Chapitre 1

Logic concepts

In this chapter we will limit ourselves to the introduction of the first elements of classical logic.

Let us define logic as the study of arguments. In otherwords, logic attempts to codify what counts as legitimate means by which to draw conclusions from given information.

1.1 Propositions (Statements)

To study arguments, one must first study sentences because they are the main parts of arguments.

Definition 1.1.
A sentence that is either true or false is called a proposition.

Not all sentences are propositions, however. Questions, exclamations, commands, or self-contradictory sentences like the following examples can neither be asserted nor be denied.

Example 1.1.

1. *Is mathematics logic ?*
2. *Hey there !*
3. *Do not panic.*
4. *This sentence is false.*

P
1
0

TABLE 1.1: Truth table of a proposition P

Statements are denoted by capital letters P, Q, R,....

If the propositions is true, we assign it the value 1, (or T) ; if it is false, we assign it the value logic 0, (or F).

There are two types of propositions.

♦ An atom is a proposition that is not comprised of other propositions.

Example 1.2.

1. *Algiers is the capital of Algeria is a true proposition.*
2. *16 is a multiple of 2 is a true proposition.*
3. *19 is a multiple of 2 is a false proposition.*

♦ A proposition that is not an atom but is constructed using other propositions is called a compound proposition.

1.2 The Propositional Calculus

Propositions may be combined in various ways to form more complicated proposition. There are five types.

Definition 1.2. (**A negation**)
A negation of a given proposition P is a proposition $\bar{P}$ $(NotP)$ such that when P is true, $\bar{P}$ is false ; when P is false, $\bar{P}$ is true.

Example 1.3.

1. *The negation of* $3+8=5$ *is* $3+8\neq 5$. *In this case, we say that* $3+8=5$ *has been negated.*
2. *Negating the proposition the sine function is periodic yields the sine function is not periodic.*

3. *The negation of "The number 5 is even" is "The number 5 is not even".*

For the proposition $\bar{P}$, we obtain the following truth table (1.2)

P	$\bar{P}$
1	0
0	1

TABLE 1.2: Truth table of the proposition $\bar{P}$

Definition 1.3. (**A conjunction**)
A conjunction is a proposition formed by combining two propositions (called conjuncts) with the word and. The conjunction of sentences A and B will be designated by A *and* B ($A \wedge B$) and has the following truth table :
$A \wedge B$ is true when and only when both A and B are true.

P	Q	$P \wedge Q$
1	1	1
0	1	0
1	0	0
0	0	0

TABLE 1.3: Truth table of the statement $P \wedge Q$

A and B are called the conjuncts of A *and* B.

Example 1.4.

1. *Let :*

P_1 : *It is sunny today.*

P_2 : *The temperature is warm.*

The conjunction of these propositions is :

$P_1 \wedge P_2$: *It is sunny today and the temperature is warm.*

2. *Let :*

Q_1 : *The car is red.*

Q_2 : *The car has alloy wheels.*

The conjunction of these propositions is :

$Q_1 \wedge Q_2$: *The car is red and it has alloy wheels.*

In natural languages, there are two distinct uses of **or** : the inclusive and the exclusive. According to the inclusive usage, A **or** B means A *or* B or both', whereas according to the exclusive usage, the meaning is A *or* B, but not both. We shall introduce a special sign, $\vee$, for the inclusive connective.

Definition 1.4. (A disjunction)
A disjunction is a proposition formed by combining two propositions (called disjuncts) with the word *or*. $A \vee B$ is false when and only when both A and B are false. $A \vee B$ called a disjunction, with the disjuncts A and B. The truth table of $A \vee B$ is as follows :

A	B	$A \vee B$
1	1	1
0	1	1
1	0	1
0	0	0

TABLE 1.4: Truth table of the proposition $A \vee B$

Example 1.5.

