1,2}| = 1$.

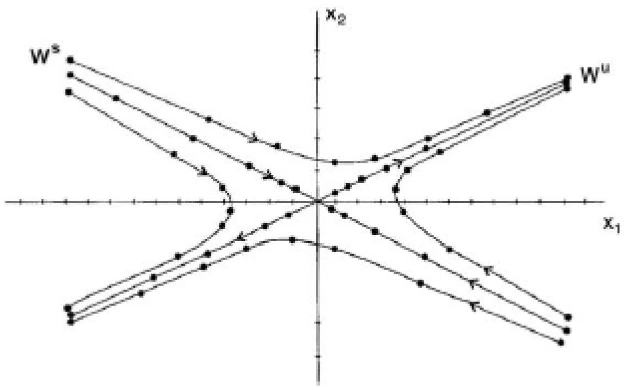


FIGURE 4.7
Saddle: $\lambda_1 > 1$, $0 < \lambda_2 < 1$. Stable and unstable subspaces: W^s, W^u.

introduced in Chapter 1. The only difference in $\mathbb{R}^2$ is that we replace the absolute value by a convenient "norm" on $\mathbb{R}^2$. Roughly speaking, a norm of a vector (point) in $\mathbb{R}^2$ is a measure of its magnitude. A formal definition follows.

Definition 1.6.1 *A real valued function on a vector space V is said to be a norm on V, denoted by $\|\;$, if the following properties hold:*

1. $|X| \geq 0$ *and* $|X| = 0$ *if and only if* $X = 0$, *for* $X \in V$.
2. $|\alpha X| = |\alpha||X|$ *for* $X \in V$ *and any scalar* α.
3. $|X + Y| \leq |X| + |Y|$ *for* $X, Y \in V$ *(the triangle inequality).*

In the sequel, we choose the ℓ_1 norm on $\mathbb{R}^2$ defined for $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ *as*

$$|X| = |x_1| + |x_2| \tag{1.32}$$

For each vector norm on $\mathbb{R}^2$ there corresponds a norm $|\;|$ on all 2×2 matrices $A = \left(a_{ij}\right)$ defined as follows

$$\|A\| = \sup\{|AX| : |X| \leq 1\} \tag{1.33}$$

It may be easily shown that for $X \in \mathbb{R}^2$,

$$|AX| \leq \|A\||X| \tag{1.34}$$

Let $\rho(A)$ be the spectral radius of A defined as $\rho(A) = \max\{|\lambda_1|, |\lambda_2| : \lambda_1, \lambda_2$ are the eigenvalues of $A\}$. Then it may be shown that for our selected vector norm

$$\|A\|_1 = \max\{(|a_{11}| + |a_{21}|), (|a_{12}| + |a_{22}|)\}.$$

For example, if $X = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $|X| = 3$. *And for the matrix* $A = \begin{pmatrix} 1 & 3 \\ -2 & -4 \end{pmatrix}$, $\|A\|_1 = \max\{3, 7\} = 7$. *The eigenvalues of A are $\lambda_1 = -2, \lambda_2 = -1$. Note that $\rho(A) = \max\{|-2|, |-1|\} = 2$, and thus $\rho(A) \leq \|A\|_1$.*

It is left to the reader to prove, in general, that $\rho(\bar{A}) \leq \|A\|_1$ for any matrix A . Without any further delay, we now give the required stability definitions.

Definition 1.6.2 *A fixed point X^* of a map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is said to be*
1. ***stable*** *if given $\varepsilon > 0$ there exists $\delta > 0$ such $|X - X^*| < \delta$ implies $\left|f^n(X) - X^*\right| < \varepsilon$ for all $n \in \mathbb{Z}^+$(see Fig. 4.8a).*
2. ***attracting*** *(sink) if there exists $\nu > 0$ such that $|X - X^*| < \nu$ implies $\lim_{n\to\infty} f^n(X) = X^*$. It is globally attracting if $\nu = \infty$ (see Fig. 4.9).*
3. ***asymptotically stable*** *if it is both stable and attracting. It is globally asymptotically stable if it is both stable and globally attracting, (see Fig. 4.12(a))*
4. ***unstable*** *if it is not stable (see Fig. 4-8b).*

Remark 1.6.1 *In [91], Sedaghat showed that a globally attracting fixed point of a continuous one-dimensional map must be stable. Kenneth Palmer pointed out to me that this result may be found in the book of Block and*

Coppel. Moreover, the proof in Block and Coppel requires only local attraction (see Appendix for a proof). The situation changes dramatically in two- or higher dimensional continuous maps, for there are continuous maps that possess a globally attracting unstable fixed point. We are going to present one of these maps and put several others as problems for you to investigate.

Example 1.6.1 *Consider the two-dimensional map in polar coordinates*

$$g\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} \sqrt{r} \\ \sqrt{2\pi\theta} \end{pmatrix}, r > 0, 0 \leq \theta \leq 2\pi.$$

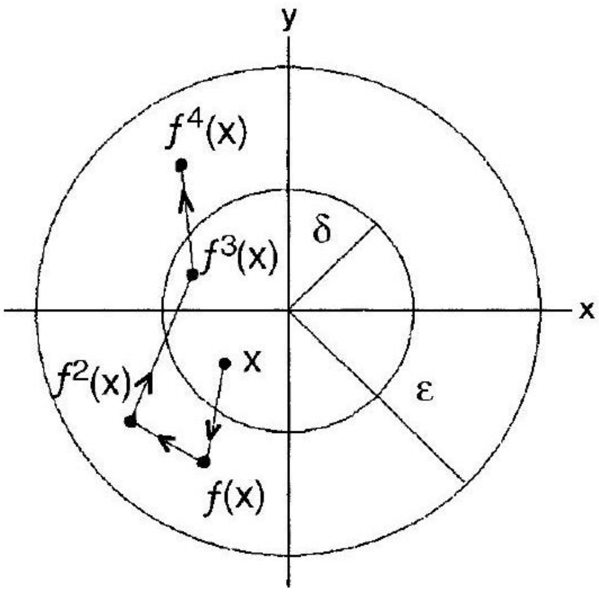


FIGURE 4.8a *(a) The fixed point $X^* = 0$ is stable.*

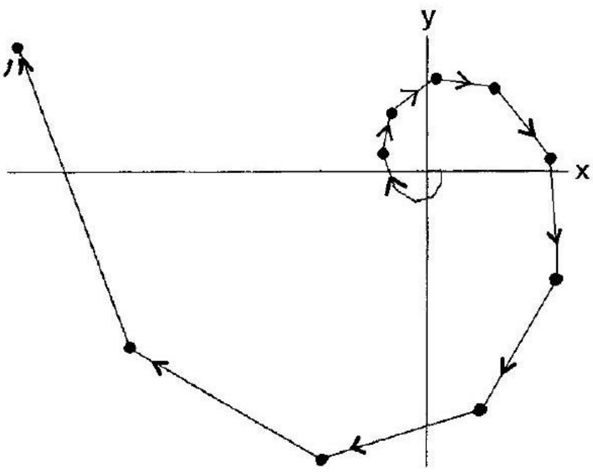


FIGURE 4.8b *(b) $X^* = 0$ is an unstable fixed point.*

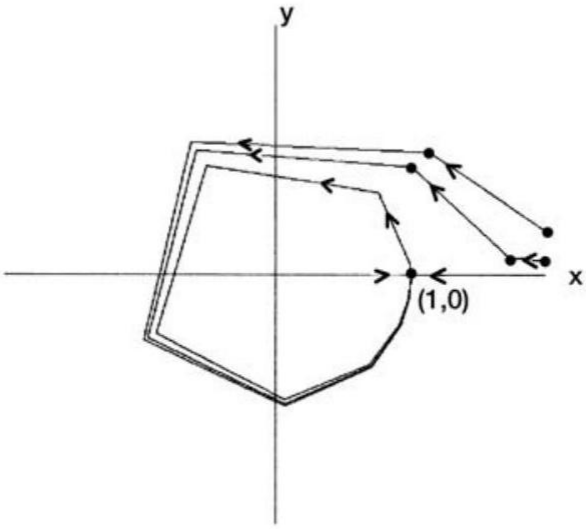


FIGURE 4.9 *$x^* = \binom{1}{0}$ is an unstable globally attracting fixed point.*

Then,

$$g^n\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r^{2^{-n}} \\ (2\pi)^{(1-2^{-n})}\theta^{2^{-n}} \end{pmatrix}$$

Clearly $\lim_{n\to\infty} g^n\binom{r}{\theta} = \binom{1}{2\pi} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. *Thus, each orbit is attracted to the fixed point* $\binom{1}{0}$. *However, if* $\theta = \delta\pi, 0 < \delta < 1$, *then the orbit of* $\begin{pmatrix} r \\ \theta \end{pmatrix}$ *will spiral clockwise around the fixed point* $\binom{1}{0}$ *before converging to it. Hence,* $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *is globally attracting but not asymptotically stable (see Fig. 4.9).*

1.7 Stability of Linear Systems

In this section, we focus our attention on linear maps where $f(X) = AX$, and A is a 2×2 matrix. Equivalently, we are interested in the difference equation

$$X(n+1) = AX(n) \tag{4.41}$$

For such linear maps, we can provide complete information about the stability of the fixed point $X^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The main result now follows.

