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Chapter 1

Field of real numbers

1.1 Definition and usual properties of the field of real numbers

Definition 1.1.1 *The field of real numbers* $(\mathbb{R}, +, \times, \leq)$ *is the set of real numbers under the two operations of addition and multiplication, with an ordering* $\leq$ *compatible with the ring structure of* $\mathbb{R}$.

The field of real numbers $\mathbb{R}$ has the following usual properties

1. $\mathbb{R}$ is a commutative field; i.e

1.1. Addition and multiplication are both commutative, which means that

$$x + y = y + x \text{ and } x \times y = y \times x, \text{ for every real numbers } x \text{ and } y.$$

1.2. Addition and multiplication are both associative, which means that

$$(x + y)+z = x+(y + z) \text{ and } (x \times y)\times z = x\times(y \times z), \text{ for every real numbers } x, y \text{ and } z.$$

1.3. There is a real number called zero and denoted 0 which is an additive identity, which means that

$$x + 0 = 0 + x = x, \text{ for every real number } x.$$

1.4. There is a real number denoted 1 which is a multiplicative identity, which means that

$$1 \times x = x \times 1 = x, \text{ for every real number } x.$$

1.5. Every real number a has an additive inverse denoted $-x$. This means that

$$x + (-x) = 0, \text{ for every real number } x.$$

1.6. Every nonzero real number a has a multiplicative inverse denoted x^{-1} or $\frac{1}{x}$. This means that

$$x \times x^{-1} = 1, \text{ for every nonzero real number } x.$$

1.7. Multiplication is distributive over addition, which means that

$$x \times (y + z) = x \times y + x \times z, \text{ for every real numbers } x, \ y \text{ and } z.$$

2. The field $\mathbb{R}$ is ordered, meaning that there is a total order $\leq$ such that

2.1. *Reflexive relationship; i.e* $x \leq x$, for every real number x.

2.2. Antisymmetric relationship; i.e $x \leq y \ \wedge \ y \leq x \Longrightarrow x = y$, for every real numbers x and y.

2.3. Transitive relationship $x \leq y \ \wedge \ y \leq z \Longrightarrow x \leq z$, for every real numbers x, y and z.

3. Many other properties can be deduced from the above ones. In particular

3.1. For all $x, y \in \mathbb{R}; x \geq 0 \wedge y \geq 0 \Longrightarrow x.y \geq 0$.

3.2. For all $x, y \in \mathbb{R}; (x \leq y)$ or $(y \leq x)$.

3.3. For all $x, y, z \in \mathbb{R}; x \leq y \Longrightarrow x + z \leq y + z$.

3.4. For all $n, m \in \mathbb{N}$ and for all $x \in \mathbb{R}_+ : \begin{cases} x \leq 1 \text{ and } n \leq m \Longrightarrow x^n \geq x^m. \\ x \geq 1 \text{ and } n \leq m \Longrightarrow x^n \leq x^m. \end{cases}$

3.5. For all $x, y \in \mathbb{R}^* : \begin{cases} 0 < x \leq y \iff 0 < \frac{1}{y} \leq \frac{1}{x}. \\ x \leq y < 0 \iff \frac{1}{y} \leq \frac{1}{x} < 0. \\ x < 0 < y \iff \frac{1}{x} < 0 < \frac{1}{y}. \end{cases}$

1.2 Newton's binomial formula

Proposition 1.2.1 *Let x and y be two real numbers and n be a non-zero natural number. The Newton's binomial formula is given by*

$$(x+y)^n = \sum_{k=0}^{n} C_n^k x^k y^{n-k} \quad \textit{where} \quad C_n^k = \frac{n!}{k!\,(n-k)!}, \quad 1! = 1 \quad \textit{and} \quad 0! = 1.$$

Notation 1.2.1 *:*

$$\mathbb{R}^* = \mathbb{R} - \{0\} \qquad , \qquad \mathbb{R}_+^* = \{x \in \mathbb{R};\ x > 0\} \qquad , \qquad \mathbb{R}_-^* = \{x \in \mathbb{R};\ x < 0\}\,.$$

$$\mathbb{R}_+ = \{x \in \mathbb{R};\ x \geq 0\} \qquad , \qquad \mathbb{R}_- = \{x \in \mathbb{R};\ x \leq 0\}\,.$$

