Abdelhafid Boussouf University Center- Mila
Institute of Sciences and Technology
Module: Analysis 2
Duration: 90 min

Academic Year: 2023/2024
First Year Engineering
Semester 2
May 2024

الإمتحان النهائي: التحليل 2
Final exam: Analysis 2

Exercise 1: (5 points)

Show that the following functions are equivalent, in the same order, in higher order, or in lower oder:

بين أن الدوال التالية، متكافئة ، عند نفس الرتبة، عند رتبة أعلى، أو عند رتبة أدنى،

1. $f(x) = \sin(x)$ and $g(x) = \tan(x)$ as x approaches 0. (لما x يقترب من 0)
2. $f(x) = \frac{1}{x^2}$ and $g(x) = \frac{1}{2x^2+3x+10}$ as x approaches infinity. (لما x يقترب من ∞)
3. $f(x) = x - 1$ and $g(x) = x^2 - 1$ as x approaches 1. (لما x يقترب من 1)
4. $f(x) = x\sin(x)$ and $g(x) = \tan(x)$ as x approaches 0. (لما x يقترب من 0)
5. $f(x) = \frac{1}{x^3}$ and $g(x) = \frac{1}{x^4}$ as x approaches ∞. (لما x يقترب من ∞)

Exercise 2 : (5 points)

1. Give the first 4 terms of Taylor series around $a = \frac{1}{2}$ for the following function:

   أعطي الحدود الأربعة الأولى لسلسلة تايلر حول $a = \frac{1}{2}$ من أجل الدالة التالية:

   $$f(x) = \sin(\pi x)$$

2. Obtain the Maclaurin series of $\frac{1}{1+x^2}$ and deduce the Maclaurin series of $\arctan(x)$. (the first 4 terms)

   أوجد سلسلة ماكلورين لـ $\frac{1}{1+x^2}$ ثم استنتج سلسلة ماكلورين لـ $\arctan(x)$. (الحدود الأربعة الأولى)

Exercise 3: (5 points)

Find the integrals:(أوجد التكاملات)

$$1)I_1 = \int \frac{x+1}{x^2+2x+2}dx$$
$$2)I_2 = \int \arctan(x)dx$$
$$3)I_3 = \int \frac{1}{\sin(x)}dx$$
$$4)I_4 = \int \frac{x}{x^2-3x+2}dx$$
$$5)I_5 = \int \frac{x^5}{x^3-1}dx$$

Exercise 4: (5 points)

Solve the following differential equations:

حل المعادلتان التفاضلتان التاليتان:

$$(1+x)^2\frac{dy}{dx} = (1+y)^2 \quad (1)$$

and

$$x^2\frac{dy}{dx} + 2xy = 5y^3 \quad (2)$$

GOOD LUCK (بالتوفيق)

التصحيح النموذجي للإمتحان النهائي
في مادة التحليل 02:

التمرين 01:

1) $\lim\limits_{x\to 0} \dfrac{\sin(x)}{\tan(x)} = \lim\limits_{x\to 0} \dfrac{\sin(x)}{\dfrac{\sin(x)}{\cos(x)}} = \lim\limits_{x\to 0} \cos(x) = 1$ (0,75)

f و g متكافئتان ومن نفس الرتبة لما $x\to 0$ (0,25)

2) $\lim\limits_{x\to +\infty} \dfrac{\dfrac{1}{x^2}}{\dfrac{1}{2x^2+3x+10}} = \lim\limits_{x\to +\infty} \dfrac{2x^2+3x+10}{x^2} = 2$ (0,75)

f و g عند نفس الرتبة لما $x\to +\infty$. (0,25)

3) $\lim\limits_{x\to +1} \dfrac{x-1}{x^2-1} = \lim\limits_{x\to +\infty} \dfrac{1}{2x} = \dfrac{1}{2}$ (0,75)

f و g عند نفس الرتبة لما $x\to 1$ (0,25)

4) $\lim\limits_{x\to +\infty} \dfrac{x\sin(x)}{\tan(x)} = \lim\limits_{x\to 0} x\cos(x) = 0$ (0,75)

f عند رتبة أعلى من g لما $x\to +\infty$ (0,25)