Theorem 1.7.1 *The following statements hold for Equation (**??**):*

(a) If $\rho(A) < 1$, then the origin is asymptotically stable.

(b) If $\rho(A) > 1$, then the origin is unstable.

(c) If $\rho(A) = 1$, then the origin is unstable if the Jordan form is of the form $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, and stable otherwise.

Proof. Suppose that $\rho(A) < 1$. Then it follows from Eqs. (**??**), (**??**), (**??**), that $\lim_{n\to\infty} X(n) = 0$. Thus, the origin is (globally) attracting. To prove stability, we consider three cases.

(i) Suppose that the solution $X(n)$ is given by Equation (**??**). This is the case when the eigenvalues of the matrix A are real and there are two linearly independent eigenvectors.

$$X(n) = P \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} P^{-1} X(0).$$

Hence,

$$\begin{aligned} |\, X(n)\| &\leq \|P\| \left\|P^{-1}\right\| \rho(A) |X(0)| \\ &\leq M|X(0)| \end{aligned}$$

where $M = \|P\| \left\|P^{-1}\right\| \rho(A)$. Now, given $\varepsilon > 0$, let $\delta = \varepsilon/M$. Then $|X(0)| < \delta$ implies that $|X(n)| < M\delta = \varepsilon$. This shows that the origin is stable.

(ii) Suppose that the solution $X(n)$ is given by Equation (**??**). This case occurs if the matrix A has a repeated eigenvalue λ and only one eigenvector.

$$X(n) = P \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix} P^{-1} X(0)$$

$$|X(n)| \leq \|P\| \left\|P^{-1}\right\| \left(n|\lambda|^{n-1} + |\lambda|^n\right) |X(0)|.$$

Since $n|\lambda|^n \to 0$ as $n \to \infty$, there exists $N \in \mathbb{Z}^+$, such that the term $\left(n|\lambda|^{n-1} + |\lambda|^n\right)$ is bounded by a positive number L. Hence,

$$|X(n)| \leq M|X(0)|$$

where $M = L\|P\| \left\|P^{-1}\right\|$. The proof of the stability of the origin may be completed by seting $\delta = \varepsilon/M$ as in part (a).

(iii) If A is not in the form $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, then it is either diagonalizable to $J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, where $|\lambda_1| < 1$ and $|\lambda_2| = 1$ or $|\lambda_2| < 1$ and $|\lambda_1| = 1$. In either case, J^n is bounded and hence the origin is stable. ■

1.8 The Trace-Determinant Plane

1.8.1 Stability Analysis

Table (4.1) provides a partial summary of everything we have done so far. In this section we provide another important way of presenting these results, namely, trace-determinant plane, where we employ pictures, rather than a table. This turns out to be a better scheme when one is interested in studying bifurcation in two-dimensional systems.

The following two results provide the framework for using the trace-determinant plane. Recall that for matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $\operatorname{tr} A = a_{11} + a_{22}$, and $\det A = a_{11}a_{22} + a_{12}a_{21}$.

Theorem 1.8.1 *Let* $A = \left(a_{ij}\right)$ *be a* 2×2 *matrix. Then* $\rho(A) < 1$ *if and only if*

$$|\operatorname{tr} A| - 1 < \det A < 1. \tag{1.35}$$

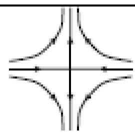
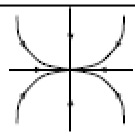
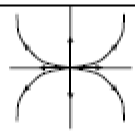
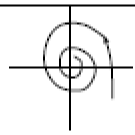
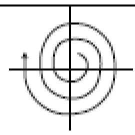
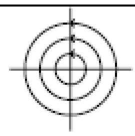
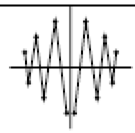
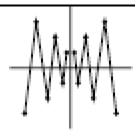
Proof.
(i) Assume that $\rho(A) < 1$. If λ_1, λ_2 are the eigenvalues of A, then $|\lambda_1| < 1$ and $|\lambda_2| < 1$. The characteristic equation of the matrix A is given by $\det(A - \lambda I) = \lambda^2 - (a_{11} + a_{22})\,\lambda - (a_{11}a_{22} - a_{12}a_{21}) = 0$, or $\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0$. Hence the eigenvalues are

$$\lambda_1 = \frac{1}{2}\left[\operatorname{tr} A + \sqrt{(\operatorname{tr} A)^2 - 4\det A}\right],$$
$$\lambda_2 = \frac{1}{2}\left[\operatorname{tr} A - \sqrt{(\operatorname{tr} A)^2 - 4\det A}\right].$$

Case (a) λ_1 and λ_2 are real roots, i.e., $(\operatorname{tr} A)^2 - 4\det A \geq 0$. Now $-1 < \lambda_1, \lambda_2 < 1$ implies that

$$-2 < \operatorname{tr} A \pm \sqrt{(\operatorname{tr} A)^2 - 4\det A} < 2$$

TABLE 4.1
Partial summary of the stability of linear systems.

Type	Eigenvalue	Phase Portrait
Saddle	$0 < \lambda_1 < 1 < \lambda_2$	
Sink	$0 < \lambda_2 < \lambda_1 < 1$	
Source	$\lambda_2 > \lambda_1 > 1$	
Spiral Sink	$\lambda = \alpha \pm i\beta$, $\lvert\lambda\rvert < 1$, $\beta \neq 0$	
Spiral Source	$\lambda = \alpha \pm i\beta$, $\lvert\lambda\rvert > 1$, $\beta \neq 0$	
Center	$\lambda = \alpha \pm i\beta$, $\lvert\lambda\rvert = 1$, $\beta \neq 0$	
"Oscillatory" Saddle	$-1 < \lambda_1 < 0$, $\lambda_2 < -1$	
"Oscillatory" Source	$\lambda_1 > 1$, $\lambda_2 > -1$	

or

$$-2 - \operatorname{tr} A < \sqrt{(\operatorname{tr} A)^2 - 4 \det A} < 2 - \operatorname{tr} A \tag{1.36}$$

$$-2 - \operatorname{tr} A < -\sqrt{(\operatorname{tr} A)^2 - 4 \det A} < 2 - \operatorname{tr} A. \tag{1.37}$$

Squaring the second inequality (**??**) yields

$$1 - \operatorname{tr} A + \det A > 0. \tag{1.38}$$

Similarly, if we square the first inequality in (**??**) we obtain

$$1 + \operatorname{tr} A + \det A > 0. \tag{1.39}$$

Now from the second inequality (**??**) and the first inequality in (**??**) we obtain $2 + \operatorname{tr} A > 0$ and $2 - \operatorname{tr} A > 0$ or $|\operatorname{tr} A| < 2$. Since $(\operatorname{tr} A)^2 - 4\det A \geq 0$, it follows that

$$\det A \leq (\operatorname{tr} A)^2/4 < 1. \tag{1.40}$$

Combining (**??**), (**??**) and (**??**) yields (**??**).