1.3 Intervals of $\mathbb{R}$

Let $a, b \in \mathbb{R}$ be two real numbers, such that $a < b$. The only intervals of $\mathbb{R}$ are

1. $\mathbb{R} = \left]-\infty, +\infty\right[$.
2. ϕ : the empty set.
3. $[a,b] = \{x \in \mathbb{R};\ a \leq x \leq b\}$.
4. $]a,b[= \{x \in \mathbb{R};\ a < x < b\}$.
5. $]a,b] = \{x \in \mathbb{R};\ a < x \leq b\}$.
6. $[a,b[= \{x \in \mathbb{R};\ a \leq x < b\}$.
7. $[a,+\infty[= \{x \in \mathbb{R}; x \geq a\}$.
8. $]a,+\infty[= \{x \in \mathbb{R};\ x > a\}$.
9. $]-\infty,b] = \{x \in \mathbb{R};\ x \leq b\}$.
10. $]-\infty,b[= \{x \in \mathbb{R};\ x < b\}$.

1.4 Completed number line $\overline{\mathbb{R}}$ (Extension of $\mathbb{R}$)

Definition 1.4.1 *We call the completed number line* $\overline{\mathbb{R}}$ *the set* $\mathbb{R} \cup \{-\infty, +\infty\}$.

1.5 Operations on $\overline{\mathbb{R}}$

1. $(+\infty) + (+\infty) = +\infty$ and $(-\infty) + (-\infty) = -\infty$.

2. For all $x \in \mathbb{R}$, we have $x + (+\infty) = +\infty$ and $x + (-\infty) = -\infty$.

3. $(+\infty) \times (+\infty) = +\infty$, $(-\infty) \times (-\infty) = +\infty$, $(+\infty) \times (-\infty) = -\infty$.

4. For all $x \in \mathbb{R}_-^*$, we have $x \times (+\infty) = -\infty$ and $x \times (-\infty) = +\infty$.

5. For all $x \in \mathbb{R}_+^*$, we have $x \times (+\infty) = +\infty$ and $x \times (-\infty) = -\infty$.

1.6 Indeterminate forms

It may often be possible to simply add, subtract, multiply, divide or exponentiate the corresponding limits of two functions. However, there are occasions where it is unclear what the sum, difference, product or power of these two limits ought to be. For example, it is unclear what the following expressions ought to evaluate to

$$\frac{0}{0}, \ \frac{\infty}{\infty}, \ \ 0 \times \infty, \ \ (+\infty) + (-\infty).$$

1.7 Bounded subset, upper and lower bounds

1. A non-empty subset $A\,(A \subseteq \mathbb{R})$ *is said to be bounded above if:*

$$\exists M \in \mathbb{R}, \forall x \in A; x \leq M.$$

In this case, the numbre M is called an upper bound of A; i.e.

$$M \in \mathbb{R} \text{ is an upper bound of } A \Longleftrightarrow \forall x \in A, x \leq M.$$

2. A non-empty subset $A\,(A \subseteq \mathbb{R})$ *is said to be bounded below if:*

$$\exists m \in \mathbb{R}, \forall x \in A; x \geq m.$$

In this case, the numbre m is called a lower bound of A; i.e.

$$m \in \mathbb{R} \text{ is a lower bound of } A \Longleftrightarrow \forall x \in A, x \geq m.$$

3. A non-empty subset $A\,(A \subseteq \mathbb{R})$ is said to be bounded if and only if it is *bounded above and below*; i.e.

$$A \subseteq \mathbb{R} \;\text{ is bounded} \Longleftrightarrow \exists m, M \in \mathbb{R}, \forall x \in A;\; m \leq x \leq M.$$

4. A non-empty subset $A\,(A \subseteq \mathbb{R})$ that is not bounded is said to be unbounded.

1.7.1 Supremum and infimum

1. Let $A \subseteq \mathbb{R}$ be bounded above. The supremum of A (abreviated $\sup A$) is the **least upper bound** of A that is greater than or equal to each element of A (i.e $\forall x \in A$; $\sup A \geqslant x$).

2. Let $A \subseteq \mathbb{R}$ be bounded below. The infimum of A (abreviated $\inf A$) is the **greatest lower bound** of A that is less than or equal to each element of A (i.e $\forall x \in A$; $\inf A \leq x$).