5) $\lim\limits_{x\to +\infty} \dfrac{\dfrac{1}{x^2}}{\dfrac{1}{x^4}} = \lim\limits_{x\to +\infty} \dfrac{x^4}{x^2} = \lim\limits_{x\to +\infty} x^2 = +\infty$ (0,75)

f عند رتبة أدنى من g لما $x\to\infty$. (0,25)

التمرين 02:

1) $f(x) = \sin(\pi x)$; $f(\frac{1}{2}) = \sin(\frac{\pi}{2}) = 1$ 0,5

$f'(x) = +\pi \cos(\pi x)$; $f'(\frac{1}{2}) = \pi \cos(\frac{\pi}{2}) = 0$ 0,5

$f''(x) = -\pi^2 \sin(\pi x)$; $f''(\frac{1}{2}) = -\pi^2 \sin(\frac{\pi}{2}) = -\pi^2$ 0,5

$f'''(x) = -\pi^3 \cos(\pi x)$; $f'''(\frac{1}{2}) = -\pi^3 \cos(\frac{\pi}{2}) = 0$ 0,5

$$f(x) = f(a)(x-a)^0 + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2$$
$$+ \frac{f'''(a)}{2!}(x-\frac{1}{2})^2 + \dots$$

$$= (x-\frac{1}{2})^0 + 0(x-a)^1 - \pi^2(x-\frac{1}{2})^2 + \pi^4(x-a)^4$$
$$+ \pi^6 (x-a)^6 + \dots$$

0,5

2)

1	$1+x^2$
$-(1+x^2)$	$1-x^2+x^4-x^6 \dots$
$-x^2$	
$-(-x^2-x^4)$	
x^4	
$-(x^4+x^6)$	
$-x^6$	

1,5

$\Rightarrow f(x) = 1 - x^2 + x^4 - x^6 \dots$

ومنه لدينا:

$$\arctan(x) = \int \frac{1}{2+x^2}$$

1

$$= x^2 - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots$$

التمرين 03:

1) $I_1 = \int \frac{x+1}{x^2+2x+2}\,dx = \ln(x^2+2x+2) + C$

(تغيير المتغير)

1

2) $I_2 = \int \arctan x \, dx$

$\begin{cases} f(x) = \arctan(x) \\ g'(x) = 1 \end{cases} \Rightarrow \begin{cases} f'(x) = \frac{1}{1+x^2} \\ g(x) = x \end{cases}$

1

بالتجزئة

$I_2 = x\arctan(x) - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx$

$= x\arctan(x) - \frac{1}{2}\ln(1+x^2) + C$

3) $I_3 = \int \frac{1}{\sin(x)}\,dx$

1

$t = \tan\left(\frac{x}{2}\right) \Rightarrow \frac{x}{2} = \arctan(t) \Rightarrow x = 2\arctan(t)$

$dx = \frac{2\,dt}{1+t^2}$

$\sin x = \frac{2t}{t^2+1}$

$I_3 = \int \frac{1}{\frac{2t}{t^2+1}} \frac{2\,dt}{1+t^2} = \int \frac{dt}{t} = \ln(t) + C$

$= \ln\left(\tan\frac{x}{2}\right) + C$

4) $$I_4 = \int \frac{x}{x^2-3x+2}\,dx$$

$$\Delta = b^2 - 4ac = (-3)^2 - 4(1)\cdot(2) = 9 - 8 = 1$$

$$x_1 = \frac{-b-\sqrt{\Delta}}{2a} = \frac{3-1}{2} = 1$$

$$x_2 = \frac{-b+\sqrt{\Delta}}{2a} = \frac{3+1}{2} = 2$$

ومنه:

$$\frac{x}{x^2-3x+2} = \frac{x}{(x-1)(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-2)}$$

$$= \frac{A(x-2)+B(x-1)}{(x-1)(x-2)}$$

بالمطابقة:

$$x = A(x-2) + B(x-1)$$

$$x=1 \Rightarrow 1 = A(1-2) \Rightarrow \boxed{A=-1}$$

$$x=2 \Rightarrow 2 = B(2-1) \Rightarrow \boxed{B=2}$$

وعليه

$$I_4 = -\int \frac{1}{(x-1)}\,dx + 2\int \frac{1}{(x-2)}\,dx$$

$$= -\ln|x-1| + 2\ln|x-2| + C$$

$$= \ln\left(\frac{|x-2|^2}{|x-1|}\right) + C$$

1

5) $$I_5 = \int \frac{x^5}{x^3-1}\,dx$$

$$\begin{array}{r|l} x^5 & x^3-1 \\ \hline -\;(x^5-x^2) & x^2 \\ \hline x^2 & \end{array}$$

