Chapter 2: Calculation of Primitives

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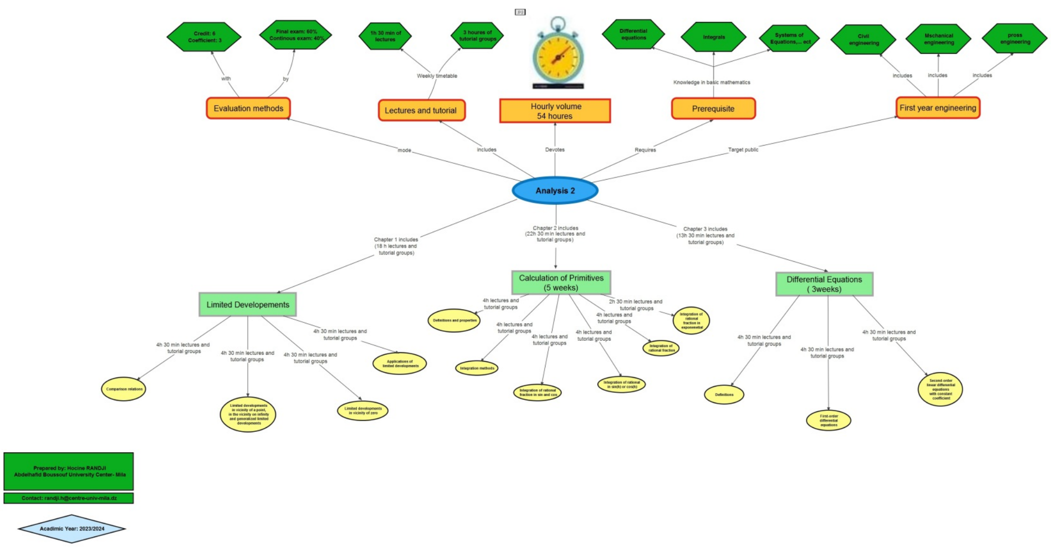
Objectives

- Understand and explain the process of integrating functions.
- Identify and apply different methods for integrating functions.
- Demonstrate the ability to simplify and solve integrals.
- Analyze and solve integrals involving rational, trigonometric,or hyperbolic functions.

Introduction



I 1. Definitions and properties (primitive, integral, and definite integral):

When delving into the foundation principles of integral and differential calculus and their practical applications, we simultaneously cultivated methodologies for addressing challenges through calculus. This chapter led to the development of strategies for solving problems utilizing the calculation of antiderivatives known also primitives.

1. Definitions:

Definition: Def 1:

Let f be a function on a closed interval $[a, b]$ in $\mathbb{R}$, and let F be a function differentiable on $[a, b]$. F is said to be an antiderivative or primitive of f on $[a, b]$ if

$\forall x \in [a, b], \quad F'(x) = f(x).$

Proposition:

If F and G are two antiderivatives of f on $[a, b]$, then

$F - G = c, \quad c \in \mathbb{R}.$

Example:E.g. 1:

The functions F and G defined on $[1, 2]$ by $F(x) = \ln x$ and $G(x) = \ln x + \alpha$, with $\alpha \in \mathbb{R}$, are two antiderivatives of the function $f(x) = \dfrac{1}{x}$ on $[1, 2]$.

Definition: Def 2:

The set of all antiderivatives of the function $f : [a, b] \to \mathbb{R}$ is called the indefinite integral of f, denoted $\int f(x)\,dx$. So if F is an antiderivative of f on $[a, b]$, we have

$$\int f(x)\,dx = F(x) + c, \quad c \in \mathbb{R}.$$

Example:E.g. 2:

For all $x \in [1, 2]$: $\int \frac{1}{x}\,dx = \ln x + c, \quad c \in \mathbb{R}$.