Case (b) λ_1 and λ_2 are complex conjugates, i.e.,

$$(\operatorname{tr} A)^2 - 4\det A < 0. \tag{1.41}$$

In this case we have $\lambda_{1,2} = \frac{1}{2}\left[\operatorname{tr} A \pm i\sqrt{4\det A - (\operatorname{tr} A)^2}\right]$ and

$$|\lambda_1|^2 = |\lambda_2|^2 = \frac{(\operatorname{tr} A)^2}{4} + \frac{4\det A}{4} - \frac{(\operatorname{tr} A)^2}{4} = \det A.$$

Hence $0 < \det A < 1$. To show that inequalities (**??**) and (**??**) hold, note that since $\det A > 0$ either (**??**) (if $\operatorname{tr} A > 0$) or (4.46) (if $\operatorname{tr} A < 0$) holds. Without loss of generality, assume that $\operatorname{tr} A > 0$. Then (**??**) holds. From (**??**), $\operatorname{tr} A < 2\sqrt{\det A}$. If we let $\det A = x$, then $x \in (0,1)$ and $f(x) = 1 + x - 2\sqrt{x} < 1 + \det A - \operatorname{tr} A$. Note that $f(0) = 1$ and $f'(0) = 1 - \frac{1}{\sqrt{x}}$ indicate that f is decreasing on $(0,1)$ with range $(0,1)$. This implies that $1 + \det A - \operatorname{tr} A > f(x) > 0$ and this completes the proof.

(ii) Conversely, assume that (**??**) holds. Then we have two cases to consider.

Case (a) $(\operatorname{tr} A)^2 - 4\det A \geq 0$. Then

$$\begin{aligned}
\left|\lambda_{1,2}\right| &= \frac{1}{2}\left|\operatorname{tr} A \pm \sqrt{(\operatorname{tr} A)^2 - 4\det A}\right| \\
&< \frac{1}{2}\left|\operatorname{tr} A \pm \sqrt{(\det A + 1)^2 - 4\det A}\right| \\
&< \frac{1}{2}\left(\det A + 1 + \sqrt{(\det A - 1)^2}\right) \\
&= \frac{1}{2}(\det A + 1 - (\det A - 1)) = 1.
\end{aligned}$$

Case (b) $(\operatorname{tr} A)^2 - 4\det A < 0$. Then

$$\begin{aligned}
\left|\lambda_{1,2}\right| &= \frac{1}{2}\left|\operatorname{tr} A \pm i\sqrt{4\det A - (\operatorname{tr} A)^2}\right| \\
&= \frac{1}{2}\sqrt{(\operatorname{tr} A)^2 + 4\det A - (\operatorname{tr} A)^2} \\
&= \sqrt{\det A} < 1.
\end{aligned}$$

■

As a by-product of the preceding result, we obtain the following criterion for asymptotic stability.

Corollary 1.8.1 *The origin in Equation (**??**) is asymptotically stable if and only if condition (**??**) holds true.*

*Note that condition (**??**) may be spelled out in the following three inequalities:*

$$1 + \operatorname{tr} A + \det A > 0, \quad \textit{or}$$

$$\det A > -\operatorname{tr} A - 1(\textbf{??})' \tag{1.42}$$

$$1 - \operatorname{tr} A + \det A > 0, \quad \textit{or}$$

$$\det A > \operatorname{tr} A - 1(\textbf{??})' \tag{1.43}$$

$$\det A < 1.(\textbf{??})' \tag{1.44}$$

Viewing $\det A$ *as a function of* $\operatorname{tr} A$*, the above three inequalities give us the stability region as the interior of the triangle bounded by the lines* $\det A = -\operatorname{tr} A - 1, \det A = \operatorname{tr} A - 1$*, and* $\det A = 1$*.*

Next we delve a little deeper into finding the exact values of the eigenvalues of the matrix A along the boundaries of the triangle enclosing the stability region. The following result provides us with the needed answers. Let $\lambda_1 = \frac{1}{2}\left(\operatorname{tr} A + \sqrt{(\operatorname{tr} A)^2 - 4\det A}\right), \lambda_2 = \frac{1}{2}\left(\operatorname{tr} A - \sqrt{(\operatorname{tr} A)^2 - 4\det A}\right)$ *be the eigenvalues of A.*

Theorem 1.8.2 *The following statements hold for any* 2×2 *matrix A.*
(i) If $|\operatorname{tr} A| - 1 = \det A$*, then we have*

(a) the eigenvalues of A are $\lambda_1 = 1$ *and* $\lambda_2 = \det A$ *if* $\operatorname{tr} A > 0$*,*

(b) the eigenvalues of A are $\lambda_2 = -1$ *and* $\lambda_1 = -\det A$ *if* $\operatorname{tr} A < 0$*..*

(ii) If $|\operatorname{tr} A| - 1 < \det A$*, and* $\det A = 1$*, then the eigenvalues of A are* $e^{\pm i\theta}$*, where* $\theta = \cos^{-1}(\operatorname{tr} A/2)$*.*

Proof.
(i) Let $|\operatorname{tr} A| - 1 = \det A$. Then $(\operatorname{tr} A)^2 - 4\det A = (\det A + 1)^2 \geq 0$. This implies that the eigenvalues are real numbers. Moreover, $\lambda_{1,2} = \frac{1}{2}\left(\operatorname{tr} A \pm \sqrt{(\operatorname{tr} A)^2 - 4\det A}\right) = \frac{1}{2}(\operatorname{tr} A \pm (\det A - 1))$.

(a) If $\operatorname{tr} A > 0$, then $\operatorname{tr} A = 1 + \det A$, and consequently,

$$\lambda_{1,2} = \begin{cases} 1 \\ \det A \end{cases}.$$

(b) If $\operatorname{tr} A < 0$, then $\operatorname{tr} A = -1 - \det A$, and consequently,

$$\lambda_{1,2} = \begin{cases} 1 \\ \det A \end{cases}.$$

(ii) Let $|\operatorname{tr} A| - 1 < \det A$, and $\det A = 1$. Then $(\operatorname{tr} A)^2 - 4\det A < (\det A + 1)^2 - 4 = 0$. Hence, the eigenvalues are complex conjugates. Moreover,

$$\lambda_{1,2} = \frac{1}{2}\left((\operatorname{tr} A)^2/4 \pm \sqrt{4 \det A - (\operatorname{tr} A)^2}\right)$$
$$= \frac{1}{2} \operatorname{tr} A \pm \sqrt{1 - (\operatorname{tr} A)^2}.$$

Thus $\left|\lambda_{1,2}\right| = \sqrt{(\operatorname{tr} A)^2/4 + 1 - (\operatorname{tr} A)^2/4} = 1$. Furthermore, $\theta = \arctan(\lambda_{1,2}) = \tan^{-1}\left(\frac{\pm\sqrt{1-(\operatorname{tr} A)^2/4}}{(\operatorname{tr} A)^2/4}\right) = \cos^{-1}(\operatorname{tr} A/2)$, which give $\lambda_{1,2} = e^{\pm i\theta} = \cos\theta \pm i \sin\theta$. ■

1.8.2 Navigating the Trace-Determinant Plane

The trace-determinant plane is effective in the study of linear systems with parameters. It provides us a chart of those locations where we can expect dramatic changes in the phase portrait. There are three critical loci. Let T denotes the trace and D denote the determinant. Then there are three critical lines: $D = \operatorname{tr} A - 1, D = -\operatorname{tr} A - 1$, and $D = 1$; they enclose the stability region in the trace-determinant planes.

We now illustrate our analysis by the following example.

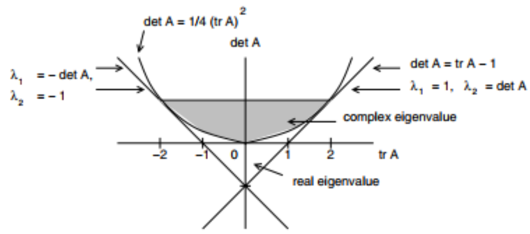
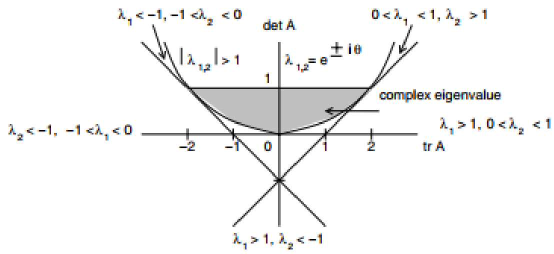


FIGURE 4.10a



(a) The stability region for Equation (**??**) is the shaded triangle. FIGURE 4.10b (b) The determination of eigenvalues in different regions.

