Solution de la Série N° 1 :

Exercice 1 :

* On suppose : $f(x) = \text{Arc}\sin x - \dfrac{x}{\sqrt{1-x^2}}$, $\forall x \in]0,1[$

f est défini et dérivable sur $]0,1[$, On a :

$$f'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{\sqrt{1-x^2} + \frac{2x^2}{2\sqrt{1-x^2}}}{1-x^2} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{(1-x^2)\sqrt{1-x^2}}$$

$$= \frac{1}{\sqrt{1-x^2}}\left(1 - \frac{1}{1-x^2}\right) = \frac{-x^2}{\sqrt{1-x^2}(1-x^2)} < 0 \ (\text{car } -x^2 < 0$$

et $1-x^2 > 0$, $\sqrt{1-x^2} \geq 0$, $\forall x \in]0,1[$). Donc $f'(x) < 0$

$\Rightarrow f$ est décroissante $\forall x \in]0,1[$.

On a : $x > 0 \Rightarrow f(x) < f(0) = 0 \Rightarrow \text{Arc}\sin x - \dfrac{x}{\sqrt{1-x^2}} < 0$

$$\Rightarrow \text{Arc}\sin x < \frac{x}{\sqrt{1-x^2}}, \ \forall x \in]0,1[.$$

* On suppose : $f(x) = \text{Arc}\tan x - \dfrac{x}{1-x^2}$ $\forall x > 0$,

f est dérivable $\Rightarrow f'(x) = \dfrac{1}{1+x^2} - \dfrac{1+x^2-2x^2}{(1+x^2)^2} \Rightarrow$

$$f'(x) = \frac{1}{1+x^2} - \frac{1-x^2}{(1+x^2)^2} = \frac{2x^2}{(1+x^2)^2} > 0 \Rightarrow f \text{ croissante}$$

On a : $x > 0 \Rightarrow f(x) > f(0) = 0 \Rightarrow \text{Arc}\tan x - \dfrac{x}{1+x^2} > 0$

$$\Rightarrow \text{Arc}\tan x > \frac{x}{1+x^2}, \ \forall x > 0$$

* On suppose : $f(x) = \text{Arc}\sin x + \text{Arc}\cos x$, $x \in]-1,1[$

f défini et dérivable $\Rightarrow f'(x) = \dfrac{1}{\sqrt{1-x^2}} - \dfrac{1}{\sqrt{1-x^2}} = 0$

$f'(x) = 0 \Rightarrow f$ est constante $\forall x \in]-1,1[$

$f(0) = \text{Arc}\sin 0 - \text{Arc}\cos 0 = \dfrac{\pi}{2}$. Alors $\forall x \in]-1,1[$

$$f(x) = \frac{\pi}{2} \Rightarrow \text{Arc}\sin x + \text{Arc}\cos x = \frac{\pi}{2}.$$

* On suppose : $f(x) = \operatorname{Arctan} x + \operatorname{Arctan} \frac{1}{x}$

$$f'(x) = \frac{1}{1+x^2} + \frac{-\left(\frac{1}{x^2}\right)}{1+\frac{1}{x^2}} = \frac{1}{1+x^2} - \frac{x^2}{x^2(1+x^2)} = 0$$

$\Rightarrow f'(x) = 0 \Rightarrow f(x)$ est constante $\forall x > 0$

$f(1) = \operatorname{Arctan} 1 + \operatorname{Arctan} 1 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$. Alors

$\forall x > 0,\ f(x) = \frac{\pi}{2} \Rightarrow \operatorname{Arctan} x + \operatorname{Arctan} \frac{1}{x} = \frac{\pi}{2}$

Exercice 02 :