2. Properties:

Fundamental

If f and g are two functions having primitives on $[a; b]$, then $f + g$ and $f \circ g$ also have primitives, and we have:

1. $\int_a^b (\lambda f)(x)\,dx = \lambda \int_a^b f(x)\,dx = \lambda \int_a^b f(x)\,dx.$
2. $\int_a^b f'(x)\,dx = f(x) + c, \ where\ c \in \mathbb{R}$
3. $\int_a^b (f + g)(x)\,dx = \int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
4. $\int_a^b f(x)\,dx\Big|_a^b = f(b) - f(a)$

Fundamental:Table of integrals:

Let us give the following table of usual integrals:

Function	Integral
$\int k\,dx$	$kx + C$
$\int x^n\,dx$	$\frac{x^{n+1}}{n+1} + C$
$\int e^x\,dx$	$e^x + C$
$\int \frac{1}{x}\,dx$	$\ln\lvert x\rvert + C$
$\int \ln x\,dx$	$x\ln x - x + C$
$\int \frac{1}{x\ln x}\,dx$	$\ln\lvert\ln x\rvert + C$
$\int \frac{1}{1+x^2}\,dx$	$\arctan x + C$
$\int \frac{1}{\sqrt{1-x^2}}\,dx$	$\arcsin x + C$
$\int \sin x\,dx$	$-\cos x + C$
$\int \cos x\,dx$	$\sin x + C$
$\int \tan x\,dx$	$-\ln\lvert\cos x\rvert + C$
$\int \sec^2 x\,dx$	$\tan x + C$
$\int \csc^2 x\,dx$	$-\cot x + C$
$\int \sec x \tan x\,dx$	$\sec x + C$
$\int \csc x \cot x\,dx$	$-\csc x + C$
$\int \frac{1}{\cos^2 x}\,dx$	$\tan x + C$
$\int \frac{1}{\sin^2 x}\,dx$	$-\cot x + C$
$\int \frac{1}{\sqrt{a^2-x^2}}\,dx$	$\arcsin \frac{x}{a} + C$
$\int \frac{1}{\sqrt{x^2\pm a^2}}\,dx$	$\ln\lvert x + \sqrt{x^2 \pm a^2}\rvert + C$
$\int \frac{1}{\sqrt{x^2-a^2}}\,dx$	$\ln\lvert x + \sqrt{x^2 - a^2}\rvert + C$

3. Exercice

[*solution* n°1 p.16]

What is the integral of $e^x \, dx$?

- [ ] $\ln(e^x) + C$
- [ ] $\frac{1}{e^x} + C$
- [ ] $e^x + C$
- [ ] $e^2 x - C$

4. Exercice

[*solution* n°2 p.16]

What is the definite integral of $\int_0^1 2x \, dx$?

- [ ] 2
- [ ] 1
- [ ] 4
- [ ] 0

5. Exercice

[*solution* n°3 p.16]

Which of the following is the indefinite integral of $3x^2 + 2x - 1$

- [ ] $x^3 + x^2$
- [ ] $x^3 + x^2 + x + C$
- [ ] $x^3 + x^2 - C$
- [ ] $x^3 + x^2 - x + C$

6. Exercice

[*solution* n°4 p.16]

What is the integral of $\sin(x) \, dx$?

- [ ] $-\cos(x) + C$
- [ ] $-\sin(x) + C$
- [ ] $\cos(x) + C$
- [ ] $\sin(x) + C$

7. Exercice

[solution n°5 p.17]

Evaluate $\int_1^3 (x^2 + 1)\, dx$.

- ☐ 8
- ☐ 12
- ☐ 10
- ☐ 6

8. Integration methods:

Integration by parts

Integration by parts is a method in calculus used to find the integral of the product of two functions u and v. The formula for differentiation of the product of u and v is expressed as:

$d(uv) = u\, dv + v\, du$

This formula is known as the integration by parts formula. It's primarily used when integrating expressions that can be represented as a product of two functions, u and du, such that finding the function v from its differential dv and evaluating the integral $\int vdu$ is simpler than directly evaluating $\int udv$.

Example:E.g.

