Lectures: Limited Developments (Taylor expansions)

By Dr. Hocine RANDJI

Abdelhafid Boussouf University Center- Mila- Algeria

Institute of Science and Technology

First Year Engineering

Module: Analysis 2

Semester 2

randji.h@centre-univ-mila.dz

1.0

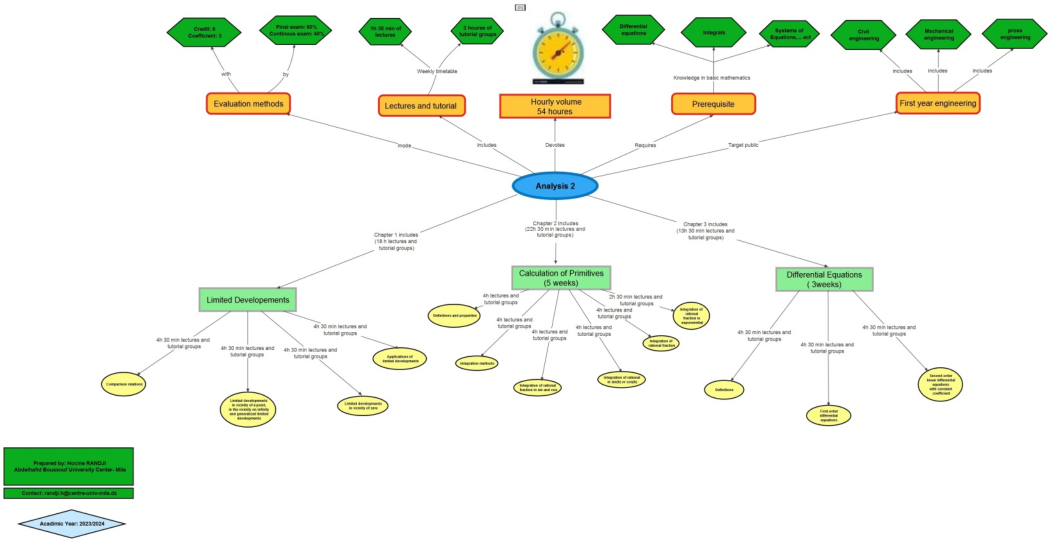
March 2024

Table of contents

Objectives

- ***Remember*** the basic concepts of integrals and derivation.
- ***Understand and Apply Key Concepts:*** Grasp fundamental concepts related to infinitesimals and Taylor series, including definitions, theorems, and their applications. Develop the ability to identify and compare different orders of infinitesimals, and apply Taylor's formula to approximate functions.
- ***Analyze and Solve Problems***: Engage in problem-solving exercises that involve comparing infinitesimals, determining their equivalence, and using Taylor series for function approximation. Analyze the behavior of functions near specific points and evaluate the accuracy of polynomial approximations.
- ***Utilize Computational Tools***: Implement computational techniques, such as using Python to generate Taylor series expansions, to enhance understanding and practical application of theoretical concepts. This includes creating, verifying, and applying series expansions to solve complex problems

Introduction



Recommended prerequisite knowledge:

Basic mathematics concepts (differential equations, integrals, systems of equations, ...)

I Exercice

Find the integral of $f(x) = 3x^2 + 2x - 5$

- ☐ $x^3 + x - 5x^2 + C$
- ☐ x^3 + x^2 - 5x + C
- ☐ $x^3 + x^2 - 5x + C$
- ☐ $x^3 + x^2 - 5x + C$

II Exercice

Calculate the integral of $g(x) = \dfrac{1}{x^2}$

- ☐ $-\dfrac{1}{x} + C$
- ☐ $\dfrac{1}{x} + C$
- ☐ $-\dfrac{1}{2x} + C$
- ☐ $\dfrac{1}{2x} + C$

III Exercice

Find the derivative of $f(x) = 3x^2 + 2x - 5$

- ☐ $6x + 2x$
- ☐ $6x^2 + 2$
- ☐ $6x + 2$
- ☐ $6x^2 + 2x$

IV Exercice

Calculate the derivative of $g(x) = \frac{1}{x^2}$.