1) $x = \ln\left(\tan\left(\frac{y}{2} + \frac{\pi}{4}\right)\right)$

* $\operatorname{ch} x = \frac{1}{2}\left(e^x + e^{-x}\right) = \frac{1}{2}\left(e^x + \frac{1}{e^x}\right)$

$$= \frac{1}{2}\left(e^{\ln \tan\left(\frac{y}{2}+\frac{\pi}{4}\right)} + \frac{1}{e^{\ln \tan\left(\frac{y}{2}+\frac{\pi}{4}\right)}}\right)$$

$$= \frac{1}{2}\left[\tan\left(\frac{y}{2}+\frac{\pi}{4}\right) + \frac{1}{\tan\left(\frac{y}{2}+\frac{\pi}{4}\right)}\right]$$

$$= \frac{1}{2}\left(\frac{\sin\left(\frac{y}{2}+\frac{\pi}{4}\right)}{\cos\left(\frac{y}{2}+\frac{\pi}{4}\right)} + \frac{\cos\left(\frac{y}{2}+\frac{\pi}{4}\right)}{\sin\left(\frac{y}{2}+\frac{\pi}{4}\right)}\right)$$

$$= \frac{1}{2}\left(\frac{1}{\cos\left(\frac{y}{2}+\frac{\pi}{4}\right)\sin\left(\frac{y}{2}+\frac{\pi}{4}\right)}\right) = \frac{1}{\sin 2\left(\frac{y}{2}+\frac{\pi}{4}\right)}$$

$$= \frac{1}{\sin\left(y+\frac{\pi}{2}\right)} = \frac{1}{\cos y}$$

* $\operatorname{sh} x = \frac{1}{2}\left(e^x - e^{-x}\right) = \frac{1}{2}\left[\tan\left(\frac{y}{2}+\frac{\pi}{4}\right) - \frac{1}{\tan\left(\frac{y}{2}+\frac{\pi}{4}\right)}\right]$

$$= \frac{1}{2}\left[\frac{\sin\left(\frac{y}{2}+\frac{\pi}{4}\right)}{\cos\left(\frac{y}{2}+\frac{\pi}{4}\right)} - \frac{\cos\left(\frac{y}{2}+\frac{\pi}{4}\right)}{\sin\left(\frac{y}{2}+\frac{\pi}{4}\right)}\right]$$

$$= \frac{1}{2}\,\frac{\sin^2\left(\frac{y}{2}+\frac{\pi}{4}\right) - \cos^2\left(\frac{y}{2}+\frac{\pi}{4}\right)}{\cos\left(\frac{y}{2}+\frac{\pi}{4}\right)\sin\left(\frac{y}{2}+\frac{\pi}{4}\right)} = \frac{-\cos 2\left(\frac{y}{2}+\frac{\pi}{4}\right)}{\cos y}$$

$$= \frac{-\cos\left(y+\frac{\pi}{2}\right)}{\cos y} = \frac{\sin y}{\cos y} = \tan y$$

$$* \; th\,x = \frac{sh\,x}{ch\,x} = \tan y \cdot \cos y = \frac{\sin y}{\cos y} \cdot \cos y = \sin y$$

2) $$* \lim_{x \to +\infty} e^{-x}(ch^3 x - sh^3 x) = \lim_{x \to +\infty} e^{-x} \frac{1}{8}\left[(e^{x}+e^{-x})^3 - (e^{x}-e^{-x})^3\right]$$

$$= \lim_{x \to +\infty} \frac{e^{-x}}{8}\left[(e^{3x}+3e^{x}+3e^{-x}+e^{-3x}) - (e^{3x}-3e^{x}+3e^{-x}-e^{-3x})\right]$$

$$= \lim_{x \to +\infty} \frac{e^{-x}}{8}(6e^{x}+2e^{-3x}) = \lim_{x \to +\infty}\left(\frac{6}{8} + \underbrace{\frac{2e^{-4x}}{8}}_{\to 0}\right) =$$

$$= \frac{6}{8} = \frac{3}{4}.$$

$$* \lim_{x \to +\infty} (x - \ln(ch\,x)) = \lim_{x \to +\infty}\left(x - \ln\left(\frac{e^{x}+e^{-x}}{2}\right)\right)$$

$$= \lim_{x \to +\infty}\left(x - \ln\frac{e^{x}}{2}(1+e^{-2x})\right)$$

$$= \lim_{x \to +\infty}\left(\cancel{x} - \cancel{x} + \ln 2 + \ln(1+\underbrace{e^{-2x}}_{\to 0})\right) = \ln 2$$

Exercice 3 :

Formule de Mac Laurin est :