For instance, consider the integral:

$$\int x \sin(x)\, dx$$

Let $u = x$ and $dv = \sin(x)\, dx$. Then, $du = dx$ and $v = -\cos(x)$. Substituting into the integration by parts formula, we have:

$$\int x \sin(x)\, dx = -x \cos(x) + \int \cos(x)\, dx = -x \cos(x) + \sin(x) + C$$

Integration by change of variable:

Let $f : [a; b] \to \mathbb{R}$ be a continuous function on $[a; b]$ and let $g : [\alpha; \beta] \to [a; b]$ be a continuously differentiable (C^1) function such that $g(\alpha) = a$ and $g(\beta) = b$. We aim to show that:

$$\int_a^b f(x)\, dx = \int_\alpha^\beta f(g(t))g'(t)\, dt.$$

Example:E.g. 1

Evaluate the integral:

$\int_0^1 e^{x^2} \cdot 2x\,dx.$

Here, we can choose $u = g(t) = t^2$. Then, $g'(t) = 2t$ and the limits are from $u = 0$ to $u = 1$. Therefore:

$\int_0^1 e^{x^2} \cdot 2x\,dx = \int_0^1 e^u\,du = [e^u]_0^1 = e - 1.$

Example:E.g. 2

Evaluate the integral:

$I = \int \cos(x) \cdot e^{\sin x} dx$

Using the substitution $u = \sin x$. Then $du = \cos x dx$, thus:

$I = \int e^u du = e^u + c = e^{\sin x} + c$ where $c \in \mathbb{R}$.

Integrals of Rational Functions

When dealing with integrals, one important aspect is identifying functions whose integrals can be expressed in terms of elementary functions. Rational functions, a significant class of functions, have integrals that often fall into this category.

Definition

A ***rational function*** is defined as a ratio of two polynomials. Specifically, if $f(x)$ and $g(x)$ are polynomials, then a rational function can be expressed as:

$$Q(x) = \frac{f(x)}{g(x)}$$

where both $f(x)$ and $g(x)$ do not share common roots, ensuring the fraction is in its simplest form.

Proper and Improper Fractions:

Rational functions can be categorized into ***proper*** and ***improper*** fractions based on the degrees of the numerator and the denominator:

Definition

A fraction is ***proper*** if the degree of the numerator is less than the degree of the denominator.

Definition

A fraction is ***improper*** if the degree of the numerator is equal to or greater than the degree of the denominator.

Simplifying Improper Fractions:

If the rational function is improper, it can be simplified by dividing the numerator by the denominator using polynomial division. This process expresses the improper fraction as the sum of a polynomial and a proper fraction:

$$Q(x) = P(x) + \frac{N(x)}{g(x)} \qquad (1)$$

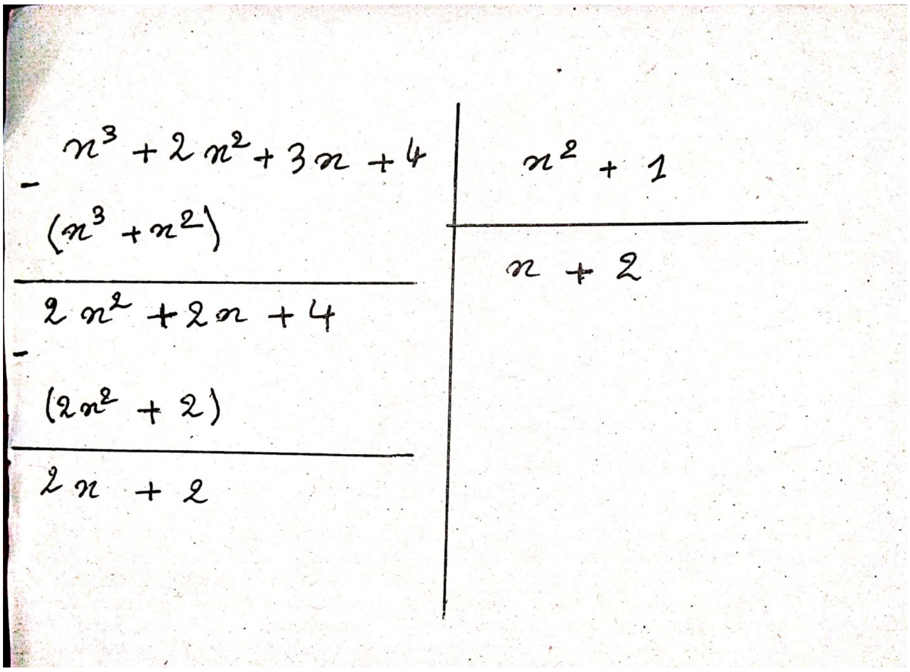
where $P(x)$ is a polynomial, and $\frac{N(x)}{g(x)}$ is a proper fraction.

