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1st Year Mathematics (Algebra 1)

Solution to Tutorial Series No. 1
Logic concepts

Solution0 1. *(Exercise 1)*

1. *The sentence "Paris is in France or Madrid is in China" is a proposition.*
   - ***Truth Value:** True.*
2. *The sentence "Open the door" is not a proposition because it is a command.*
3. *The sentence "The moon is a satellite of the Earth" is a proposition.*
   - ***Truth Value:** True.*
4. *The equation $x + 5 = 7$ is not a proposition.*
5. *The inequality $x + 5 > 9$ for every real number x is a proposition.*
   - ***Truth Value:** False for $x = 3$.*

Solution0 2.
Determine whether each of the following implications is true or false.

1. *If 0.5 is an integer, then $1 + 0.5 = 3$.*
   True
2. *If $5 > 2$, then cats can fly.*
   False
3. *If $3 \times 5 = 15$, then $1 + 2 = 3$.*
   True
4. *For any real $x \in \mathbb{R}$, if $x \leq 0$, then $(x - 1) < 0$.*
   True

Solution0 3.
Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function. Negate the following propositions:

1. *$\exists x \in \mathbb{R}$ such that $f(x) = 0$.*
   Negation: *$\forall x \in \mathbb{R},\ f(x) \neq 0$.*
2. *$\exists M > 0$ such that $\forall A > 0,\ \exists x \geq A$ with $f(x) \leq M$.*
   Negation: *$\forall M > 0,\ \exists A > 0$ such that $\forall x \geq A,\ f(x) > M$.*
3. *$\exists x \in \mathbb{R}$ such that $f(x) > 0$ and $x > 0$.*
   Negation: *$\forall x \in \mathbb{R},\ (f(x) \leq 0$ or $x \leq 0)$.*
4. *$\forall \epsilon > 0,\ \exists \eta > 0,\ \forall (x, y) \in I^2,\ (|x - y| \leq \eta \Rightarrow |f(x) - f(y)| > \epsilon)$.*
   Negation: *$\exists \epsilon > 0$ such that $\forall \eta > 0,\ \exists (x, y) \in I^2$ with $|x - y| \leq \eta$ and $|f(x) - f(y)| \leq \epsilon$.*

Solution0 4.

Consider the statement "for all integers a and b, if $a+b$ is even, then a and b are even":

1. *Write the contrapositive of the statement.*
   Contrapositive: *For all integers a and b, if a or b is odd, then $a+b$ is odd.*

2. *Write the converse of the statement.*
   Converse: *For all integers a and b, if a and b are even, then $a+b$ is even.*

3. *Write the negation of the statement.*
   Negation: *There exist integers a and b such that $a+b$ is even, but a or b (or both) are odd.*

4. *Is the original statement true or false? Prove your answer.*
   Answer: *The original statement is **true**.*

   Proof: *Assume a and b are integers such that $a+b$ is even. We need to show that a and b are even.*

   If a and b were both odd, then $a+b$ would be even (since sum of two odd numbers is even), contradicting our assumption that $a+b$ is even. Therefore, a and b cannot both be odd simultaneously, implying that a and b must be even.

   Hence, the original statement is true.

5. *Is the contrapositive of the original statement true or false? Prove your answer.*
   Answer: *The contrapositive of the original statement is **true**.*

   Proof: *Assume a or b is odd. We need to show that $a+b$ is odd.*

   If a is odd and b is even, then $a+b$ is odd (since sum of an odd and even number is odd). Similarly, if a is even and b is odd, then $a+b$ is odd. If both a and b are odd, then $a+b$ is even.

   In all cases where a or b is odd, $a+b$ turns out to be odd. Hence, the contrapositive statement is true.

6. *Is the converse of the original statement true or false? Prove your answer.*
   Answer: *The converse of the original statement is **false**.*

   Counterexample: *Take $a=1$ and $b=2$. Here, $a+b=3$, which is odd. However, $a=1$ is odd and $b=2$ is even. Hence, a and b are not both even, yet $a+b$ is even.*

   Therefore, the converse statement does no

Solution0 5.

1. *Prove that if n is an even integer, then n^2 is also an even integer.*

   Proof: *Suppose n is an even integer. By definition, this means $n=2k$ for some integer k. Now, consider n^2:*

   $$n^2 = (2k)^2 = 4k^2$$

   Since k is an integer, $4k^2$ is clearly divisible by 2 (specifically, $4k^2 = 2(2k^2)$). Therefore, n^2 is even. □

2. *Prove that for all integers a and b, if $a+b$ is even, then both a and b are even.*

   Proof: *Assume $a+b$ is even. We want to show that both a and b are even.*

   Suppose a is odd (so $a = 2m+1$ for some integer m). Then b must be odd as well, because the sum of an odd and an even number is odd. Thus, $b = 2n+1$ for some integer n.

   Now, let's compute $a+b$:

   $$a+b = (2m+1) + (2n+1) = 2m+2n+2 = 2(m+n+1)$$

   This expression shows that $a+b$ is even, which contradicts our initial assumption that $a+b$ is odd.

   Therefore, our assumption that a is odd must be false. Hence, a (and similarly b) must be even.

   Therefore, if $a+b$ is even, then both a and b are even. □

Solution0 6.

Suppose, for the sake of contradiction, that $\sqrt{2}$ is rational. This means we can express $\sqrt{2}$ as $\frac{p}{q}$, where p and q are integers with no common factors (i.e., $\frac{p}{q}$ is in its simplest form).