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}$$

$$0 < \theta < 1$$

Montrons que : $\forall x > 0, \; x - \frac{x^2}{2} < \ln(1+x) < x.$

$* \; \underline{x - \frac{x^2}{2} < \ln(1+x)}$:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(\theta x)}{3!}x^3 \;, \; 0<\theta<1$$

$$f(x) = \ln(1+x) \Rightarrow f(0) = 0.$$

$$f'(x) = \frac{1}{1+x} \Rightarrow f'(0) = 1$$

$$f''(x) = -\frac{1}{(1+x)^2} \Rightarrow f'(0) = -1$$

$$f^{(3)}(x) = \frac{2}{(1+x)^3} \Rightarrow f^{(3)}(\theta x) = \frac{2}{(1+\theta x)^3} \cdot \text{Alors :}$$

$$f(x) = \ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3!}\,\frac{2}{(1+\theta x)^3}\,x^3 .$$

On a : $\frac{1}{3(1+\theta x)^3}\,x^3 > 0$ (car $x > 0$ et $0 < \theta < 1$)

$$\Rightarrow x - \frac{1}{2}x^2 + \frac{1}{3(1+\theta x)^3}\,x^3 > x - \frac{x^2}{2} \Rightarrow$$

$$\ln(1+x) > x - \frac{x^2}{2}.$$

* $\underline{\ln(x+1) < x}$:

$$f(x) = \ln(x+1) = f(0) + \frac{f'(0)}{1!}\,x + \frac{f''(\theta x)}{2!}\,x^2, \quad 0 < \theta < 1$$

$$f(0) = 0, \quad f'(0) = 1 \; ; \; f''(x) = -\frac{1}{(1+x)^2} \Rightarrow f''(\theta x) = -\frac{1}{(1+\theta x)^2}$$

$$f(x) = \ln(x+1) = x - \frac{1}{2(1+\theta x)^2}\,x^2 \; ; \; 0 < \theta < 1$$

On a : $-\frac{1}{2(1+\theta x)^2}\,x^2 < 0$ (car $x > 0$ et $0 < \theta < 1$)

$$\Rightarrow x - \frac{1}{2(1+\theta x)^2}\,x^2 < x \Rightarrow \ln(1+x) < x$$

Alors : $x - \frac{x^2}{2} < \ln(1+x) < x, \; \forall x > 0$

<u>Exercice 04</u> :

$$\lim_{x \to 0} \frac{\ln(x+1) - x + \frac{x^2}{4}}{(\sin x)^2} = \lim_{x \to 0} \frac{x - \frac{x^2}{2} + x^2\varepsilon(x) - x + \frac{x^2}{4}}{(\sin x)^2}$$

$$= \lim_{x \to 0} \frac{-\frac{1}{4}x^2 + x^2\varepsilon(x)}{(\sin x)^2} = \lim_{x \to 0} \frac{x^2\left(-\frac{1}{4} + \varepsilon(x)\right)}{(\sin x)^2}$$

$$= \lim_{x \to 0} \frac{-\frac{1}{4} + \overbrace{\varepsilon(x)}^{\to 0}}{\underbrace{\left(\frac{\sin x}{x}\right)^2}_{\to 1}} = -\frac{1}{4}$$

Exercice 05 :

1) * $f(x) = e^x = f(0) + \frac{f'(0)}{1!}x + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k + \frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}, \quad 0<\theta<1$$

On a : $f^{(n)}(x) = e^x \Rightarrow f^{(n)}(0) = 1$ et $f^{(n+1)}(\theta x) = e^{\theta x}$. Alors :

$$f(x) = e^x = 1 + \frac{x}{1!} + \frac{1}{2!}x^2 + \cdots \frac{1}{n!}x^n + \frac{e^{\theta x}}{(n+1)!}x^{n+1} \; ; \; 0<\theta<1$$

$$= \sum_{k=0}^{n} \frac{1}{k!}x^k + \frac{e^{\theta x}}{(n+1)!}x^{n+1} \; ; \; 0<\theta<1.$$

* Montrer que : $1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} < e < 1 + \frac{1}{1!} + \cdots + \frac{1}{n!} + \frac{e}{(n+1)!}$