1. Any nonempty subset A of $\mathbb{R}$ and bounded above admits an least upper bound ($\sup A$).

2. Any nonempty subset A of $\mathbb{R}$ and bounded below admits a greatest lower bound ($\inf A$).

Remark 1.7.1 *Denote $A = \{x \in \mathbb{R}; x \in A\}$ and $-A = \{x \in \mathbb{R}; -x \in A\}$. We have*

$$\begin{aligned} \sup(A) = M &\Longleftrightarrow \inf(-A) = -M \\ &\Longleftrightarrow \inf(-A) = -\sup(A) \\ &\Longleftrightarrow \sup(-A) = -\inf(A). \end{aligned}$$

1. The upper bounds of $A =]-2, 8]$ is $[8, +\infty[$, thus $\sup A = 8$.

2. The lower bounds of $A =]-2,8]$ is $]-\infty,-2]$, thus $\inf A = -2$.

3. The upper bounds of $A =]-\infty,3]$ is $[3,+\infty[$, thus $\sup A = 3$ and $\inf A$ does not exist.

4. The lower bounds of $A =]1,+\infty[$ is $]-\infty,1]$, thus $\inf A = 1$ and $\sup A$ does not exist.

1.7.2 Upper bound and lower bound characteristic properties

Theorem 1.7.1 *Let $A \subset \mathbb{R}$ be non-empty and let $M \in \mathbb{R}$.*

$$M = \sup A \Longleftrightarrow \begin{cases} 1) & \forall x \in A;\ x \leq M, \\ 2) & \forall \varepsilon > 0,\ \exists x \in A; M - \varepsilon < x. \end{cases}$$

Theorem 1.7.2 *Let $A \subset \mathbb{R}$ be non-empty and let $m \in \mathbb{R}$.*

$$m = \inf A \Longleftrightarrow \begin{cases} 1) & \forall x \in A;\ x \geq m, \\ 2) & \forall \varepsilon > 0,\ \exists x \in A;\ m + \varepsilon > x. \end{cases}$$

1.7.3 Maximal and minimal elements

Definition 1.7.1 *:*

1. Let $A \subset \mathbb{R}$ be non-empty and let $M \in \mathbb{R}$ be an upper bound of A, if $M \in A$ then we say that M is maximal of A and we note $\max A$.

2. Let $A \subset \mathbb{R}$ be non-empty and let $m \in \mathbb{R}$ be a lower bound of A, if $m \in A$ then we say that m is minimal of A and we note $\min A$

Remark 1.7.2 *:*

- *If* $\max A$ *exist, then* $\sup A$ *exist and* $\max A = \sup A$.
- *If* $\min A$ *exist, then* $\inf A$ *exist and* $\min A = \inf A$.
- *If* $\sup A$ *exist and* $\sup A \notin A$, *then* $\max A$ *does not exist.*
- *If* $\inf A$ *exist and* $\inf A \notin A$, *then* $\min A$ *does not exist.*

Example 1.7.1 *:*

1) Let $A =]-\infty, 3]$. *We have* $\sup A = 3$ *and* $3 \in A$*, then* $\max A = 3$.

2) Let $A = [-1, +\infty[$. We have $\inf A = -1$ *and* $-1 \in A$*, then* $\min A = -1$.

3) $A =]-2, 5[$. We have $\sup A = 5 \notin A$ *and* $\inf A = -2 \notin A$*, then* $\min A$ *and* $\max A$ *do not exist*

1.8 Archimedes axiom

For all $x \in \mathbb{R}$, it exist $n \in \mathbb{N}$ *such that* $n > x$; (i.e. the set $\mathbb{N}$ is not *bounded from above* in $\mathbb{R}$).

1.9 Rational and irrational numbers

Definition 1.9.1 *We note by* $\mathbb{Q} = \left\{ \frac{a}{b};\ a \in \mathbb{Z}\ ,\ b \in \mathbb{Z}^* \right\}$ *and* $\mathbb{Q}^* = \mathbb{Q} - \{0\}$. *The elements of* $\mathbb{Q}$ *are called rational numbers*

- *The set* $\mathbb{Q}$ *which contains* $\mathbb{Z}$*, is stable for the* $+$ *and* $\times$ laws.
- Provided with the restrictions of these laws; it is itself a commutative field.
- In particular the inverse of any element of $\mathbb{Q}^*$ *is still in* $\mathbb{Q}^*$.