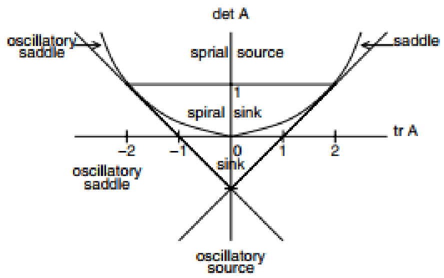


FIGURE 4.10c (c) Description of the dynamics of Equation (**??**) in all the regions in the det-trace plane.

Example 1.8.1 *Consider the one-parameter family of linear systems $X(n+1) = AX(n)$, where*

$$A = \begin{pmatrix} -1 & a \\ -2 & 1 \end{pmatrix}$$

which depends on the parameter a. As a varies, the determinant of the matrix, $\det A$, is always $2a - 1$, while the trace of the matrix, $\operatorname{tr} A$, is always 0 . As we vary the parameter a from negative to positive values, the corresponding point (T, D) moves vertically along the line $T = 0$. Now if $D < -1$, which occurs if $2a - 1 < -1$ or $a < 0$, we have a degenerate case, $\lambda_1 = 1$ and $\lambda_2 = -1$ with corresponding eigenvectors $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus every point on the y-axis is a fixed point and every other point in the plane is periodic of period 2. For $0 < a \leq \frac{1}{2}$, we have a sink, and for $\frac{1}{2} < a < 1$ we have a spiral sink. At exactly $a = 1$ we have a center, and if $a > 1$ we have a spiral source (see Fig 4.10b).

The values of a where critical dynamical changes occur are called bifurcation values. In this example, the bifurcation values of a are $0, \frac{1}{2}, 1$.]

1.9 Liapunov Functions for Nonlinear Maps

In 1892, the Russian mathematician A. M. Liapunov (sometimes transliterated as Lyapunov) introduced a new method to investigate the stability of

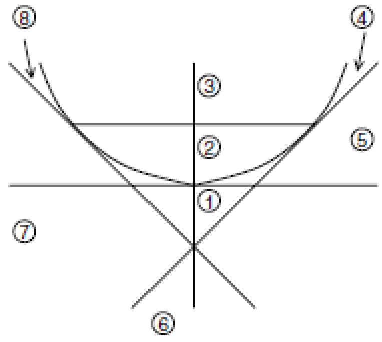


FIGURE 4.10d

nonlinear differential equations. This method, now known as Liapunov's second method, allows one to determine the stability of solutions to a differential equation without actually solving it.

In this section, we will adapt Liapunov's second method to two-dimensional maps/difference equations. The adaptation process is more or less straightforward and follows closely to LaSalle and Elaydi. Consider the difference equation

$$X(n+1) = f(X(n)) \tag{1.45}$$

where $f : G \to \mathbb{R}^2, G \subset \mathbb{R}^2$, is continuous. Let X^* be a fixed point of f, that is, $f(X^*) = X^*$. For $V : \mathbb{R}^2 \to \mathbb{R}$, we define the variation ΔV of V relative to Equation (4.52) as

$$\Delta V(X) = V(f(X)) - V(X).$$

Hence,

$$\Delta V(X(n)) = V(X(n+1)) - V(X(n)).$$

So, if $\Delta V \leq 0$, then V is nonincreasing along the orbits of f.

Definition 1.9.1 *A real valued function $V : G \to \mathbb{R}, G \subset \mathbb{R}^2$, is said to be a Liapunov function on G if*

1. *V is continuous on G*
2. *$\Delta V(X) \leq 0$, whenever X and $f(X) \in G$.*

Let $B(X, \gamma) = \left\{Y \in \mathbb{R}^2 : |Y - X| < \gamma\right\}$ denote the open ball around X. Then, we say that the Liapunov function is positive definite at X^* if $V(X) > 0$

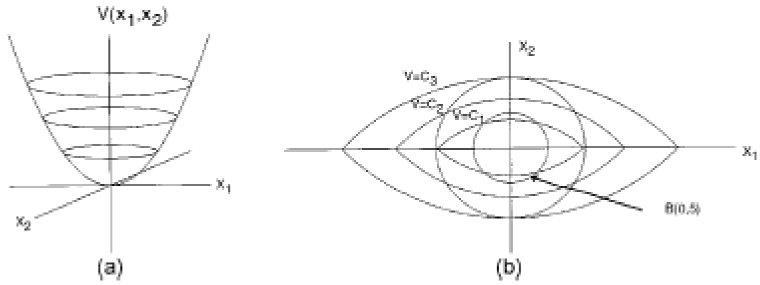


FIGURE 4.11
A Liapunov function V and its level curves.

for all $X \in B(X^*, \delta)$, for some $\delta > 0, X \neq X^*$, and $V(X^*) = 0$. The function V is said to be negative definite if $-V$ is positive definite. We now present the reader with an informal geometrical discussion on the first Liapunov stability theorem. Without loss of generality, we focus our attention on the stability of the fixed point $X^* = 0$. Suppose that there exists a positive definite Liapunov function V defined on $B(0, \eta), \eta > 0$. Figure 4.11 (a) illustrates the graph of V in a three-dimensional coordinate system, while Fig. 4.11(b) depicts the level curves $V\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c$ in the plane. Assume that for some $\varepsilon > 0, B(0, \varepsilon)$ contains the level curve $V\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \tilde{c}$ and this level curve, in turn, contains the ball $B(0, \delta), 0 < \delta \leq \varepsilon$.

Now, if $X \in B(0, \delta)$, then $V(X) \leq \tilde{c}$. Since $\Delta V \leq 0$, it follows that $V(f^n(X)) \leq V(X) \leq \tilde{c}$, for all $n \in \mathbb{Z}^+$. Consequently, the orbit of X stays indefinitely in $B(0, \varepsilon)$, and hence 0 is a stable fixed point. On the other hand, if $\Delta V < 0$, then $V(f^n(X)) < V(X) < \tilde{c}$ for all $n \in \mathbb{Z}^+$, which intuitively leads to the conclusion that $f^n(X) \to 0$ as $n \to \infty$. This is the essence of the proof of the next theorem. A more rigorous proof follows.

Theorem 1.9.1 *Suppose that V is a positive definite Liapunov function defined on an open ball $G = B(X^*, \gamma)$ around a fixed point X^* of a continuous map f on $\mathbb{R}^2$. Then, X^* is stable. If, in addition, $\Delta V(X) < 0$, whenever X and $f(X) \in G, X \neq X^*$, then X^* is asymptotically stable on G. Moreover, if $G = \mathbb{R}^2$ and $V(X) \to \infty$ as $|X| \to \infty$, then X^* is globally asymptotically stable.*

Proof. Choose $\alpha_1 > 0$ such that $B(X^*, \alpha_1) \subset G \cap H$. Since f is continuous, there is $\alpha_2 > 0$ such that if $X \in B(X^*, \alpha_2)$ then $f(X) \in B(X^*, \alpha_1)$.
Let $0 < \varepsilon \leq \alpha_2$ be given. Define $\psi(\varepsilon) = \min\{V(X) | \varepsilon \leq |X - X^*| \leq \alpha_1\}$. By the intermediate value theorem, there exists $0 < \delta < \varepsilon$ such that $V(\bar{X}) < \psi(\varepsilon)$ whenever $|X - X^*| < \delta$.

Realize now that if $X_0 \in B(X^*, \delta)$, then $X(n) \in B(X^*, \varepsilon)$ for all $n \geq 0$. This claim is true because, if not, there exists $X_0 \in B(X^*, \delta)$ and a positive integer m such that $X(r) \in B(X^*, \varepsilon)$ for $1 \leq r \leq m$ and $X(m+1) \notin B(X^*, \varepsilon)$. Since $X(m) \in B(X^*, \varepsilon) \subset B(X^*, \alpha_2)$, it follows that $X(m+1) \in B(X^*, \alpha_1)$. Consequently,

$V(X(m+1)) \geq \psi(\varepsilon)$. However, $V(X(m+1)) \leq \ldots, \leq V(X_0) < \psi(\varepsilon)$ and we thus have a contradiction. This establishes stability.