On a : pour $x = 1$ dans la formule précédente :

$$f(1) = e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \frac{e^\theta}{(n+1)!} \; ; \; 0<\theta<1$$

$$= \sum_{k=0}^{n} \frac{1}{k!} + \frac{e^\theta}{(n+1)!} \; ; \; 0<\theta<1.$$

On a : $0<\theta<1 \Rightarrow 1 < e^\theta < e \Rightarrow \frac{1}{(n+1)!} < \frac{e^\theta}{(n+1)!} < \frac{e}{(n+1)!}$

$$\Rightarrow \frac{1}{(n+1)!} < e - \sum_{k=0}^{n} \frac{1}{k!} < \frac{e}{(n+1)!}$$

$$\Rightarrow 0 < e - \sum_{k=0}^{n} \frac{1}{k!} < \frac{e}{(n+1)!}$$

$$\Rightarrow \sum_{k=0}^{n} \frac{1}{k!} < e < \frac{e}{(n+1)!} + \sum_{k=0}^{n} \frac{1}{k!}$$

$$\Rightarrow 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} < e < 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \frac{e}{(n+1)!}$$

* Déduire la limite de la suite $U_n = \sum_{k=0}^{n} \frac{1}{k!}$

On a : $0 < e - \sum_{k=0}^{n} \frac{1}{k!} < \frac{e}{(n+1)!}$

passage à la limite :

$$0 < \lim_{n\to+\infty}\left(e - \sum_{k=0}^{n} \frac{1}{k!}\right) < \lim_{n\to+\infty} \frac{e}{(n+1)!} = 0 \Rightarrow$$

$$\lim_{n\to+\infty}\left(e-\sum_{k=0}^{n}\frac{1}{k!}\right)=0 \Rightarrow \lim_{n\to+\infty}\sum_{k=0}^{n}\frac{1}{k!}=e.$$

Alors $\lim_{n\to+\infty} U_n = \lim_{n\to+\infty}\sum_{k=0}^{n}\frac{1}{k!}=e.$

Exercice 06 :

On a : $f(x)=\ln(x+1)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\dots+\frac{f^{(n)}(0)}{n!}x^n+\frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}$

$0<\theta<1$

et on a : $f^{(n)}(x)=(-1)^{n-1}\frac{(n-1)!}{(1+x)^n} \Rightarrow$

$$f(x)=\ln(x+1)=x-\frac{1}{2}x^2+\dots+(-1)^{n-1}\frac{1}{n}x^n+(-1)^n\frac{1}{(n+1)}\frac{x^{n+1}}{(1+\theta x)^{n+1}}$$

pour $x=1$:

$$f(1)=\ln 2=1-\frac{1}{2}+\dots+(-1)^{n-1}\frac{1}{n}+(-1)^n\frac{1}{n+1}\frac{1}{(1+\theta)^{n+1}} \Rightarrow$$

$$1-\frac{1}{2}+\dots+(-1)^{n-1}\frac{1}{n}=\ln 2-(-1)^n\frac{1}{n+1}\frac{1}{(1+\theta)^{n+1}} \Rightarrow$$

$$\lim_{n\to+\infty}\left(1-\frac{1}{2}+\dots+(-1)^{n-1}\frac{1}{n}\right)=\lim_{n\to+\infty}\left(\ln 2-(-1)^n\underbrace{\frac{1}{n+1}}_{\to 0}\underbrace{\frac{1}{(1+\theta)^{n+1}}}_{\to 0}\right)$$

$$=\ln 2.$$

Exercice 07 :

$U_n=\prod_{k=1}^{n}\left(1+\frac{k}{n^2}\right)$; $n\in\mathbb{N}^*$

On pose : $V_n=\ln U_n \Rightarrow V_n=\ln\prod_{k=1}^{n}\left(1+\frac{k}{n^2}\right)=\sum_{k=1}^{n}\ln\left(1+\frac{k}{n^2}\right)$