1. *Let :*

$$P_1 : \textit{It is raining.}$$
$$P_2 : \textit{It is windy.}$$

   The disjunction of these propositions is :

$$P_1 \vee P_2 : \textit{It is raining or it is windy.}$$

2. *Let :*

$$Q_1 : \textit{The temperature is above 25 C.}$$
$$Q_2 : \textit{The humidity level is high.}$$

   The disjunction of these propositions is :

$$Q_1 \vee Q_2 : \textit{The temperature is above 25 C or the humidity level is high.}$$

3. *« 10 is divisible by 2» is a true statement. « 10 is divisible by 3 » is a false statement. So, $P \wedge Q$ is a false statement. On the other hand, $P \vee Q$ is true.*

A statement of the form "If P, then Q" is called a conditional statement. The statement "P" is called the antecedent or the hypothesis, and "Q" is called the

P	Q	$P \Longrightarrow Q$
1	1	1
1	0	0
0	1	1
0	0	1

TABLE 1.5: Truth table of the statement $P \Longrightarrow Q$

P	Q	$P \Leftrightarrow Q$
1	1	1
0	1	0
1	0	0
0	0	1

TABLE 1.6: Truth table of the statement $P \Leftrightarrow Q$

consequent or the conclusion. If P, then Q is also expressed by saying that Q is a necessary condition for P. An other way to express it is to say that P is a sufficient condition for Q.

Definition 1.5. (Implication, Equivalence)
Let P and Q be two statements.

1. A statement of the form P implies Q (in symbol $P \Rightarrow Q$) is called an implication. The statement $P \Rightarrow Q$ and the statement 'If P, then Q' are logically same, for (as we shall see) the truth values of both the statements are always same. Again, "P" is called the antecedent or the hypothesis, and "Q" is called the consequent or the conclusion.
2. A statement of the form P if and only if Q (briefly P iff Q) is called an equivalence. P implies Q and Q implies P (in symbol $P \Leftrightarrow Q$) and read as "P is equivalent to Q" is an statement that is true when P and Q are simultaneously true or false, and false in all other cases.

The truth tables for the two logical connectors "$\Rightarrow$" and "$\Leftrightarrow$" are presented in tables (1.5) and (1.6).

Remarque 1.6.

1. In practice, if P, Q, and R denote three statements, then the composite statement ($P \Rightarrow Q$ and $Q \Rightarrow R$) is written as : ($P \Rightarrow Q \Rightarrow R$).

   Similarly, the composite statement ($P \Leftrightarrow Q$ and $Q \Leftrightarrow R$) is written as : ($P \Leftrightarrow Q \Leftrightarrow R$).

2. The implication $Q \Rightarrow P$ is called the converse of $P \Rightarrow Q$.

1.2.1 Properties

Definition 1.7.
Let P and Q be two propositions (either composite or simple).

1. If P is true when Q is true and if P is false when Q is false, then we say that P and Q have the same truth table or that they are logically equivalent, and we denote this by $P \Leftrightarrow Q$.
2. In the opposite case, we denote it by $P \not\Leftrightarrow Q$.

Example 1.6.
Let P, Q, and R be three statements, then :

1. $\bar{\bar{P}} \Leftrightarrow P$.
2. $(P \wedge P) \Leftrightarrow P$. *Similarly,* $(P \vee P) \Leftrightarrow P$.
3. $(P \wedge Q) \Leftrightarrow (Q \wedge P), \qquad (P \vee Q) \Leftrightarrow (Q \vee P)$.
4. $(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R), \qquad (P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$.
5. $(P \wedge (Q \vee P) \Leftrightarrow P$.
6. $(P \Leftrightarrow Q \Leftrightarrow (Q \Leftrightarrow P)$.
7. $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R), \qquad P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$.
8. $\overline{P \wedge Q} \Leftrightarrow \overline{P} \vee \overline{Q}, \qquad \overline{P \vee Q} \Leftrightarrow \overline{P} \wedge \overline{Q}$.
9. *The compound proposition* $((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$ *is true regardless of the truth values of* P, Q *and* R.
10. *The compound proposition* $((P \Leftrightarrow Q) \wedge (Q \Leftrightarrow R)) \Rightarrow (P \Leftrightarrow R)$ *is true regardless of the truth values of* P, Q *and* R.
11. $(P \Rightarrow Q) \Leftrightarrow (\overline{P} \vee Q)$.
12. $\overline{(P \Rightarrow Q)} \Leftrightarrow (P \wedge \overline{Q})$.
13. $(P \Rightarrow Q) \Leftrightarrow (\overline{Q} \Rightarrow \overline{P})$.
14. $(P \Leftrightarrow Q) \Leftrightarrow ((P \Rightarrow Q) \wedge (Q \Rightarrow P))$.