To prove asymptotic stability, assume that $X_0 \in B(X^*, \delta)$. Then $X(n) \in B(X^*, \varepsilon)$ holds true for all $n \geq 0$. If $\{X(n)\}$ does not converge to X^*, then it has a subsequence $\{X(n_i)\}$ that converges to $Y \in R^k$. Let $E \subset B(X^*, \alpha_1)$ be an open neighborhood of Y with $X^* \notin E$. Having already defined on E the function $h(x) = V(f(X))/V(X)$, we may consider h as well defined, continuous, and $h(X) < 1$ for all $X \in E$. Now if $\eta \in (h(Y), 1)$ then there exists $\delta > 0$ such that $X \in B(Y, \delta)$ implies $h(X) \leq \eta$. Thus for sufficiently large n_i,

$$V(f(X(n_i))) \leq \eta V(X(n_i - 1)) \leq \eta^2 V(X(n_i - 2)), \ldots, \leq \eta^{n_i} V(X_0).$$

Hence,

$$\lim_{n_i \to \infty} V(X(n_i)) = 0.$$

But, since $\lim_{n_i \to \infty} V(X(n_i)) = V(Y)$, this statement implies that $V(Y) = 0$, and consequently $Y = X^*$.

To prove the global asymptotic stability, it suffices to show that all solutions are bounded and then repeat the above argument. Begin by assuming there exists an unbounded solution $X(n)$, and then some subsequence $\{X(n_i)\} \to \infty$ as $n_i \to \infty$. Since $V(X) \to \infty$, as $|X| \to \infty$, this assumption implies that $V(X(n_i)) \to \infty$ as $n_i \to \infty$, which is a contradiction since $V(X_0) > V(X(n_i))$ for all i. This concludes the proof. ■

Example 1.9.1 *Consider the two-dimensional system*

$$\begin{aligned} x(n+1) &= \frac{ay(n)}{1+x^2(n)} \\ y(n+1) &= \frac{bx(n)}{1+y^2(n)} \end{aligned} \tag{1.46}$$

or the map

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ay/\left(1+x^2\right) \\ bx/\left(1+y^2\right) \end{pmatrix}.$$

*Discuss the stability of the zero solution of Equation (**??**)*

SOLUTION Take $V(x, y) = x^2 + y^2$. Then V is positive definite. Moreover,

$$\begin{aligned}
\Delta V\begin{pmatrix}x\\y\end{pmatrix} &= V\left(F\begin{pmatrix}x\\y\end{pmatrix}\right) - V\begin{pmatrix}x\\y\end{pmatrix} \\
&= \frac{a^2y^2}{(1+x^2)^2} + \frac{b^2x^2}{(1+y^2)^2} - x^2 - y^2 \\
&= \left(\frac{b^2}{(1+y^2)^2} - 1\right)x^2 + \left(\frac{a^2}{(1+x^2)^2} - 1\right)y^2 \\
&\leq \left(b^2 - 1\right)x^2 + \left(a^2 - 1\right)y^2.
\end{aligned} \tag{1.47}$$

Now, we have three cases to consider. The first case is if $a^2 < 1$ and $b^2 < 1$, then $\Delta V < 0$ and thus we conclude from Theorem 4.3 that the origin is asymptotically stable. Furthermore, since $V\begin{pmatrix}x\\y\end{pmatrix} \to \infty$ as $\left\|\begin{pmatrix}x\\y\end{pmatrix}\right\| \to \infty$, the origin is globally asymptotically stable (see Fig. 4.12(a)).

However, in the second case, if $a^2 \leq 1$ and $b^2 \leq 1$, then $\Delta V \leq 0$ and we can only conclude from Theorem 4.3 that the origin is stable.

In the final case, when $a^2 > 1$ and $b^2 > 1$, Theorem 4.3 fails to provide us with information about the stability (or lack thereof) of the origin.

It is now evident that finer analysis is needed to fully understand the stability in the last two cases. Subsequently, we are led to an important result due to LaSalle which is commonly known as LaSalle's invariance principle. To prepare for such important results, we should become familiar with certain terminology, some old and some new.

Recall that a set H is (positively) invariant under a map $f : \mathbb{R}^2 \to \mathbb{R}^2$ if $f(H) \subset H$. The (positive) limit set $\Omega(x)$ of $x \in \mathbb{R}^2$ is defined to be the set of all limit points of its positive orbit $O(x)$. It may be shown (Problem 2) that

$$\Omega(x) = \bigcap_{i=0}^{\infty} \overline{\bigcup_{n=i}^{\infty} f^n(x)}. \tag{1.48}$$

Furthermore, $\Omega(x)$ is closed and (positively) invariant (Problem 3). A closed set H is said to be invariantly connected if it is not the union of two nonempty disjoint closed invariant sets.

The nagging question still persists as to whether or not $\Omega(X)$ is nonempty for a given $X \in \mathbb{R}^2$. The next lemma settles this question.

Lemma 1.9.1 *If $O(x)$ is bounded, then $\Omega(x)$ is nonempty, compact, and invariant.*

Proof. The proof is left to the reader as Problem 2. ■

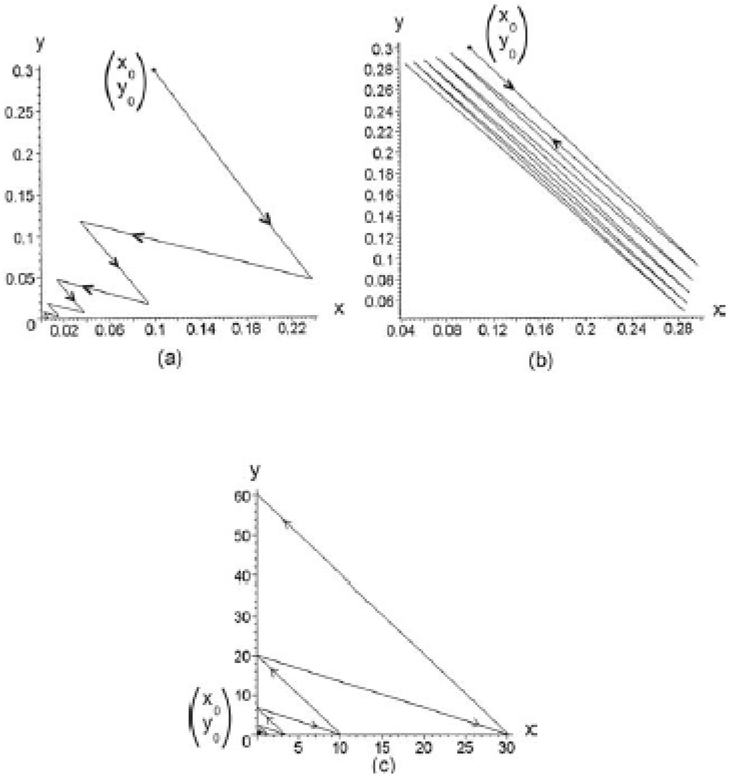


FIGURE 4.12
(a) $a^2 < 1$, $b^2 < 1$, the origin is globally asymptotically stable. (b) $a^2 = 1$, $b^2 = 1$, an orbit approaching a 4-cycle. (c) $a^2 > 1$, $b^2 > 1$, the origin is unstable.

Now, let V be a Liapunov function on a subset $G \subset \mathbb{R}^2$. Define

$$E = \{X \in \bar{G} : \Delta V(X) = 0\}.$$

Let M be the maximal invariant subset of E, and for $c \in \mathbb{R}^+$, $V^{-1}(c) = \{X : \ V(X) = c, X \in \mathbb{R}^2\big\}$.

Theorem 1.9.2 ***(LaSalle's Invariance Principle)***
Suppose that V is a positive definite Liapunov function defined on an open ball $G = B(X^*, \gamma)$ around a fixed point X^* of a two-dimensional map f. If for $X \in G$, $O(X)$ is bounded and contained in G, then for some $c \in \mathbb{R}^+$, $f^n(X) \to M \cap V^{-1}(c)$ as $n \to \infty$.

Proof. Let $X \in \mathbb{R}^2$ such that $O(x)$ is bounded. Then, by Lemma 4.1, $\Omega(X) \neq \phi$. Now, since $V(f^n(X))$ is nonincreasing and bounded below, $\lim_{n\to\infty} V(f^n(X)) = c$, for some $c \in \mathbb{R}^+$. For $Y \in \Omega(X)$, $f^{n_i}(X) \to Y$ as

$n_i \to \infty$, for some subsequence of positive integers n_i. By the contimuity of V, it follows that

$$\lim_{n_i \to \infty} V\left(f^{n_i}(X)\right) = V(Y) = c$$

Hence, $\Omega(X) \subset V^{-1}(c)$. Furthermore, since $\Omega(X)$ is invariant, $V(f(Y)) = V(Y)$ and consequently, $\Delta V(Y) = 0$. Hence $\Omega(X) \subset E$. But, $\Omega(X)$ is invariant, which implies that it must be contained in M. Therefore, $f^n(X) \to M \cap V^{-1}(c)$.