D'après Ex 03 :

$$\frac{k}{n^2}-\frac{k^2}{2n^4}<\ln\left(1+\frac{k}{n^2}\right)<\frac{k}{n^2} \Rightarrow$$

$$\sum_{k=1}^{n}\frac{k}{n^2}-\sum_{k=1}^{n}\frac{k^2}{2n^4}<\sum_{k=1}^{n}\ln\left(1+\frac{k}{n^2}\right)<\sum_{k=1}^{n}\frac{k}{n^2} \Rightarrow$$

$$\frac{1}{n^2}\sum_{k=1}^{n}k-\frac{1}{2n^4}\sum_{k=1}^{n}k^2<\sum_{k=1}^{n}\ln\left(1+\frac{k}{n^2}\right)<\frac{1}{n^2}\sum_{k=1}^{n}k$$

On a : $\sum_{k=1}^{n}k=\frac{n(n+1)}{2}$

et on a: $k^2 \leq n^2 \Rightarrow \sum_{k=1}^{n} k^2 \leq \sum_{k=1}^{n} n^2 = n.n^2 \Rightarrow$

$$\frac{1}{2n^4}\sum_{k=1}^{n} k^2 \leq \frac{1}{2n^4}\cdot n.n^2 = \frac{1}{2n} \Rightarrow -\sum_{k=1}^{n} \frac{k^2}{2n^4} \geq -\frac{1}{2n}$$

Alors: $\frac{1}{n^2}\frac{n(n+1)}{2} - \frac{1}{2n} < \sum_{k=1}^{n} \ln\left(1+\frac{k}{n^2}\right) < \frac{1}{n^2}\cdot\frac{n(n+1)}{2}$

passage à la limite:

$$\lim_{n\to+\infty}\left(\underbrace{\frac{n(n+1)}{2n^2}}_{\frac{1}{2}} - \underbrace{\frac{1}{2n}}_{0}\right) < \lim_{n\to+\infty}\sum_{k=1}^{n} \ln\left(1+\frac{k}{n^2}\right) < \lim_{n\to+\infty}\underbrace{\frac{n(n+1)}{2n^2}}_{\frac{1}{2}}$$

$$\Rightarrow \lim_{n\to+\infty}\sum_{k=1}^{n} \ln\left(1+\frac{k}{n^2}\right) = \frac{1}{2} \Rightarrow \lim V_n = \frac{1}{2}$$

$$\Rightarrow \lim_{n\to+\infty} U_n = \lim_{n\to+\infty} e^{V_n} = e^{\frac{1}{2}}.$$

Exercice 08:

1) $f(x) = x(\operatorname{ch}x)^{\frac{1}{x}} = x\, e^{\frac{1}{x}\ln(\operatorname{ch}x)}$; $x_0 = 0$, $n = 3$

on a: $\operatorname{ch}x = 1 + \frac{x^2}{2} + O(x^2) \Rightarrow \ln(\operatorname{ch}x) = \ln\left(1+\frac{x^2}{2} + O(x^2)\right)$

on pose $y = \frac{x^2}{2} + O(x^2)$ $(x\to 0 \Rightarrow y \to 0)$. Donc

$$\ln(1+y) = y - \frac{1}{2}y^2 + O(y^2) \Rightarrow$$

$$\ln\left(1+\frac{x^2}{2}+O(x^2)\right) = \ln(\operatorname{ch}x) = \frac{x^2}{2} + \frac{1}{8}x^4 + O(x^4)$$

$$\Rightarrow \frac{1}{x}\ln(\operatorname{ch}x) = \frac{x}{2} - \frac{1}{8}x^3 + O(x^3)$$

$$\Rightarrow e^{\frac{1}{x}\ln(\operatorname{ch}x)} = e^{\frac{x}{2} - \frac{1}{8}x^3 + O(x^3)}$$. On pose $z = \frac{x}{2} - \frac{1}{8}x^3 + O(x^3)$