1.3 Mathematical quantifiers

Definition 1.8.
Let E be a set. A predicate on E is a statement containing letters called variables

such that when each of these variables is replaced by an element of E, we obtain proposition.

A predicate containing the variable x will be denoted $P(x)$.

Example 1.7.

The statement $P(n)$ defined by " n is a multiple of 2" is a predicate on $\mathbb{N}$. It becomes an proposition when an integer value is assigned to n. For example,

1. *the proposition $P(10)$ defined by "10 is a multiple of 2" obtained by replacing n with 10 is true ;*
2. *the proposition $P(11)$ defined by "11 is a multiple of 2" obtained by replacing n with 11 is false.*

From a predicate $P(x)$ defined on a set E, we can construct new propositions called quantified propositions using the quantifiers "there exists ($\exists$)" and "for all ($\forall$)".

Definition 1.9.

Let $P(x)$ be a predicate defined on a set E.

1. (**Universal quantifier**)The quantifier "for all" (also called "for every"), denoted by $\forall$, allows us to define the quantified proposition "$\forall x\ E, in\ P(x)$" which is true when all elements x of E satisfy $P(x)$.
2. (**Existential quantifier**) The quantifier "there exists," denoted by $\exists$, allows us to define the quantified proposition "$\exists x \in E,\ P(x)$" which is true when we can find (at least) one element x belonging to E that satisfies the statement $P(x)$.

Example 1.8.

1. *The statement "$x^2+2x-3 \leq 0$" is a predicate defined on $\mathbb{R}$. It can be true or false depending on the value of x. The statement "$\forall x \in [-3,1]\ x^2+2x-3 \leq 0$" is a quantified proposition. It is true because the quantity x^2+2x-3 is negative or zero for every x belonging to the closed interval $[-3,1]$.*
2. *The quantified proposition "$\forall x \in \mathbb{N}\ (n-3)n > 0$" is false because there exists an element n in $\mathbb{N}$ (taking $n=0$, $n=1$, $n=2$, or $n=3$) for which the statement "$(n-3)n > 0$" is false.*
3. *The quantified proposition "$\exists x \in \mathbb{R}\ x^2 = 4$" is true because there exists (at least) one element in $\mathbb{R}$ that satisfies $x^2 = 4$. This is the case for the two real numbers -2 and 2.*

1.3.1 The rules of negation for a quantified proposition

1. The negation of "for every element x in E, the statement $P(x)$ is true" is "there exists an element x in E for which the statement $P(x)$ is false."»

$$\overline{(\forall x \in E \quad P(x))} \quad \Leftrightarrow \quad (\exists x \in E \quad \overline{P(x)})$$

.

2. La négation de « il existe un élément x de E pour lequel l'énoncé $P(x)$ est vrai » est « pour tout élément x de E, l'énoncé $P(x)$ est faux ». ».

$$\overline{(\exists x \in E \quad P(x))} \quad \Leftrightarrow \quad (\forall x \in E \quad \overline{P(x)})$$

.

Here are some examples :

Example 1.9.

1. *The negation of* $(\forall x \in [0, +\infty[\quad (x^2 \geq 1))$ *is* $(\exists x \in [0, +\infty[\quad (x^2 < 1))$.
2. *The negation of* $(\exists z \in \mathbb{C} \quad z^2 + z + 1 = 0)$ *is* $(\forall z \in \mathbb{C} \quad z^2 + z + 1 \neq 0)$.
3. *It is not more difficult to write the negation of complex sentences. For the proposition :*

$$\forall x \in \mathbb{R} \quad \exists y > 0 \quad (x + y > 10),$$

its negation

$$\exists x \in \mathbb{R} \quad \forall y > 0 \quad (x + y \leq 10).$$

Remarque 1.10.
The order of quantifiers is very important. For example, the two logical sentences

$$\forall x \in \mathbb{R} \quad \exists y > 0 \ \ (x + y > 10) \quad \textit{et} \ \ \exists y > 0 \quad \forall x \in \mathbb{R} \ \ (x + y > 10).$$

are different. The first one is true, the second one is false.