Example

Consider an improper rational function:

$$Q(x) = \frac{x^3 + 2x^2 + 3x + 4}{x^2 + 1}$$

By performing polynomial division, we divide $x^3 + 2x^2 + 3x + 4$ by $x^2 + 1$):



This can be rewritten as:

$$Q(x) = x + 2 + \frac{2x + 2}{x^2 + 1} \qquad (1)$$

Here, $x + 2$ is the polynomial part, and $\frac{2x + 2}{x^2 + 1}$ is the proper fraction.

Integrating Rational Functions

The integral of a rational function can be found by integrating its polynomial and proper fraction components separately.

Example

Using the example above:

$$\int Q(x)\,dx = \int \left(x + 2 + \frac{3x + 4}{x^2 + 1} \right) dx$$

The integral can be broken down into simpler parts:

$$\int x\,dx + \int 2\,dx + \int \frac{2x}{x^2+1}\,dx + \int \frac{2}{x^2+1}\,dx$$

Each of these integrals can be solved using standard integration techniques. So we have:

$$\int Q(x)\,dx = \frac{x^2}{2} + 2x + \ln|x^2+1| + 2\arctan(x) + C$$

By understanding these principles, one can effectively handle the integration of rational functions.

Integration of a rational fraction in sin and cos:

For integrals involving trigonometric functions, several cases are distinguished.

Method

Integrals of the Form $\int R(\cos x)\sin x\,dx$***:***

In this case, we make the change of variables $t = \cos x$, hence $dt = -\sin x\,dx$.

Integrals of the Form $\int R(\sin x)\cos x\,dx$***:***

Here, we make the change of variables $t = \sin x$, hence $dt = \cos x\,dx$.

Example

$$I = \int \sin^2 x \cos x dx$$

Change of variables: $t = \sin x$, $dt = \cos x\,dx$. Therefore,

$$I = \int t^2 dt = \frac{1}{3}t^3 + C = \frac{1}{3}\sin^3(x) + C$$

where C is a real number.

Method

Integrals of the Form $\int R(\sin x, \cos x)\,dx$***:***

Here, we make the change of variables $t = \tan\frac{x}{2}$, hence:

$$x = 2\arctan t \quad \text{and} \quad dx = \frac{2}{1+t^2}\,dt$$

We have:

$$\sin x = \frac{2t}{1+t^2}$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

Example

$$I = \int \frac{dx}{\sin x}$$

Change of variables: $t = \tan \frac{x}{2}$, $x = 2 \arctan t$, hence $dx = \frac{2}{1+t^2}\, dt$ and $\sin x = \frac{2t}{1+t^2}$. Therefore,

$$I = \int \frac{dt}{t} = \ln|t| + c, \quad c \in \mathbb{R}$$

Thus,

$$I = \ln\left|\tan \frac{x}{2}\right| + c, \quad c \in \mathbb{R}$$

Integration of a rational fraction in exponential:

Integrals involving the exponential function e^x can often be tackled using substitution methods. One common approach is to let $t = e^x$, which simplifies the integral.

Method

For integrals of the form $\int R(e^x)\, dx$, we use the substitution $t = e^x$. Then, we have:

$$dt = e^x\, dx \quad \text{or} \quad dx = \frac{dt}{t}$$

This substitution transforms the integral into a function of t.