- ☐ $-\frac{1}{x^3}$
- ☐ $-\frac{2}{x^3}$
- ☐ $\frac{2}{x^3}$
- ☐ $-\frac{1}{2x^3}$

V Lecture 1: Comparison relations

Consider multiple infinitesimal quantities $\alpha, \beta, \gamma, \ldots$ which depend on the same variable x and tend towards zero as x approaches a certain limit, either a or infinity. The manner in which these variables approach zero will be examined when we analyze their ratios. We will adopt the following definitions.

1. Definitions

Definition: Def. 1:

If the ratio $\frac{\beta}{\alpha}$ has a finite non-zero limit, denoted by A, meaning $\lim \frac{\beta}{\alpha} = A \neq 0$, and consequently $\lim \frac{\alpha}{\beta} = \frac{1}{A} \neq 0$, then the infinitesimals β and α are known as infinitesimals of the same order.

Example:E.g 1:

Let us take $\alpha = x$ and $\beta = \sin(2x)$, then we have

$$\lim_{x \to 0} \frac{\beta}{\alpha} = \lim_{x \to 0} \frac{\sin 2x}{x} = 2.$$

So α and β are of the same order where $x \to 0$.

Example:E.g 2:

As x approaches zero, the quantities $x, \sin(3x), \tan(2x)$, and $7\ln(1+x)$ exhibit similar behavior, indicating they are of the same order.

Definition: Def. 2:

When the ratio of two infinitesimals α and β approaches zero; which means $\lim_{x \to 0} \frac{\beta}{\alpha} = 0 (or \lim_{x \to 0} \frac{\alpha}{\beta} = \infty)$, the infinitesimal β is considered of higher order than the infinitesimal α, while α is of `lower order` than β.

Example:E.g. 3:

Given $\alpha = x$ and $\beta = x^n$, where $n > 1$ and $x \to 0$, β is of a higher order compared to α since x^n approaches zero faster than x; so that we have $\lim\limits_{x \to 0} \frac{x^n}{x} = 0$. We can also say α is of a lower order compared to β.

Definition: Def. 3:

An infinitesimal β is considered of the k th order relative to another infinitesimal α if the infinitesimals α^k and β are of the same order.

Example:E.g. 4:

With $\alpha = x$ and $\beta = x^4$, β is a forth-order infinitesimal relative to α; since x^4 approaches zero faster than x, that is, $\lim\limits_{x \to 0} \frac{\beta}{\alpha^4} = \lim\limits_{x \to 0} \frac{x^4}{x^4} = 1$.

Definition: Def. 4:

Let α and β be two infinitesimals. We say that α is equivalent to β as x tends to x_0, denoted by $\alpha \sim \beta$, if $\lim \frac{\beta}{\alpha} = 1$.

Example:E.g.5:

As $x \to 0$ we have $\frac{1}{\sin x} \sim x$, $\tan x \sim x$, $e^x - 1 \sim x$, $\ln(x+1) \sim x$, $1 - \cos x \sim \frac{x^2}{2}$.

2. Theorems:

Fundamental:Theorem 1:

When comparing two infinitesimals α and β, if they are equivalent, their difference $\alpha - \beta$ is an infinitesimal with a higher order than both of them.

Proof:

By evaluating the limit

$$\lim \frac{\alpha - \beta}{\alpha} = \lim \left(1 - \frac{\beta}{\alpha}\right) = 1 - 1 = 0.$$

Fundamental:Theorem 1:

If the difference between two infinitesimals $\alpha - \beta$ is an infinitesimal of higher order than both α and β are equivalent infinitesimals.

Proof:

If $\lim \frac{\alpha - \beta}{\alpha} = 0 \iff 1 - \lim \frac{\beta}{\alpha} = 0$ so $1 = \lim \frac{\beta}{\alpha}$. Thus we have $\alpha \sim \beta$.