1.4 Proof methods

The purpose of the propositional logic of Chapter 1 is to model the basic reasoning that one does in mathematics. Here are some classical methods of reasoning.

1.4.1 Universal Proofs

Our first paragraph proofs will be for propositions with universal quantifiers. To prove $\forall x \; E, \; P(x)$ (interpreted to mean that $P(x)$ holds for all x from a given universe E) from a given set of premises, we show that every object x satisfies $P(x)$ assuming those premises. For this we follow the following diagram

$$\text{Let a be an object in the universe } E \longrightarrow \text{Prove } P(a).$$

Example 1.10. *To prove that for all real numbers x,*

$$(x-1)^3 = x^3 - 3x^2 + 3x - 1$$

we introduce a real number and then check the equation.

Proof 1.11.
Let a be a real number. Then,

$$(x-1)^3 = (x-1)(x-1)^2 = (x-1)(x^2 - 2x + 1) = x^3 - 3x^2 + 3x - 1$$

1.4.2 Existential Proofs

Suppose that we want to write a paragraph proof for $\exists x \in E, \; P(x)$. This means that we must show that there exists at least one object of the universe E that satisfies the formula $P(x)$. It will be our job to find that object. To do this directly, we pick an object that we think will satisfy $P(x)$. This object is called a candidate. We then check that it does satisfy $P(x)$. This type of a proof is called a direct existential proof, and its structure is illustrated as follows :

Choose a candidate from the universe. $\longrightarrow$ Check that the candidate satisfies $P(x)$.

1.4.3 Direct reasoning

The phrase translates to : "We want to show that the proposition ' $P \Rightarrow Q$ ' is true. We assume that P is true and then demonstrate that Q is true as well by following these steps :

1. **assume the antecedent,**
2. **translate the antecedent,**
3. **translate the consequent so that the goal of the proof is known,**
4. **deduce the consequent.**

Example 1.11.
We use Direct Proof to write a paragraph proof of the proposition

$$\textit{for all integers } x, \textit{if } 4 \textit{ divides } x, \textit{then } x \textit{ is even.}$$

Proof 1.12.
Assume that 4 divides the integer a. This means $a = 4k$ for some integer k. We must show that $a = 2l$ for some integer l, but we know that $a = 4k = 2.(2k)$. Hence, let $l = 2k$.

1.4.4 Reasoning by contraposition

Sometimes it is difficult to prove a conditional directly. An alternative is to prove the contrapositive. This is sometimes easier or simply requires fewer lines.

Contrapositive reasoning is based on the following equivalence :

$$(P \Rightarrow Q) \Leftrightarrow (\overline{Q} \Rightarrow \overline{P}). \tag{1.1}$$

Therefore, if we want to show the proposition $P \Rightarrow Q$, we actually demonstrate that if $\overline{Q}$ is true, then $\overline{P}$ is true.

Example 1.12.
Let us show that

$$\textit{for all integers } n, \textit{if } n^2 \textit{ is odd, then } n \textit{ is odd.}$$

A direct proof of this is a problem. Instead, we prove its contrapositive,

if n is not odd, then n^2 is not odd.

Proof 1.13.
Let n be an even integer. This means that $n = 2k$ for some integer k. To see that n^2 is even, calculate to find $n^2 = (2k)^2 = 4k^2 = 2.(2k^2)$.

1.4.5 Proof by Cases

Suppose that we want to prove $P \Rightarrow Q$ and this is difficult for some reason. We notice, however, that P can be broken into cases. Namely, there exist $P_1, P_2, ..P_n$ such that $P \Leftrightarrow P_1 \bigvee P_2 \bigvee ... \bigvee P_n$.
If we can prove $P_i \Rightarrow Q$ for each i, we have proved $P \Rightarrow Q$.