ومنه

$$I_5 = \int x^2\,dx + \int \frac{x^2}{x^3-1}\,dx$$

$$= \frac{1}{3}x^3 + \frac{1}{3}\ln|x^3-1| + C$$

1

التمرين 04:

$$(1+x^2)\frac{dy}{dx} = (1+y)^2$$

$$\int \frac{1}{(1+y)^2}\,dy = \int \frac{1}{(1+x)^2}\,dx$$

$$\int (1+y)^{-2}\,dy = \int (1+x)^{-2}\,dx$$ 1

$$u = 1+x \Rightarrow du = dx$$

$$-(1+y)^{-1} = -(1+x)^{-1} + C_1$$

$$\frac{1}{1+y} = \frac{1}{1+x} - C_1$$

ومنه

$$\frac{1}{1+y} = \frac{1}{1+x} - C_1\frac{(1+x)}{(1+x)}$$ 1

$$= \frac{1 - C_1(1+x)}{(1+x)}$$

$$1+y = \frac{1+x}{1-C_1(1+x}$$ 0,5

$$y = \frac{1+x}{1-C_1(1+x)^{-1}}$$

$$C = -C_1$$

$$y = \frac{1+x}{1+C(1+x)} - 1$$

2

$$x^2\frac{dy}{dx} + 2xy = 5y^3$$

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{5}{x^2}y^3$$

$$z = y^{1-3}$$

$$\frac{dy}{dx} = -\frac{1}{2}z^{\frac{3}{2}}\frac{dz}{dx}$$

$$\Rightarrow -\frac{1}{2}\, y^{-\frac{3}{2}} \frac{dy}{dx} + \frac{2}{x}\left(y^{-\frac{1}{2}}\right) = \frac{5}{x^2}\left(y^{-\frac{1}{2}}\right)^3$$

نضرب الطرفين في $-2\, y^{\frac{3}{2}}$:

$$\frac{dy}{dx} + \frac{2}{x}\, y^{-\frac{1}{2}}\left(-2\, y^{\frac{3}{2}}\right) = \frac{-10}{x^2}$$

$$\Rightarrow \frac{dy}{dx} - \frac{4}{x}\, y = -\frac{10}{x^2} \qquad 1$$

الحل العام

$$y = y_H + y_p$$

$$\frac{dy_H}{dx} - \frac{4}{x}\, y_H = 0$$

$$\Rightarrow \frac{dy_H}{y_H} = \frac{4}{x}\, dx \Rightarrow \ln|y_H| = 4\ln|x| + C_1$$
$$= \ln|x|^4 + C_1$$

$$|y_H| = e^{C_1}|x|^4 \Leftrightarrow y_H = \pm\, e^{C_1} x^4$$

ومنه $\boxed{y_H = k\, x^4}$ 0.5

نستعمل طريقة تغيير الثابت لإيجاد الحل الخاص y_p، نعوض في المعادلة:

$$\frac{dy_p}{dx} - \frac{4}{x}\, y_p = \frac{-10}{x^2}$$

حيث $y_p = k(x)\, x^4$

$$\frac{dk(x)}{dx} = -\frac{10}{x^6} = -10\, x^{-6}$$

$$\Rightarrow dk(x) = -10\, x^{-6}$$

$$\Rightarrow k(x) = -10\left(-\frac{1}{5}\, x^{-5}\right)$$

$$\boxed{k(x) = 2x^{-5} + c} \qquad 0.5$$

$$z_p = (2x^{-5} + c)\, x^{4} = (2x^{-1} + c\, x^{4})$$

ومنه الحل العام:

$$z = z_H + z_p = k\, x^4 + 2x^{-1} + c\, x^4$$

$$= (k + c)\, x^4 + 2x^{-1}$$

$$= A x^4 + 2x^{-1}$$

$$z = y^{-2} = A x^4 + 2x^{-1}$$

$$y^2 = \frac{1}{A x^4 + 2x^{-1}}$$

ومنه: $$\boxed{y = \pm \sqrt{\frac{x}{A x^5 + 2}}}$$ 0,5