Example:E.g. 7:

Consider the infinitesimals $\alpha = x$ and $\beta = x + x^2$ where x approaches zero. Then we have

$$\lim_{x\to 0} \frac{\beta - \alpha}{\alpha} = \lim_{x\to 0} \frac{x^2}{x} = \lim_{x\to 0} x = 0$$

and

$$\lim_{x\to 0} \frac{\alpha - \beta}{\beta} = \lim_{x\to 0} \frac{-x^2}{x + x^2} = \lim_{x\to 0} \frac{-x}{1 + x} = 0$$

Note

If the ratio of two infinitesimals has no limit and does not approach infinity, then they are not comparable in the given context.

3. Equivalent of some known functions in vicinity of zero:

3.1. Table 1:

Function	Equivalent in vicinity of zero
$\sin(x)$	x
$1 - \cos(x)$	$\frac{x^2}{2}$
$\tan(x)$	x
$\ln(1 + x)$	x
$(1 + x)^a$	$1 + ax$
$(1 - x)^a$	$1 - ax$

4. Exercises

4.1. Exercise 01:

From the following functions (as $x \to 0$), what are those of the same order as x, and also of higher order and lower order than x:

1) x^2 2) $\sqrt{x(1-x)}$ 3) $\sin 3x$ 4) $2x \cos x \sqrt[3]{\tan^2 x}$ 5) xe^{2x}

4.2. Exercise 02:

From the following infinitesimal functions, what are those which equivalent to x as $x \to 0$:

1) $2\sin x$ 2) $\frac{1}{2}\tan 2x$ 3) $x - 3x^2$ 4) $\sqrt{2x^2 + x^3}$ 5) $\ln(1+x)$ 6) $x^3 + 3x^4$

4.3. Exercise 03:

Check that, the infinitesimal functions $1 - x$ and $1 - \sqrt[3]{x}$ are of the same order. Are they equivalent?

4.4. Exercise 04:

Using equivalent functions, calculate the following limits:

$$1) l_1 = \lim_{x \to 0} \frac{x \ln(x)}{\sin(2x)},$$

$$2) l_2 = \lim_{x \to 0} \frac{\ln(1 + \sin^2(x))}{\tan(\frac{x}{2})},$$

$$3) l_3 = \lim_{x \to \pi/2} e^{\frac{1}{2x - \pi} \ln(\sin(x))}$$

$$4) l_4 = \lim_{x \to +\infty} \left(\frac{\ln(1 + x)}{\ln(x)}\right)^{x \ln(x)}$$

VI Lecture 2: Taylor's Formula

Expressing functions as infinite polynomials, known as power series, is a valuable technique. Polynomial functions simplify analysis as they only entail basic arithmetic operations. If a complex function can be represented as an infinite polynomial, we can utilize this representation for differentiation, integration, and approximation. Thus, the crucial inquiry arises: under what conditions can a function be represented by a power series? To delve into this question, let's revisit the concept of geometric series.

1. Illustrative example:

Let us consider the following geometric series:

$$1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{+\infty} x^n$$

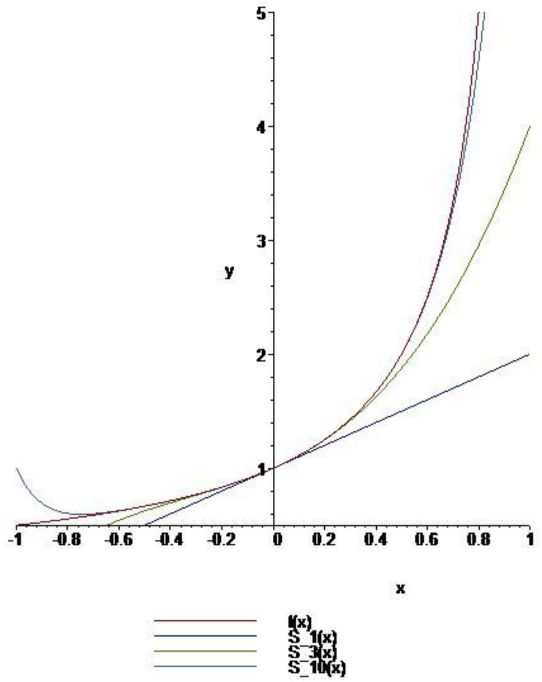
We recall that

$$a + ar + ar + ar^2 + ar^3 + \ldots$$

is convergent if and only if : $|r| < 1$. In our case, we have

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

So in this example, we have written the function $f(x) = \dfrac{1}{1-x}$ as infinite polynomial