A remark is now in order. Note that if $M = \{X^*\}$ is a singleton, then the preceding theorem tells us that X^* is definitely an asymptotically stable fixed point. This observation leads to the complete analysis of part (b) of Example 3.9.1. ■

Example 1.9.2 ***(Example 4.8 Revisited).***
Let us reexamine case (b) in Example 3.9.1 in light of LaSalle's invariance principle. Here are the two subcases to consider.

Case one is where $a^2 \leq 1, b^2 \leq 1$, *and* $a^2 + b^2 < 2$. *Without loss of generality, we may assume that* $a^2 < 1$ *and* $b^2 = 1$. *Then,* $\Delta V \leq \left(a^2 - 1\right) y^2$ *which is zero when* $y = 0$. *Thus* E *is the* x *axis. To find the largest invariant subset* M *of* E, *note that for* $\begin{pmatrix} x \\ 0 \end{pmatrix} \in E, F\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ bx \end{pmatrix}$. *Hence,* $M = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}$. *Consequently, the origin is asymptotically stable.*

In case two, $a^2 = 1$ *and* $b^2 = 1$. *It follows from Equation (**??**) that* $\Delta V = 0$ *if* $x = 0$ *or* $y = 0$. *Thus* $E = M =$ *the union of the two coordinate axes. LaSalle's invariance now tells us that there exists* $c > 0$ *such that each orbit* $O^+(u)$ *approaches the set* $\left\{\begin{pmatrix} \pm c \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \pm c \end{pmatrix}\right\}$.

Now, if $c \neq 0, F^4\begin{pmatrix} c \\ 0 \end{pmatrix} = \begin{pmatrix} c \\ 0 \end{pmatrix}$, *and that* $\begin{pmatrix} c \\ 0 \end{pmatrix}$ *is a point of period 4 with the cycle* $\left\{\begin{pmatrix} c \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ bc \end{pmatrix}, \begin{pmatrix} abc \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ ac \end{pmatrix}\right\}$. *Similarly, we may show that* $\left\{\begin{pmatrix} 0 \\ c \end{pmatrix}, \begin{pmatrix} ac \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ abc \end{pmatrix}, \begin{pmatrix} bc \\ 0 \end{pmatrix}\right\}$ *is also a 4-cycle. Since* $\Omega(u)$ *is invariantly connected, each orbit must approach only one of these 4-cycles [Fig. 4.12(b)]. Finally, we observe that if* $ab = 1$, *then we have 2-cycles instead.*

We now end this section by giving a result about instability. This will enable us to treat the remaining case of Example 3.9.1.

Theorem 1.9.3 *Let* $V : G \subset \mathbb{R}^2 \to \mathbb{R}$ *be a continuous function such that relative to Equation (**??**),* ΔV *positive definite (negative definite) on a neighborhood of a fixed point* x^*. *If there erists a sequence* $x_i \to x^*$ *as* $i \to \infty$ *with* $V(x_i) > 0\,(V(x_i) < 0\,)$, *then* x^* *is unstable.*

Proof. Assume that $\Delta V(x) > 0$ for $x \in B(\eta), x \neq 0, \Delta V(0) = 0$. We will prove Theorem 3.9.3 by contradiction, first assuming that the zero solution is stable. Then for $\varepsilon < \eta$, there will exist $\delta < \varepsilon$ such that $\|x_0\| < \delta$ implies $\|x(n, 0, x_0)\| < \varepsilon, n \in Z^+$.

Since $a_i \to 0$, pick $x_0 = a_j$ for some j with $V(x_0) > 0$, and $\|x_0\| < \delta$. Hence $\overline{0(x_0)} \subset \overline{B(\varepsilon)} \subset B(\eta)$ is closed and bounded (compact). Since its domain is compact, $V(x(n))$ is also compact and therefore bounded above. Since $V(x(n))$ is also increasing, it follows that $V(x(n)) \to c$. Following the proof of LaSalle's invariance principal, it is easy to see that $\lim_{n\to\infty} x(n) = 0$. Therefore, we would be led to believe that $0 < V(x_0) < \lim_{n\to\infty} V(x(n)) = 0$. This statement is unfeasible-so the zero solution cannot be stable, as we first assumed. The zero solution of Equation (**??**) is thus unstable.

The conclusion of the theorem also holds if ΔV is negative definite and $V(a_i) < 0$. ■ We are now in a position to tackle the remaining case in Example 4.8. Assume now that $a^2 > 1$ and $b^2 > 1$. Let δ be sufficiently small such that $b^2 > \left(1+\delta^2\right)^2$ and $a^2 > \left(1+\delta^2\right)^2$. Define a function $V\begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y^2$ on the open disc $B(0,\delta)$ centered at the origin and with radius δ. Then V is clearly positive definite. Moreover, for $\begin{pmatrix} x \\ y \end{pmatrix} \in B(0,\delta)$ we have

$$\begin{aligned} \Delta V\begin{pmatrix} x \\ y \end{pmatrix} &= \left(\frac{b^2}{(1+y^2)^2} - 1\right)x^2 + \left(\frac{a^2}{(1+x^2)^2} - 1\right)y^2 \\ &\geq \left(\frac{b^2}{1+\delta^2)^2} - 1\right)x^2 + \left(\frac{a^2}{(1+\delta^2)^2} - 1\right)y^2 \\ &> 0 \text{ if } \begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Hence by Theorem 3.7.1, the origin is unstable (see Fig. 4.12(c)).

1.10 Linear Systems Revisited

In Sec. 3.3, we have settled most of the questions concerning the stability of second-order linear systems:

$$X(n+1) = AX(n). \tag{1.49}$$

In this section, we are going to construct suitable Liapunov functions for System (**??**). This is important for our program since by modifying such Liapunov functions we can find appropriate Liapunov functions for a large class of nonlinear equations of the form

$$Y(n+1) = AY(n) + g(Y(n))$$

which we will study in the next section. But, before embarking on our task we need to introduce a few preliminaries about definite matrices.

Let $X \in \mathbb{R}^2, B = \left(b_{ij}\right)$ a real symmetric 2×2 matrix, and consider the quadratic form $V : \mathbb{R}^2 \to \mathbb{R}$

defined by

$$V(X) = X^T BX = \sum_{i=1}^{2} \sum_{j=1}^{2} b_{ij} X_i X_j \tag{1.50}$$

where X^T denotes the transpose of vector X. A matrix B is said to be positive (negative) definite if the corresponding $V(X)$ as defined in Equation (**??**) is positive (negative) for all $0 \neq X \in \mathbb{R}^2$. If, however, $V(X) \geq 0 (V(X) \leq 0)$ for all $X \in \mathbb{R}^2$, then B is positive (negative) semidefinite. The following result, due to Sylvester, gives a complete characterization of the notion of definiteness of V.