Definition 1.9.2 *The elemnts of* $\mathbb{R} \backslash \mathbb{Q}$ *are called irrational numbers.*

1.9.1 Density of $\mathbb{Q}$ in $\mathbb{R}$

Theorem 1.9.1 *Let* x, y *be two real numbers, such that* $x < y$*; there exists a rational number* $q \in \mathbb{Q}$*, such that* $x < q < y$*; i.e. between two real numbers there is always a rational number. We translate this property by saying that* $\mathbb{Q}$ *is dense in* $\mathbb{R}$*.*

1.10 *Absolute value*

Definition 1.10.1 *We call absolute value of a real number x any application $|x| : \mathbb{R} \longrightarrow \mathbb{R}_+$ defined by*

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } \ x \leq 0, \end{cases}$$

which satisfied the following properties

1) $|x| = 0 \Leftrightarrow x = 0$, for every real number x.

2) $|x + y| \leq |x| + |y|$, for every real numbers x and y.

3) $|x - y| \geq ||x| - |y||$, for every real numbers x and y.

4) $|x \times y| = |x| \times |y|$, for every real numbers x and y.

1.11 Integer part of a real number

For all $x \in \mathbb{R}$ there exists an unique integer $(n \in \mathbb{Z})$ noted $E(x)$ (or $[x]$) which is called integer part of x, satisfying:

$$\forall x \in \mathbb{R}, E(x) \leq x \leq E(x) + 1.$$

In other words $E(x)$ is the largest integer less than or equal to x $(n \leq x \leq n + 1; n \in \mathbb{Z})$. Especially, *if* $x \in \mathbb{Z}$, then $E(x) = x$.

Example 1.11.1 *:*

$E(0.3) = 0; \quad (0 \leq 0.3 \leq 0 + 1 = 1)$.
$E(3.3) = 3; \quad (3 \leq 3.3 \leq 3 + 1 = 4)$.
$E(-1.5) = -2; \quad (-2 \leq -1.5 \leq -2 + 1 = -1)$.
$E(-4) = -4$.
$E(5) = 5$.
Let $A = \left\{ x_n = \frac{1}{2} + \frac{n}{2n+1}; \ n \in \mathbb{N} \right\}$.

1. Prouve that: $\forall x_n \in A, \frac{1}{2} \leq x_n < 1$.

2. Find $\inf(A)$ and $\sup(A)$.

3. Show that: $\sup(A) = 1$.

1. We proue that $\frac{1}{2} \leq x_n < 1, \forall n \in \mathbb{N}$.

   We have $\forall n \in \mathbb{N}, x_n = \frac{1}{2} + \frac{n}{2n+1}$. So

$$\begin{aligned} \forall n \in \mathbb{N}, 0 \leq 2n < 2n+1, \quad &\Longleftrightarrow \quad 0 \leq \frac{2n}{2n+1} < 1, \forall n \in \mathbb{N} \\ &\Longleftrightarrow \quad 0 \leq \frac{n}{2n+1} < \frac{1}{2}, \forall n \in \mathbb{N} \\ &\Longleftrightarrow \quad \frac{1}{2} \leq \frac{1}{2} + \frac{2n}{2n+1} < 1, \forall n \in \mathbb{N} \end{aligned}$$

   then

$$\frac{1}{2} \leq x_n < 1, \forall n \in \mathbb{N}.$$

2. *Since* $\frac{1}{2} \leq x_n < 1$, *then* A *is bounded; i.e.* $\inf A$ *and* $\sup A$ *exist.*

   $x_0 = \frac{1}{2}$ *is the a lower bound of* A *and* $x_0 = \frac{1}{2} \in A$, *thus* $\min A = \frac{1}{2}$, *this means that* $\inf A = \frac{1}{2}$.

   And 1 is the least upper bound of A, so $\sup A = 1$.

3. We will now show that $\sup A = 1$. To do this, we must check the second property of supremum of the set A.

   Let $x_n = \frac{1}{2} + \frac{n}{2n+1} > 1 - \varepsilon$ and we find n as a function of ε.