Example 1.13.
Our example of a proof by cases is a well-known one :

for all integers a and b, if $a = \pm b$, then a divides b and b divides a.

Proof 1.14.
Let a and b be integers and suppose $a = \pm b$. To show a divides b and b divides a, we have two cases to prove.

1. *Assume $a = b$. Then, $a = b.1$ and $b = a.1$.*
2. *Next assume $a = -b$. This means that $a = b.(-1)$ and $b = a.(-1)$.*

In both the cases, we have proved that a divides b and b divides a.

1.4.6 Counterexamples

To show that an proposition of the type $\forall x \in E \;\; P(x)$ is true, one must demonstrate that $P(x)$ is true for each x in E.
On the other hand, to show that this proposition is false, it is enough to find an $a \in E$ such that $P(a)$ is false (i.e., $\overline{P(a)}$ is true). (Remember, the negation of $\forall x \in E \;\; P(x)$ is $\exists x \in E \;\; \overline{P(x)}$).
To find such an a is to find a counterexample to the proposition $\forall x \in E \;\; P(x)$.

Example 1.14.
Show false : $x+2 = 7$ for all real numbers x. To do this, we show that $\exists x$, $(x+2 \neq$

7) *is true by noting that the real number* 0 *satisfies* $x + 2 \neq 7$*. Hence,* 0 *is a counterexample to* $\forall x \in \mathbb{R}, x + 2 = 7$*.*

1.4.7 Proof by Contradiction :

Proof by contradiction, also known as reasoning by the absurd, involves demonstrating the truth of a proposition by showing that its opposite leads to a contradiction. To use this type of reasoning, follow these steps :

1. **Assume the negation :** Suppose that the proposition you want to prove is false.
2. **Develop the consequences :** Develop the logical consequences of this assumption.
3. **Identify the contradiction :** Show that these consequences lead to a contradiction.
4. **Conclusion :** Conclude that the initial assumption must be false, which means that the proposition you want to prove is true.

Example 1.15.
Prove that :

for all integers n*, if* n^2 *is odd, then* n *is odd.*

Proof 1.15.
Take an integer n *and let* n^2 *be odd. In order to obtain a contradiction, assume that* n *is even. So,* $n = 2k$ *for some integer* k*. Substituting, we have* $n^2 = (2k)^2 = 2(2k^2)$*, showing that* n^2 *is even. This is a contradiction. Therefore,* n *is an odd integer.*

1.4.8 Proof by Induction

Proof by induction is a method used in mathematics to prove that a property $P(n)$ holds for all integers n starting from a certain initial value n_0. It consists of two steps :

1. **Base Case :** Verify that the property $P(n)$ is true for the initial value $n = n_0$.
2. **Inductive Step :** Assume that the property is true for a certain integer k (inductive hypothesis), and then prove that the property is true for $k + 1$.

Example 1.16.

Prove that the sum of the first n positive integers is given by the formula :

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

1. ***Step 1*** *: Base Case*

   First, we need to verify the formula for the initial value $n = 1$.

   When $n = 1$:

   $$1 = \frac{1(1+1)}{2} = \frac{1 \cdot 2}{2} = 1$$

   So, the formula holds for $n = 1$.

2. ***Step 2 :*** *Inductive Step*

   Next, we assume that the formula holds for some integer k. That is, we assume :

   $$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$$

   This assumption is called the inductive hypothesis.

   We need to show that if the formula holds for $n = k$, then it also holds for $n = k + 1$.

   Consider the sum of the first $k + 1$ positive integers :

   $$1 + 2 + 3 + \cdots + k + (k+1)$$

   Using the inductive hypothesis, we can write :

   $$1 + 2 + 3 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1)$$

   We need to simplify the right-hand side :

   $$\frac{k(k+1)}{2} + (k+1)$$

   Factor $k + 1$ out of the terms on the right :

   $$\frac{k(k+1)}{2} + (k+1)$$

   $$= \frac{k(k+1) + 2(k+1)}{2}$$

   $$= \frac{(k+1)(k+2)}{2}$$