University Center Abdelhafid Boussouf Mila
Institute of Mathematics and Computer Science
1st Year Mathematics (Algebra 1)

Series of Tutorial No. 2
Sets, relations and functions

Exercise 1.
Consider the following sets: $A = \{1, 2, 3, 4, 5\}$, $B = \{3, 4, 5, 6, 7\}$ *and* $C = \{2, 4, 6, 8, 10\}$ *Find the following sets:* $(A \cup B)$, $(B \cap C)$, $(A - B)$ *and* $(A \Delta C)$

Exercise 2.
Let $A, B,$ *and* C *be three subsets of a set* E. *Show that:*

1. $A \setminus B = A \cup B^c$
2. $A \cup B = A \setminus B$

Where A^c *denotes the complement of* A *in* E. ***Simplify the following sets:***

1. $A \cup B \setminus C \cup A$
2. $A \setminus B \cup (C \setminus A)$

Exercise 3.
Let $E = \{a,\ b,\ c\}$ *be a set. Can we write: 1)* $a \in E$, *2)* $a \subset E$, *3)* $\{a\} \subset E$, *4)* $\emptyset \in E$,*5)* $\emptyset \subset E$, *6)* $\{\emptyset\} \subset E$?

Exercise 4.

1. *What is the image of the sets* $\mathbb{R}$, $[0, 2\pi]$, $[0, \frac{\pi}{2}]$, *and the inverse image of the sets* $[0, 1]$, $[3, 4]$, $[1, 2]$ *under the function* $f(x) = \sin(x)$?
2. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be defined by* $f(x) = x^2 + 1$. *Consider the sets* $A = [-3, 2]$ *and* $B = [0, 4]$. *Compare* $f(A \setminus B)$ *and* $f(A) \setminus f(B)$.
3. *What condition must function* f *satisfy so that* $f(A \setminus B) = f(A) \setminus f(B)$?

Exercise 5. *(Power set)*
Let $E = \{a, b, c, d\}$. *Find the power set* $\mathcal{P}(E)$ *of* E, *which is the set of all subsets of* E. *Give an example of a partition of* E, *which is a collection of non-empty disjoint subsets of* E *whose union equals* E.

Exercise 6.
Let A, B, *and* C *be three subsets of a set* E. *Prove that:*

1. $A \setminus B = A \cap B^c$.
2. $A \triangle B = (A \cup B) \setminus (A \cap B)$.
3. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
4. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Exercise 7.
Determine whether the following relations are reflexive, symmetric, antisymmetric, and transitive:

1. $E = \mathbb{Z}$ *and* $x\mathcal{R}y \Leftrightarrow |x| = |y|$.
2. $E = \mathbb{R} \setminus \{0\}$ *and* $x\mathcal{R}y \Leftrightarrow xy > 0$.
3. $E = \mathbb{Z}$ *and* $x\mathcal{R}y \Leftrightarrow x - y$ *is even.*

Identify among the above examples which relations are orders and which are equivalence relations.

Exercise 8.
Determine whether the following relations are reflexive, symmetric, antisymmetric, and transitive:

1. $E = \mathbb{R}$ *and* $x\mathcal{R}y \Leftrightarrow x = -y$.
2. $E = \mathbb{R}$ *and* $x\mathcal{R}y \Leftrightarrow \cos^2(x) + \sin^2(y) = 1$.
3. $E = \mathbb{N}$ *and* $x\mathcal{R}y \Leftrightarrow \exists p, q \geq 1$ *such that* $y = px^q$ *(where* p *and* q *are integers).*

Identify among the above examples which relations are orders and which are equivalence relations.

Exercise 9.
Let $E = \mathbb{Z}$*, the set of all integers. Consider the following relations on* E*:*

1. *Relation* $\sim_1$ *defined by* $x \sim_1 y$ *if and only if* $x + y$ *is even.*
2. *Relation* $\sim_2$ *defined by* $x \sim_2 y$ *if and only if* x *and* y *have the same remainder when divided by 5.*
3. *Relation* $\sim_3$ *defined by* $x \sim_3 y$ *if and only if* $x - y$ *is a multiple of 7.*

For each relation $\sim_i$ *(where* $i = 1, 2, 3$*):*

1. *Determine if* $\sim_i$ *is an equivalence relation on* $\mathbb{Z}$*. Explain why or why not.*
2. *If* $\sim_i$ *is an equivalence relation, identify the equivalence classes of* $\mathbb{Z}$ *under* $\sim_i$*.*

Exercise 10.
Let $\mathcal{R}$ *be an equivalence relation on a non-empty set* E*. Show that*

$$\forall x, y \in E, \quad x\mathcal{R}y \quad \Leftrightarrow \quad \dot{x} = \dot{y}.$$

Exercise 11.
Let $\mathbb{N}^*$ *denote the set of positive integers. Define the relation* $\mathcal{R}$ *on* $\mathbb{N}^*$ *by* $x\mathcal{R}y$ *if and only if* x *divides* y*.*

1. *Show that* $\mathcal{R}$ *is a partial order relation on* $\mathbb{N}^*$*.*
2. *Is* $\mathcal{R}$ *a total order relation?*
3. *Describe the sets* $\{x \in \mathbb{N}^* \mid x\mathcal{R}5\}$ *and* $\{x \in \mathbb{N}^* \mid 5\mathcal{R}x\}$*.*

4. *Does* $\mathbb{N}^*$ *have a least element? A greatest element?*

Exercise 12.
Let f *be the function from* $\mathbb{R}$ *to* $\mathbb{R}$ *defined by* $f(x) = x^2 + x - 2$.

1. *Give the definition of* $f^{-1}(\{4\})$. *Calculate* $f^{-1}(\{4\})$.
2. *Is the function* f *bijective?*
3. *Give the definition of* $f([-1, 1])$. *Calculate* $f([-1, 1])$.
4. *Give the definition of* $f^{-1}([-2, 4])$. *Calculate* $f^{-1}([-2, 4])$.

Exercise 13.
Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ *be defined by* $f(x) = \frac{2x}{1+x^2}$.

1. *Is* f *injective? Is* f *surjective?*
2. *Show that* $f(\mathbb{R}) = [-1, 1]$.
3. *Show that the restriction* $g : [-1, 1] \longrightarrow [-1, 1]$ *defined by* $g(x) = f(x)$ *is a bijection.*

Exercise 14.
Let $f : E \to F$, $g : F \to G$, *and* $h = g \circ f$.

1. *Show that if* h *is injective, then* f *is injective. Also, show that if* h *is surjective, then* g *is surjective.*
2. *Show that if* h *is surjective and* g *is injective, then* f *is surjective.*
3. *Show that if* h *is injective and* f *is surjective, then* g *is injective.*