We have

$$\begin{aligned}
x_n = \frac{1}{2} + \frac{n}{2n+1} > 1 - \varepsilon \quad &\Longrightarrow \quad -\frac{1}{2} + \frac{n}{2n+1} > -\varepsilon \\
&\Longrightarrow \quad \frac{1}{2} - \frac{n}{2n+1} < \varepsilon \\
&\Longrightarrow \quad \frac{2n+1-2n}{2\left(2n+1\right)} < \varepsilon \\
&\Longrightarrow \quad \frac{1}{2\left(2n+1\right)} < \varepsilon \\
&\Longrightarrow \quad \frac{1}{2n+1} < 2\varepsilon \\
&\Longrightarrow \quad 2n+1 > \frac{1}{2\varepsilon} \\
&\Longrightarrow \quad 2n > \frac{1}{2\varepsilon} - 1 \\
&\Longrightarrow \quad n > \frac{1}{4\varepsilon} - \frac{1}{2}
\end{aligned}$$

thus

$$\exists n = E\left(\frac{1}{4\varepsilon} - \frac{1}{2}\right) + 1, \text{ such that } x_n > 1 - \varepsilon,$$

which implies that $\sup A = 1$.

Chapter 2

Field of complex numbers

2.1 Definitions and general notions

The set of complex numbers $\mathbb{C}$ is written in the form

$$\mathbb{C} = z = x + iy; \quad x, y \in \mathbb{R}, \quad i^2 = -1,$$

wher $x = Re(z)$ is the real part of z and $y = Im(z)$is its imaginary part.

If $y = 0$, z is said to be real., and ff $x = 0$, z is said to be imaginary.

Example 2.1.1 *:*

$$1 + 3i \quad , \quad \sqrt{2} - \pi i \quad , \quad 4 \quad , \quad i \quad , \quad \sqrt{6}i \quad , \quad \frac{2}{3}i \quad , \quad 1 - \frac{4}{3}i$$

2. *Let $z = x + iy$ be a complex nuber. The number $\overline{z} = x - iy$ is called the conjugate of z.*

We have the following properties
$\overline{z + z'} = \overline{z} + \overline{z'}$; $\quad \overline{z.z'} = \overline{z}.\overline{z'}$; $\quad z - \overline{z} = 2yi$;
$z - \overline{z} = 2x$ $\quad \overline{\overline{z}} = z$.

2.2 Module of a complex number

3. We call module of a complex number z the positive real number $|z|$ defined by

$$|z| = \sqrt{z.\overline{z}} = \sqrt{x^2 + y^2}.$$

The number $|z|$ atisfies the following properties. For all $z, z' \in \mathbb{C}$, *we have*

a) $|z| \geq 0,$ *and* $|z| = 0 \Leftrightarrow z = 0.$

b) $|z.z'| = |z| . |z'| .$

c) $|z + z'| \leq |z| + |z'| .$

d) $\left|\frac{z}{z'}\right| = \frac{|z|}{|z'|};\ \ z' \neq 0.$

e) $\left|\frac{1}{z}\right| = \frac{1}{|z|};\ \ z \neq 0.$

2.3 Argument of a complex number

Let $z = x + iy$ be a nonzero complex numbers.*On appelle argument de* z *le nombre réel* θ *défini d'un multiple entier de* 2π *près par.*

$\cos\theta = \frac{x}{|z|};\ \ \sin\theta = \frac{y}{|z|}$ *on note* $\theta = \arg(z)$.

i.e: $\arg(z) = \cos\theta = \frac{x}{|z|}\sin\theta = \frac{y}{|z|}$

Proposition 2.3.1 *:*

Let z_1 *and* z_2 *be two complex numbers. We have*

- $\arg(z_1 \times z_2) = \arg(z_1) + \arg(z_2)$.
- $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$.

2.4 Tregonometric form of a complex number

Let $z = x + iy;\ \ |z| = \sqrt{x^2 + y^2}$ *and* $\theta = \arg(z)$.

We have $x = |z|\cos\theta, \quad y = |z|\sin\theta$. *Thus*

$$\begin{aligned} z &= x + iy = |z|\cos\theta + i\,|z|\sin\theta \\ &= |z|\,(\cos\theta + i\sin\theta) \end{aligned}$$

.