Example

Consider another integral:

$$I = \int \frac{1}{1+e^x}\, dx$$

Using the substitution $t = e^x$, we have $dx = \frac{dt}{t}$. Therefore,

$$I = \int \frac{1}{1+t} \cdot \frac{dt}{t} = \int \frac{dt}{t(1+t)}$$

We can further simplify this using partial fractions:

$$\frac{1}{t(1+t)} = \frac{A}{t} + \frac{B}{1+t}$$

Solving for A and B, we find $A = 1$ and $B = -1$. Hence,

$$\frac{1}{t(1+t)} = \frac{1}{t} - \frac{1}{1+t}$$

So the integral becomes:

$$I = \int \left(\frac{1}{t} - \frac{1}{1+t} \right) dt = \int \frac{dt}{t} - \int \frac{dt}{1+t}$$

Evaluating these integrals gives:

$$I = \ln|t| - \ln|1+t| + C = \ln\left|\frac{t}{1+t}\right| + C$$

Substituting back $t = e^x$, we get:

$$I = \ln\left|\frac{e^x}{1+e^x}\right| + C$$

Simplifying further, we obtain:

$$I = \ln\left|\frac{e^x}{1+e^x}\right| + C$$

Thus,

$$I = x - \ln|1 + e^x| + C$$

Integration of a rational in sin(h) or cos(h)

For integrals of the form $\int R(\sinh(x), \cosh(x))\, dx$, we often use substitutions such as $t = \sinh(x)$ or $t = \cosh(x)$.

Example

Consider another integral:

$$I = \int \frac{dx}{\cosh(x)}$$

Using the substitution $t = \sinh(x)$, we have $dt = \cosh(x)\, dx$. Therefore,

$$dx = \frac{dt}{\cosh(x)}$$

Since $\cosh^2(x) - \sinh^2(x) = 1$, we can write $\cosh(x) = \sqrt{1+t^2}$. Hence,

$$I = \int \frac{dt}{\sqrt{1+t^2}}$$

This is a standard integral:

$$I = \sinh^{-1}(t) + C$$

Substituting back $t = \sinh(x)$, we get:

$$I = \sinh^{-1}(\sinh(x)) + C = x + C$$

II Exercice

[solution n° 6 p.17*]*

An antiderivative for the function $g(x) = x\sin(x^2)$ is:

- ☐ $G(x) = \frac{1}{3}\cos(x^2)$
- ☐ $G(x) = -\frac{1}{4}\cos(x^2)$
- ☐ $G(x) = \cos(x^2)$
- ☐ $G(x) = \frac{1}{3}\sin(x^2)$
- ☐ $G(x) = -\frac{1}{2}\cos(x^2)$

III Final exam:

Final exam:

[cf. res]

Exercises solution

> **Solution** n°1

Exercice p. 7

What is the integral of $e^x\,dx$?

- ☐ $\ln(e^x) + C$
- ☐ $\frac{1}{e^x} + C$
- ☑ $e^x + C$
- ☐ $e^2x - C$

> **Solution** n°2

Exercice p. 7

What is the definite integral of $\int_0^1 2x\,dx$?

- ☐ 2
- ☑ 1
- ☐ 4
- ☐ 0

> **Solution** n°3

Exercice p. 7

Which of the following is the indefinite integral of $3x^2 + 2x - 1$

- ☐ $x^3 + x^2$
- ☐ $x^3 + x^2 + x + C$
- ☐ $x^3 + x^2 - C$
- ☑ $x^3 + x^2 - x + C$

> **Solution** n°4

Exercice p. 7

What is the integral of $\sin(x)\,dx$?

- [x] $-\cos(x) + C$
- [ ] $-\sin(x) + C$
- [ ] $\cos(x) + C$
- [ ] $\sin(x) + C$

> **Solution** n°5

Exercice p. 8

Evaluate $\int_1^3 (x^2 + 1)\,dx$.

- [ ] 8
- [ ] 12
- [x] 10
- [ ] 6

> **Solution** n°6

Exercice p. 14

An antiderivative for the function $g(x) = x\sin(x^2)$ is:

- [ ] $G(x) = \frac{1}{3}\cos(x^2)$
- [ ] $G(x) = -\frac{1}{4}\cos(x^2)$
- [ ] $G(x) = \cos(x^2)$
- [ ] $G(x) = \frac{1}{3}\sin(x^2)$
- [x] $G(x) = -\frac{1}{2}\cos(x^2)$

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