The above Fig represents the graphs of

$$f(x) = \frac{1}{1-x}$$

$$S_1(x) = 1 + x$$

$$S_3(x) = 1 + x + x^2 + x^3$$

and

$$S_{10} = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10}$$

2. An overview on Taylor's formula::

Let's suppose that the function $y = f(x)$ possesses derivatives up to the (n + 1)th order within an interval containing $x = a$. We aim to find a polynomial $y = P_n(x)$ of degree not exceeding n, such that at $x = a$, its value matches $f(a)$ and its derivatives up to the nth order match those of f(x) at that point: $P_n(a) = f(a), P_n'(a) = f'(a), P_n''(a) = f''(a), \ldots, P_n^{(n)}(a) = f^{(n)}(a)$. It is reasonable to assume that such a polynomial approximates f(x) closely. We seek this polynomial in the form of a polynomial in terms of (x-a) with coefficients to be determined. We determine the coefficients $C_1, C_2, \ldots, C$_n to satisfy the given conditions.

Let's start by finding the derivatives of the polynomial $P_n(x)$ with respect to x:

$$P_n'(x) = C_1 + 2C_2(x-a) + 3C_3(x-a)^2 + \cdots + nC_n(x-a)^{n-1}$$

$$P_n''(x) = 2C_2 + 3 \cdot 2C_3(x-a) + \cdots + n(n-1)C_n(x-a)^{n-2}$$

By substituting the value of a for x and replacing $P_n'(a)$ with $f'(a)$, etc., we derive:

$$f(a) = C_0$$

$$f'(a) = C_1$$

$f''(a) = 2 \cdot 1 \cdot C_2$

$f'''(a) = 3 \cdot 2 \cdot 1 \cdot C_3$

$\ldots$

$f^{(n)}(a) = n(n-1)(n-2)\cdots 2 \cdot 1 \cdot C_n$

From which we obtain:

$C_0 = f(a)$

$C_1 = f'(a)$

$C_2 = \dfrac{f''(a)}{2\dot{1}}$

$\ldots$

$C_n = \dfrac{f^{(n)}(a)}{n(n-1)(n-2)\ldots 2\dot{1}}$

Substituting the found values of $C_1, C_2, \ldots, C_n$ into the polynomial $P_n(x)$, we obtain:

$$P_n(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a)$$

Let $R_n(x)$ denote the difference between the function f(x) and the constructed polynomial $P_n(x)$:

$R_n(x) = f(x) - P_n(x)$

Therefore:

$f(x) = P_n(x) + R_n(x)$

Expanding this expression, we get:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + R_n(x)$$

The term $R_n(x)$ is called the remainder. For those values of x where the remainder $R_n(x)$ is small, the polynomial $P_n(x)$ yields an approximate representation of the function f(x) . Thus, we can replace the function $y = f(x)$ by the polynomial $y = P_n(x)$ to an appropriate degree of accuracy equal to the value of the remainder $R_n(x)$.

The Reminder can be written; for example, in the following forms:

Lagrange's form: $R_n = \dfrac{f^{(n+1)}(\xi)(x-a)^{n+1}}{(n+1)!}$

Cauchy's form: $R_n = \dfrac{f^{(n+1)}(\xi)(x-\xi)^n(x-a)}{n!}$

where $a < \xi < x$.

We can also use the Taylor-Young formula :

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + x^n\epsilon(x)$$

where $\lim\limits_{x\to 0} \epsilon(x) = 0$, and in Landau notation we can write $x^n\epsilon(x) = o(x^n)$.