Then,

$$\sqrt{2} = \frac{p}{q}$$

Squaring both sides gives:

$$2 = \left(\frac{p}{q}\right)^2 = \frac{p^2}{q^2}$$

Multiplying through by q^2 gives:

$$2q^2 = p^2$$

This implies that p^2 is even (since $2q^2$ is even). From this, we conclude that p itself must be even (because the square of an odd number is odd, and the square of an even number is even).

Let $p = 2k$ for some integer k. Substituting in $p^2 = (2k)^2 = 4k^2$, we get:

$$2q^2 = 4k^2 \implies q^2 = 2k^2$$

This shows that q^2 is even, hence q must also be even (similar reasoning as for p). Now, both p and q are even. However, this contradicts our initial assumption that $\frac{p}{q}$ is in its simplest form (no common factors). If both p and q are even, then $\frac{p}{q}$ can be further reduced by dividing both numerator and denominator by 2, which contradicts the assumption that $\frac{p}{q}$ is already in its simplest form.

Therefore, our initial assumption that $\sqrt{2}$ is rational must be false. Hence, $\sqrt{2}$ is irrational. □

Solution0 7.

1. *Prove that if n^2 is an even integer, then n is also an even integer.*

   Proof: *Assume n is not even, i.e., n is odd. Then n can be expressed as $n = 2k+1$ for some integer k. Now, compute n^2:*

   $$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2+2k) + 1$$

   This shows that n^2 is odd, because it leaves a remainder of 1 when divided by 2.

   Therefore, if n^2 is even, then n must be even. □

2. *Prove that if a and b are integers and ab is odd, then both a and b are odd.*

   Proof: *Assume a is not odd (i.e., a is even) or b is not odd (i.e., b is even). We will show that in both cases ab cannot be odd.*

   If a is even, then $a = 2m$ for some integer m. Similarly, if b is even, then $b = 2n$ for some integer n. Therefore,
   $$ab = (2m)(2n) = 4mn$$
   which is clearly even.

   Hence, if either a or b is even, then ab cannot be odd. Therefore, if ab is odd, then both a and b must be odd. □

Solution0 8.

1. *Prove that for all positive integers n, $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.*

   Proof: *We will prove the statement by mathematical induction.*

   Base Case: *For $n = 1$,*
   $$1 = 1^2$$
   which is true.

   Inductive Step: *Assume the statement holds for some arbitrary positive integer k, i.e.,*
   $$1 + 3 + 5 + \cdots + (2k - 1) = k^2.$$
   We need to prove it holds for $k + 1$:
   $$1 + 3 + 5 + \cdots + (2k - 1) + (2(k + 1) - 1) = (k + 1)^2.$$
   Adding $2(k + 1) - 1 = 2k + 1$ to both sides gives:
   $$k^2 + 2k + 1 = (k + 1)^2.$$
   Hence, the statement holds for $k + 1$.

   Therefore, by mathematical induction, for all positive integers n, $1+3+5+\cdots+(2n-1) = n^2$. □

2. *Prove that for all positive integers n, $2^n > n$.*

   Proof: *We will prove the statement by mathematical induction.*

   Base Case: *For $n = 1$,*
   $$2^1 = 2 > 1,$$
   which is true.

   Inductive Step: *Assume the statement holds for some arbitrary positive integer k, i.e.,*
   $$2^k > k.$$
   We need to prove it holds for $k + 1$:
   $$2^{k+1} = 2 \cdot 2^k > 2 \cdot k = k + k \geq k + 1.$$
   Therefore, $2^{k+1} > k + 1$.

   Hence, by mathematical induction, for all positive integers n, $2^n > n$. □

Solution0 9.

1. *Prove that for all integers n, $n^2 \geq 0$.*

   Proof: *We will consider two cases for n:*

   Case 1: $n \geq 0$.
   $$n^2 = n \cdot n = n^2 \geq 0 \quad (\textit{since } n \geq 0)$$

   Case 2: $n < 0$. *If* $n < 0$, *then* $n^2 = (-n)^2 = n^2 \geq 0$.

   Therefore, in both cases, $n^2 \geq 0$ holds for all integers n. □

2. *Prove that for any integer n, n^3 is either even or odd.*

   Proof: *We will consider two cases for n:*

   Case 1: *n is even. If n is even, then $n = 2k$ for some integer k. Thus,*
   $$n^3 = (2k)^3 = 8k^3 = 2(4k^3),$$
   which is even.

   Case 2: *n is odd. If n is odd, then $n = 2k + 1$ for some integer k. Thus,*
   $$n^3 = (2k + 1)^3 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1,$$
   which is odd.

   Therefore, n^3 is either even or odd for any integer n. □

Solution0 10.

Prove that the following statement is false: "Every positive integer is the sum of three squares."
Proof: *We will provide a counterexample to show that not every positive integer can be expressed as the sum of three squares.*

Consider the integer $n = 7$.

Now, we will check if 7 can be expressed as $a^2 + b^2 + c^2$ where a, b, c are integers.

Testing possible combinations:

$$1^2 + 1^2 + 1^2 = 1 + 1 + 1 = 3,$$
$$1^2 + 1^2 + 2^2 = 1 + 1 + 4 = 6,$$
$$1^2 + 2^2 + 2^2 = 1 + 4 + 4 = 9.$$

None of these combinations yield 7.

Therefore, 7 cannot be expressed as the sum of three squares.

Since we have found at least one positive integer (specifically 7) that cannot be written as the sum of three squares, the statement "Every positive integer is the sum of three squares" is false. □