Theorem 1.10.1 *Let V be a quadrutic form as defined in Equation (**??**). Then the following statements hold true:*
1. V is positive definite if and only if all principle minors of B are positive, i.e., if and only if

$$b_{11} > 0 \textit{ and } \det B > 0.$$

2. V is negative definite if and only if

$$b_{11} < 0 \textit{ and } \det B > 0.$$

3. V is positive (negative) definite if and only if all eigenvalues of B are nonzero and positive (negative), respectively.
4. If λ_1, λ_2 are the eigenvalues of B, $\lambda_m = \min_i |\lambda_i|$, $\lambda_M = \max_i |\lambda_i|$, $i = 1, 2$, then

$$\lambda_m |X|^2 \leq V(X) \leq \lambda_M |X|^2.$$

5. V is semidefinite (positive or negative) if and only if the nonzero eigenvalues of B have the same sign.

Let $B = \begin{pmatrix} 3 & 2 \\ 2 & 5 \end{pmatrix}$. Then, the principal minors are 3 and $\begin{pmatrix} 3 & 2 \\ 2 & 5 \end{pmatrix}$ where both have positive determinants. Hence, B is positive definite by Theorem 3.10.1. Notice that the eigenvalues of B are $\lambda_1 = 4\sqrt{5}$, and $\lambda_2 = 4 - \sqrt{5}$, which are also positive. Moreover, if we let $V(X) = X^T BX$, then

$$\begin{aligned} V(X) &= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= 3x_1^2 + 4x_1 x_2 + 5x_2^2. \end{aligned}$$

Let us now go back to Equation (**??**) and consider the function $V(X) = X^T BX$, where B is positive definite as a prospective candidate for a Liapunov function. Then,

$$\begin{aligned} \Delta V(X(n)) &= X^T(n+1) BX(n+1) - X^T(n) BX(n) \\ &= X^T(n) A^T BAX(n) - X^T(n) BX(n) \\ &= X^T(n) \left(A^T BA - B \right) X(n). \end{aligned}$$

Thus, $\Delta V(X(n)) < 0$ if and only if

$$A^T BA - B = -C \tag{1.51}$$

for some positive definite matrix C. Equation (**??**) is called the Liapunov equation of System (**??**). The preceding argument established part of the following result whose complete proof is omitted.

Theorem 1.10.2 $\rho(A) < 1$ *if and only if for every symmetric and positive definite matrix C, Equation (**??**) has a unique solution B, which is also symmetric and positive definite.*

An immediate corollary of Theorem 3.10.2 , which will be useful in the next section, now follows.

Corollary 1.10.1 *If $\rho(A) > 1$, then there erists a real symmetric matrix B that is not positive semidefinite such that Equation (**??**) holds for some symmetric positive definite matrix C.*

1.11 Stability via Linearization

In Chapter 1, we saw how the values of the derivative $f'(o^*)$ of a nonlinear, one-dimensional map f at a hyperbolic fixed point o^* determines completely the stability of o^*. For if $|f'(o^*)| < 1$, then o^* is asymptotically stable or a sink if $|f'(o^*)| > 1$, then o^* is unstable or a repeller. In essence, what we are saying is that the behavior of the linear difference equation

$$Y(n+1) = f'(X^*)\, Y(n) \tag{1.52}$$

determines the behavior of the original equation

$$X(n+1) = f(X(n)) \tag{1.53}$$

near the fixed points. In the language of maps, this amounts to saying that the behavior of the linear map $g(X) = f'(X^*)\, X$ determines the behavior of the nonlinear map $f(X)$ near the fixed point X^*. Such a process is commonly called a linearization of the nonlinear map f or the difference equation (**??**). Now, consider a two-dimensional map $f : G \subset \mathbb{R}^2 \to \mathbb{R}^2$, where G is an open subset of $\mathbb{R}^2$. Then f is said to be contimuously differentiable, (or a C^1 map) if its partial derivatives $\dfrac{\partial f}{\partial x_1}$ and $\dfrac{\partial f}{\partial x_2}$ exist and are continuous. If there is a 2×2 matrix A such that

$$\lim_{X \to Q} \frac{|f(X) - f(Q) - A(X - Q)|}{|X - Q|} = 0,$$

then the derivative $Df(Q)$ of f at Q is defined as $Df(Q) = A$. Hence, if $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, then

$$Df(Q) = A = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}_Q$$

The matrix A is often called the Jacobian matrix of f. By the mean value theorem, we have

$$f(X) = f(Q) + A(X - Q) + g(X, P) \tag{1.54}$$

where

$$\lim_{X \to Q} \frac{g(X, Q)}{|X - Q|} = 0 \tag{1.55}$$

Statement (**??**) may be expressed in the little "o" language as $g(X, Q) = o(|X - Q|)$ as X tends to Q.

Suppose now that $Q = X^*$ is a fixed point of f, that is, $f(X^*) = X^*$. Then Equation (**??**) yields

$$f(X) - X^* = A(X - X^*) + g(X, X^*). \tag{1.56}$$

It is clear that $g(X^*, X^*) = 0$. To simplify our exposition, we make the change of variables $Y = X - X$. Equation (**??**) becomes

$$f(Y + X^*) - X^* = AY + g(Y) \tag{1.57}$$

If we now let $h(Y) = f(Y + X^*) - X^*$ in Equation (**??**), we get

$$h(Y) = AY + g(Y) \tag{1.58}$$

with $g(Y) = o(|Y|)$ as Y tends to 0 .

We now make two important observations concerning the relationship between the maps f and h. First, since $h(0) = f(X^*) - X^* = 0$, it follows that 0 is a fixed point of h if and only if X^* is a fixed point of f. Second, note that $h^n(Y) \to 0$ as $n \to \infty$ if and only if $f^n(X) = f^n(Y + X^*) \to X^*$ as $n \to \infty$. Hence, 0 is stable (asymptotically stable) under h if and only if X^* is stable (asymptotically stable) under f. A similar statement can be made about instability. Hence, without loss of generality, we may work directly with the map h in Equation (**??**). In other words, it suffices to consider the nonhomogeneous system

$$Y(n + 1) = AY(n) + g(Y(n)) \tag{1.59}$$

with $A = Df(X^*), g(Y) = o(|Y|)$, and $g(0) = 0$. The linear part of Equation (**??**) is the homogeneous equation

$$X(n+1) = AX(n). \tag{1.60}$$

The main result of this section now follows.

Theorem 1.11.1 *Let $f : G \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map, where G is an open subset of $\mathbb{R}^2$, X^* is a fixed point of f, and $A = Df(X^*)$. Then the following statements hold true:*
1. If $\rho(A) < 1$, then X^* is asymptotically stable.
2. If $\rho(A) > 1$, then X^* is unstable.
3. If $\rho(A) = 1$, then X^* may or may not be stable.

Proof.
1. Assume that $\rho(A) < 1$. Then by virtue of Theorem 4.10 , there exists a real symmetric and positive definite matrix B such that $A^TBA - B = -C$, where C is positive definite. Now, consider the Liapunov function $V(Y) = Y^TBy$. Then the variation of V relative to Equation (**??**) is given by

$$\Delta VY = -Y^TCY + 2Y^TA^TBg(Y) + V(g(Y)). \tag{1.61}$$

Now, Equation (**??**) allows us to pick a $\gamma > 0$ such that $Y^TCY \geq 4\gamma|Y|^2$ for all $Y \in \mathbb{R}^2$. There exists $\delta > 0$ such that if $|Y| < \delta$, then $\left|A^TBg(Y)\right| \leq \gamma|Y|$ and $V(g(Y)) \leq \gamma|Y|$. Hence, it follows from Equation (**??**) that $\Delta V(Y(n)) \leq -\gamma|Y(n)|^2$ which implies by Theorem 3.7.1 that the zero solution of Equation (**??**) is asymptotically stable.
2. Assume that $\rho(A) > 1$. Then, we use Corollary 3.10.1 to choose a real, symmetric 2×2 matrix B such that $B^TAB - B = -C$ is negative definite, where B is not positive semidefinite. Thus, the function $V(Y) = Y^TBY$ is negative at points arbitrarily close to the origin. Now, as in part 1 , $\Delta V(Y(n)) \leq -\gamma|Y(n)|^2$. Thus, by Theorem 3.10.2, the zero solution of Equation (**??**) is unstable.
3. We prove this part by using the following example.

Example 1.11.1 *1. Consider the system*

$$\begin{aligned} x_1(n+1) &= x_1(n) + x_2^2(n) + x_1^2(n) \\ x_2(n+1) &= x_2(n). \end{aligned}$$

The linear part has the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

with $\rho(A) = 1$. To determine the stability of the origin we use the Liapunov function $V(X) = x_1 + x_2$. Then, V is not positive and

$$\Delta V(X(n)) = x_1^2(n) + x_2^2(n) > 0 \text{ if } (x_1, x_2) \neq (0,0).$$

Hence, by Theorem 3.10.2, the origin is unstable.