Example

The Taylor series expansion of the sine function, \sin(x) , around a point a is given by:

$$\sin(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Where $f^{(n)}(a)$ denotes the n -th derivative of $\sin(x)$ evaluated at a .

To derive the Taylor series expansion of \sin(x) , let's first calculate the derivatives of $\sin(x)$:

1. $f(x) = \sin(x)$

2. $f'(x) = \cos(x)$

3. $f''(x) = -\sin(x)$

4. $f'''(x) = -\cos(x)$

5. $f^{(4)}(x) = \sin(x)$

and so on.

Now, let's evaluate these derivatives at $a = 0$, because $\sin(0) = 0$, which simplifies the calculations:

1. $f(0) = \sin(0) = 0$

2. $f'(0) = \cos(0) = 1$

3. $f''(0) = -\sin(0) = 0$

4. $f'''(0) = -\cos(0) = -1$

5. $f^{(4)}(0) = \sin(0) = 0$

6. $f^{(5)}(0) = \cos(0) = 1$

7. $f^{(6)}(0) = -\sin(0) = 0$

8. $f^{(7)}(0) = -\cos(0) = -1$

We observe a pattern in the derivatives:

- For even derivatives, the result is either 0 or -1 .

- For odd derivatives, the result is either 1 or -1 .

Now, let's substitute these values into the Taylor series formula:

$$\sin(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

$$= \frac{0}{0!}x^0 + \frac{1}{1!}x^1 + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \frac{0}{6!}x^6 + \frac{-1}{7!}x^7 + \dots$$

Simplifying this expression, we get:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

This is the Taylor series expansion of \sin(x) centered at $a = 0$, also known as the Maclaurin series.

See ""

Below is a Python code that generates the Taylor series expansion of $\sin(x)$ centered at $a = 0$:

```python
import sympy as sp

# Define the variable and function
x = sp.Symbol('x')
f = sp.sin(x)

# Compute the Taylor series expansion
taylor_series = sp.series(f, x, 0, 10)  # Expand up to the 10th term

# Print the Taylor series
print(taylor_series)
```

This code uses the `sympy` library to compute the Taylor series expansion. The `series` function computes the Taylor series expansion of the given function (`sin(x)` in this case) up to the specified order (`10` in this case), centered at the specified point (`0` in this case). Adjust the order as needed for more or fewer terms in the expansion.

3. Exercice : Properties of Taylor series expansions

If we know a power series expansion of a function f to the order 5 and we know a power series expansion of a function g to the order 3 to what order can we calculate a power series expansion of f times g?

- 15
- 8
- 7
- 14

4. Exercice

Let f be a function defined on open interval containing 0 such that $f(x) = 1 + 2x + o(x^2)$. What are the true properties among the following?

- [ ] $(f(x))^2 = 1 + 4x^2 + o(x)$
- [ ] $f(x^2) = 1 + 2x^2 + o(x^2)$
- [ ] $f(2x) = 1 + 4x + o(x)$
- [ ] $4f(x) = 1 + 4x + o(x)$

5. Exercice

Let f be a function satisfying ; in vicinity of 0, $f(x) = x + x^2 + x^3 + o(x^3)$. Th value of $f''(0)$ is .

References

1)

Murray R. Spiegel, Schaum's outline of theory and problems of advanced calculus, Mcgraw-Hill (1968).

2) Glyn James, Modern Engineering Mathematics, Pearson (2020).

3)

BOUHARIS Epouse, OUDJDI DAMERDJI Amel, Cours et exercices corrigés d'Analyse 1, Première année Licence MI Mathématiques et Informatique, U.S.T.O, 2020-2021.

4)

Benzine BENZINE, Analyse réelle cours et exercices corriges, première année maths et informatique (2016),

5)

N. PISKUNOV, DIFFERENTIAL and INTEGRAL CALCULUS, MIR PUBLISHERS Moscow (1969).

Web bibliography

https://math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)/10%3A_Power_Series/10.01%3A_Power_Series_and_Functions#:~:text=More%20specifically%2C%20if%20the%20variable,also%20to%20define%20new%20functions.