Mila University Center — Academic Year 2024/2025

Institute of Mathematics and Computer Science — Field: MI-Mathematics (1^{st} Year)

Analysis I — Semester I

Exercise Series N°01

Exercise 1:

Let $A, B \subset \mathbb{R}$ be two non-empty bounded sets. We define: $-A = \{-x \ / \ x \in A\}$.

1) Show that: $\sup(-A) = -\inf(A)$ and $\inf(-A) = -\sup(A)$

2) Show that if for all $a \in A$ and $b \in B$ we have $a \leq b$, then $\sup A \leq \inf B$.

3) Prove that $A \cup B$ is a bounded subset of $\mathbb{R}$ and

a) $\sup(A \cup B) = \max(\sup A, \sup B)$.

b) $\inf(A \cup B) = \min(\inf A, \inf B)$. (*)

4) Prove that if $A + B = \{z = x + y;\ x \in A,\ y \in B\}$ and $A - B = \{z = x - y; x \in A, y \in B\}$

then: a) $\sup(A + B) = \sup A + \sup B$.

b) $\inf(A + B) = \inf A + \inf B$.

c) $\sup(A - B) = \sup A - \inf B$.

d) $\inf(A - B) = \inf A - \sup B$. (*)

Exercise 2:

1) Show that if $r \in \mathbb{Q}$ and $x \notin \mathbb{Q}$ then $r + x \notin \mathbb{Q}$; and if $r \neq 0$ then $r.x \notin \mathbb{Q}$.

2) Prove that $\sqrt{2} \notin \mathbb{Q}$.

3) Assume that $\sqrt{2}$ and $\sqrt{3}$ are irrational. Show that $\sqrt{2} + \sqrt{3} \notin \mathbb{Q}$.

4) Show that $\dfrac{\ln 3}{\ln 2}$ is irrational..

Exercice 3:

1) Prove that:

a) $|x + y| \leq |x| + |y|, \quad \forall x, y \in \mathbb{R}$.

b) $|x - y| \geq ||x| - |y||, \quad \forall x, y \in \mathbb{R}$. (*)

Exercise 4:

Let A and B be two subsets of $\mathbb{R}$ such that $B \subset A$. Show that:

1) A is bounded $\Longrightarrow B$ is bounded.

2) $\inf(A) \leq \inf(B)$ and $\sup(A) \geq \sup(B)$.

Exercise 5:

Let $A = \left\{ a_n \in \mathbb{R} / \ a_n = \dfrac{n+3}{\frac{n}{4}+1}; n \in \mathbb{N} \right\}$ and $B = \left\{ b_n \in \mathbb{R} / \ b_n = \dfrac{1}{n^2} + \dfrac{2}{n} + 4; n \in \mathbb{N}^* \right\}$

1) Prove that A and B are bounded in $\mathbb{R}$ and that: $\sup(A) = \inf(B)$.

2) Determine $\sup(B)$ and $\inf(A)$.

Exercise 6:

Determine the supremum and infimum, if they exist, of the sets:

$A = \left\{ \sin(\frac{2n\pi}{7}); n \in \mathbb{Z} \right\}$; $B = \{\alpha x + \beta;\ x \in [-2, 1],\ \alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}\}$;

$C = [0, 1] \cap \mathbb{Q}$; $D = \left\{ \frac{n-1}{2n+1}, n \in \mathbb{N}^* \right\}$; $F = \left\{ (-1)^n + \frac{1}{n}; n \in \mathbb{N}^* \right\}$.

Exercise 7:

1) Express the following numbers in the form $a + ib$ (where $a, b \in \mathbb{R}$):

$z_1 = \frac{3+6i}{3-4i}$; $z_2 = \frac{2}{1-i\sqrt{3}}$; $z_3 = \left(\frac{1+i}{2-i}\right)^2 + \frac{3+6i}{3-4i}$; $z_4 = \frac{2+5i}{1-i} + \frac{2-5i}{1+i}$ (*)

2) Solve the following equations in $\mathbb{C}$:

$z^2 + z + 1 = 0$; $z^6 = 1$; $z^2 = \frac{1}{4}(-1+i)$ (*) ; $z^2 - \sqrt{3}z - i = 0$ (*).

Exercise 8:

Let z be a complex number with modulus ρ and argument θ, and let $\overline{z}$ be its conjugate.

Calculate the expression $(z + \overline{z})(z^2 + \overline{z}^2) \ldots\ldots (z^n + \overline{z}^n)$ in terms of ρ and θ.

Application:

Let α and $\overline{\alpha}$ be the solutions of the equation $z^2 - 2z + 2 = 0$.

- Find the trigonometric form of α and $\overline{\alpha}$.
- Show that: $\prod_{k=1}^{n} \left(\alpha^k + \overline{\alpha}^k\right) = 0; \quad \forall n \geq 2$.

Exercise 9:

1) Let $z \in \mathbb{C}$, Calculate the sum:

$$S_n = 1 + z + z^2 + \ldots.. + z^n.$$

2) - Solve the equation $z^n = 1$ in $\mathbb{C}$.

- Show that the roots can be expressed in the form $1; \alpha; \alpha^2; \ldots\ldots; \alpha^{n-1}$.
- Deduce the roots of the equation:

$$P(z) = 1 + z + z^2 + \ldots.. + z^{n-1} = 0.$$

3) Calculate for $p \in \mathbb{N}$;

$$Q(z) = 1 + z^p + z^{2p} + \ldots\ldots + z^{(n-1)p}.$$

Exercise 10:

1) Calculate the sum: $S(x) = \sum_{p=0}^{n-1} e^{ipx}; p \in \mathbb{N}, x \in \mathbb{R}$.

2) Deduce the following sums:

$$S_1(x) = \sum_{p=0}^{n-1} \cos(px) \quad (*) \qquad ; \qquad S_2(x) = \sum_{p=0}^{n-1} \sin(px) \quad (*).$$

N B: *The exercises (*) are left to the students.*