The number z is entirely determined by its modulus and its argument. The trigonometric form of z is then given by

$$\begin{aligned} z &= [|z|\,;\theta] = |z|\,(\cos\theta + i\sin\theta) \\ &= |z|\,e^{i\theta}. \end{aligned}$$

This representation is very useful for multiplication and division of complex numbers; i.e

$$\begin{aligned} z_1 \times z_2 &= |z_1|\,e^{i\theta_1} \times |z_2|\,e^{i\theta_2} \\ &= |z_1| \times |z_2|\,e^{i(\theta_1+\theta_2)}. \end{aligned}$$

and

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{|z_1|\,e^{i\theta_1}}{|z_2|\,e^{i\theta_2}} \\ &= \frac{|z_1|}{|z_2|}e^{i(\theta_1-\theta_2)}. \end{aligned}$$

As an immediate application, we have the following relation (Moivre's formula)

$$\left(e^{i\theta}\right)^n = e^{in\theta} \Longleftrightarrow (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta,$$

where

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \quad ; \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}, \quad \theta \in \mathbb{R}.$$

2.5 Application

By expanding the Moivre's formula using Newton's binomial formula, and identifying the real and imaginary parts of polynomials as a function of $\cos\theta$ and $\sin\theta$, we ge

$$(\cos\theta + i\sin\theta)^n = \sum_{k=0}^{n} C_n^k\,(\cos\theta)^{n-k}\,(i\sin\theta)^k,$$

$$where\ C_n^k = \frac{n!}{k!\,(n-k)!},\ C_n^n = \frac{n!}{n!0!} = 1\ ,\ \ C_n^0 = \frac{n!}{0!n!} = 1\quad and\ 0! = 1.\ \text{Thus}$$

$$\begin{aligned}(\cos\theta + i\sin\theta)^n &= C_n^0 (\cos\theta)^n (i\sin\theta)^0 + C_n^1 (\cos\theta)^{n-1} (i\sin\theta)^1 + ... + C_n^k (\cos\theta)^{n-k} (i\sin\theta)^k \\ &\quad +... + C_n^n (\cos\theta)^0 (i\sin\theta)^n\end{aligned}$$

Wich implies that

$$\cos n\theta = (\cos\theta)^n - C_n^2 (\cos\theta)^{n-2} (\sin\theta)^2 + C_n^4 (\cos\theta)^{n-4} (\sin\theta)^4 + ...$$
$$\sin n\theta = C_n^1 (\cos\theta)^{n-1} (\sin\theta)^1 - C_n^3 (\cos\theta)^{n-3} (\sin\theta)^3 + ...$$

So

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

- For $n = 2$, we have

$$\begin{aligned}(\cos\theta + i\sin\theta)^2 &= \sum_{k=0}^{2} C_2^k (\cos\theta)^{2-k} (i\sin\theta)^k \\ &= C_2^0 (\cos\theta)^2 (i\sin\theta)^0 + C_2^1 (\cos\theta)^1 (i\sin\theta)^1 \\ &\quad + C_2^2 (\cos\theta)^0 (i\sin\theta)^2 \\ &= \underbrace{C_2^0 \cos^2\theta - C_2^2 \sin^2\theta}_{\cos 2\theta} + i\underbrace{C_2^1 \cos\theta\sin\theta}_{\sin 2\theta} \\ &= \cos 2\theta + i\sin 2\theta.\end{aligned}$$

2.6 $n - th$ roots of a complex number

Let $z = r(\cos\theta + i\sin\theta) = r\exp(i\theta)$be a complex number and n an integer such that $n \geq 1$.

We are looking for all the complex numbers $w = \rho(\cos\delta + i\sin\delta) = \rho\exp(i\delta)$ that satisfy$w^n = z$. *We have*

$$\begin{aligned}\rho^n \exp(in\delta) &= r\exp(i\theta) \\ &\Leftrightarrow\ \rho^n = r \text{ and } n\delta = \theta + 2k\pi,\quad k \in \mathbb{Z} \\ &\Leftrightarrow\ \rho = \sqrt[n]{r} \text{ and } \delta = \frac{\theta}{n} + \frac{2k\pi}{n},\quad k \in \mathbb{Z}.\end{aligned}$$

Hence the $n-$th roots of z are

$$z_k = \sqrt[n]{r}\left(\cos\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)+i\sin\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)\right); \qquad 0 \le k \le n-1$$

Special case: the $n-th$ roots of $z = 1$ are

$$z_k = \cos\left(\frac{2k\pi}{n}\right)+i\sin\left(\frac{2k\pi}{n}\right); \qquad 0 \le k \le n-1$$