2. Let us now consider the system

$$x_1(n+1) = x_1(n) - x_1^3(n)x_2^2(n)$$
$$x_2(n+1) = x_2(n)$$

with the linear part as in (a). This time we use the Liapunov function $V(X) = x_1^2 + x_2^2$. *Then*

$$\Delta V(X(n)) = x_1^4(n)x_2^2(n)\left[-2 + x_1^2(n)x_2^2(n)\right].$$

Hence, $\Delta V \leq 0$ *if* $x_1^2 x_2^2 < 2$. *Thus, the origin is stable by Theorem 3.7.1.*

■ **Example 4.11**

(Pielou Logistic Delay Equation). One of the most popular continuous models for the growth of a population is the well-known Verhulst-Pearl differential equation given by

$$x'(t) = x(t)[a - bx(t)], a, b > 0$$

where $x(t)$ is the size of the population at time $t, x'(t) = \frac{dx}{dt}, a$ is the rate of growth of the population if the resources were unlimited and the individuals did not affect one another, and $-bx^2(t)$ represents the negative effect on the growth of the population due to crowdedness and limited resources. The solution to this equation may be obtained by separation of variables x and t, and then integrating both sides. Hence,

$$x(t) = \frac{a/b}{1 + (e^{-at}/cb)}.$$

This implies that

$$x(t+1) = \frac{a/b}{1 + (e^{-a(t+1)}/cb)}$$
$$= \frac{(a/b)e^a}{1 + (e^{-at}/cb) + (e^a - 1)}.$$

Dividing by the quantity $[1 + (e^{-at}/cb)]$ both the numerator and the denominator on the right-hand side we obtain

$$x(t+1) = \frac{e^a x(t)}{1 + \frac{b}{a}(e^a - 1)\, x(t)}$$

or

$$x(t+1) = \frac{\alpha x(t)}{1 + \beta x(t)}.$$

This equation is called the Pielou logistic equation. Now, if we assume that there is a delay of time period 1 in the response of the growth rate per individual to density change, then we obtain the difference equation (replace t by n)

$$x(n+1) = \frac{\alpha x(n)}{1 + \beta x(n-1)}. \tag{1.62}$$

As an example of a population that can be modeled by Equation (**??**) is the blowfly (Lucilia cuprina). We now write Equation (**??**) in system form. Let $x_1(n) = x(n-1)$, and $x_2(n) = x(n)$. Then,

$$\begin{pmatrix} x_1(n+1) \\ x_2(n+1) \end{pmatrix} = \begin{pmatrix} x_2(n) \\ \frac{\alpha x_2(n)}{1-\beta x_1(n)} \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix} \tag{1.63}$$

There are two fixed points $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} (\alpha-1)/\beta \\ (\alpha-1)/\beta \end{pmatrix}$.

1. The fixed point $Z_1^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Here,

$$A = Df(0) = \begin{pmatrix} 0 & 1 \\ 0 & \alpha \end{pmatrix}$$

with eigenvalues 0 and α. Since $\alpha > 1$, the origin is unstable by Theorem 3.11.1.

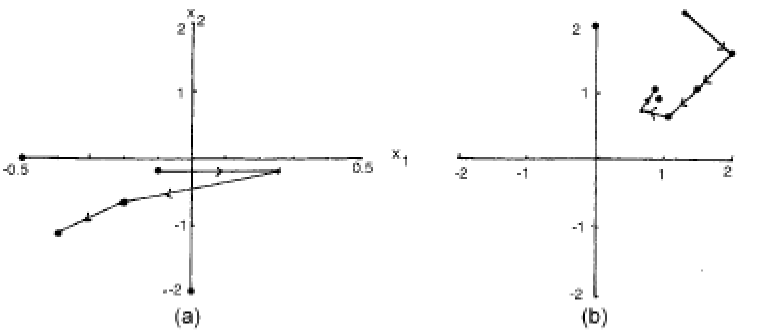


FIGURE 4.13
For the Pielou Logistic equation, (a) the trivial solution is unstable. (b) The equilibrium $Z_2^* = ((\alpha-1)/\beta,\ (\alpha-1)\beta)$ is asymptotically stable.

2. The fixed point $Z_2^* = \begin{pmatrix} (\alpha-1)/\beta \\ (\alpha-1)/\beta \end{pmatrix}$. In this case,

$$A = Df\left(z_2^*\right) = \begin{pmatrix} 0 & 1 \\ \frac{1-\alpha}{\alpha} & 1 \end{pmatrix}.$$

By Theorem 3.11.1 , $\rho(A) < 1$ if and only if

$$|\operatorname{tr} A| < 1 + \det A < 2$$

if and only if

$$1 < 1 + \frac{\alpha - 1}{\alpha} < 2$$

if and only if

$$0 < \frac{\alpha - 1}{\alpha} < 1$$

Clearly this is satisfied if $\alpha > 1$. Hence, by Theorem 4.11, z_2^* is asymptotically stable (see Fig. 4.13).

1.11.1 The Hartman-Grobman Theorem

Similar to the one-dimensional case, we say a fixed point x^* of a planar map f is hyperbolic if $|\lambda| \neq 1$, for all eigenvalues of $A = Df(x^*)$. Theorem 3.11.1 has given us almost complete information about the stability of hyperbolic fixed points.

An extension of this result to periodic points is straightforward and will be left to the reader. In proving Theorem 3.11.1, we have relied heavily on Liapunov function techniques. One may arrive at the same conclusion by looking at matters through conjugacy. This is the essence of the so-called Hartman-Grobman theorem, which roughly states that near a hyperbolic fixed point, a map is conjugate to the linear map induced by its derivative at the fixed point. But this comes with a price, we require the map f to be C^1 diffeomorphism, that is a homeomorphism such that f, f', and their inverses are continuously differentiable.

Theorem 1.11.2 ***(Hartman-Grobman Theorem).***
Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 diffeomorphism with a hyperbolic fixed point X^*. Then there exist open neighborhoods G of X^* and H of the origin and a homeomorphism $h : H \to G$ such that $f(h(X)) = h(AX)$ for all $X \in H$, where $A = Df(X^*)$.

In fact, Hartman showed that the conjugacy map h is C^1 if f is C^2. As a corollary of the Hartman-Grobman theorem, one may easily establish Theorem 3.11.1

1.11.2 The Stable Manifold Theorem

Finally, our discussion of the stability of nonlinear maps will not be complete without the stable manifold theorem. Roughly speaking, this theorem states that if the origin is a saddle under the linear map induced by the derivative of a planar map f, then under f the origin exhibits a saddle-like behavior.

An accurate statement of the theorem now follows. However, its proof, is omitted and the interested reader is referred to [25, 85]. Here we assume that the eigenvalues of $A = Df(X^*)$ are $|\lambda_1| < 1$ and $|\lambda_2| > 1$ with corresponding eigenvectors V_1 and V_2.

Theorem 1.11.3 ***(The Stable Manifold Theorem).***
Let X^* be a hyperbolic fixed point of a C^1 map $f : G \subset \mathbb{R}^2 \to \mathbb{R}^2$. *Then there exists $\varepsilon > 0$ and C^1 curves $\gamma_1 : (-\varepsilon, \varepsilon) \to \mathbb{R}^2$ and $\gamma_2 : (-\varepsilon, \varepsilon) \to \mathbb{R}^2$ such that*
1. $\gamma_1(0) = \gamma_2(0) = X^*$.
2. $\gamma_1'(0) = V_1$, and $\gamma_2'(0) = V_2$.

3. *If* $Q = \gamma_1(t)$, *then* $f^n(Q) \to X^*$ *as* $n \to \infty$.
4. *If* $Q \in \gamma_2(t)$, *then* $f^{-n}(Q) \to X^*$ *as* $n \to \infty$.

The curve γ_1 is called the stable manifold and is usually denoted by $W^*(X^*)$; likewise the curve γ_2 is called the unstable manifold and is denoted by $W'^u(X^*)$. Note that for the linear map $A = Df(X^*)$, the stable manifold is the line in the direction of the eigenvector V_1, whereas the unstable manifold is the line in the direction of the eigenvector V_2. Moreover, at the origin, the stable

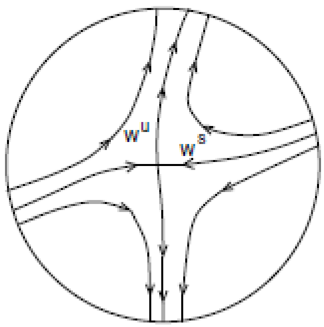


FIGURE 4.14
The stable manifold W^s and the unstable manifold W^u.

(unstable) manifold of the map f is tangent to the eigenvector $V_1(V_2)$ (see Fig. 4.14)

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