$(x \to 0 \Rightarrow z \to 0)$

$$e^{z} = 1 + z + \frac{1}{2}z^2 + \frac{1}{3!}z^3 + O(z^3) \Rightarrow$$

$$e^{\frac{1}{x}\ln(\operatorname{ch}x)} = 1 + \frac{x}{2} - \frac{1}{8}x^3 + \frac{1}{2}\left(\frac{x}{2} - \frac{1}{8}x^3\right)^2 + \frac{1}{6}\left(\frac{x}{2} - \frac{1}{8}x^3\right)^3 + O(x^3)$$

$$= 1 + \frac{1}{2}x + \frac{1}{8}x^2 - \frac{5}{48}x^3 + O(x^3) \Rightarrow$$

$$f(x) = x\, e^{\frac{1}{x}\ln(\operatorname{ch}x)} = x\left(1 + \frac{1}{2}x + \frac{1}{8}x^2 - \frac{5}{48}x^3 + O(x^3)\right)$$

$$= x + \frac{1}{2}x^2 + \frac{1}{8}x^3 + O(x^3).$$

2) $f(x) = \ln(1+\sin x)$, $x_0 = 0$, $n = 3$

On a : $\sin x = x - \frac{1}{3!}x^3 + O(x^3) \Rightarrow$

$f(x) = \ln(1 + x - \frac{1}{6}x^3 + O(x^3))$, On pose $y = x - \frac{1}{6}x^3 + O(x^3) \Rightarrow$

$f(x) = \ln(1+y) = y - \frac{1}{2}y^2 + \frac{1}{3}y^3 + O(y^3) \Rightarrow$

$f(x) = \ln(1+\sin x) = x - \frac{1}{6}x^3 - \frac{1}{2}(x - \frac{1}{6}x^3)^2 + \frac{1}{3}(x - \frac{1}{6}x^3)^3 + O(x^3)$

$= x - \frac{1}{6}x^3 - \frac{1}{2}x^2 + \frac{1}{3}x^3 + O(x^3) = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + O(x^3)$

3) $f(x) = \dfrac{1+x+x^2}{1-x+x^2}$, $x_0 = 0$, $n = 4$

$$\begin{array}{l|l}
1 + x + x^2 & 1 - x + x^2 \\ \cline{2-2}
\underline{1 - x + x^2} & 1 + 2x + 2x^2 - 2x^4 \\
2x & \\
\underline{2x - 2x^2 + 2x^3} & \\
2x^2 - 2x^3 & \\
\underline{2x^2 - 2x^3 + 2x^4} & \\
-2x^4 &
\end{array}$$

Alors $f(x) = \dfrac{1+x+x^2}{1-x+x^2} = 1 + 2x + 2x^2 - 2x^4 + O(x^4)$

4) $\cos(\sin x)$, $x_0 = 0$, $n = 4$

On a : $\sin x = x - \frac{1}{6}x^3 + O(x^3)$, On pose $y = x - \frac{1}{6}x^3$

$\Rightarrow \cos y = 1 - \frac{1}{2}y^2 + \frac{1}{24}y^4 + O(y^4)$

$\Rightarrow \cos(\sin x) = 1 - \frac{1}{2}(x - \frac{1}{6}x^3)^2 + \frac{1}{24}(x - \frac{1}{6}x^3)^4 + O(x^4)$

$= 1 - \frac{1}{2}x^2 + \frac{1}{6}x^4 + \frac{1}{24}x^4 + O(x^4) \Rightarrow$

$f(x) = \cos(\sin x) = 1 - \frac{1}{2}x^2 + \frac{5}{24}x^4 + O(x^4)$.

5) $f(x) = \dfrac{x}{1-x^2}$, $x_0 = 0$, $n = 4$

$f(x) = x \cdot \dfrac{1}{1-x^2} = x(1 + x^2 + O(x^2)) = x + x^3 + O(x^3)$.

6) $f(x) = \ln\left(\dfrac{\sin x}{x}\right)$; $x_0 = 0$, $n = 4$

On a : $\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + O(x^5) \Rightarrow$

$$\frac{\sin x}{x} = 1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 + O(x^4) \Rightarrow$$

$$f(x) = \ln\left(\frac{\sin x}{x}\right) = \ln(1 \underbrace{- \frac{1}{6}x^2 + \frac{1}{120}x^4 + O(x^4)}_{y}) \Rightarrow$$

$$\ln(1+y) = y - \frac{1}{2}y^2 + O(y^2) \Rightarrow$$

$$f(x) = \ln\left(\frac{\sin x}{x}\right) = -\frac{1}{6}x^2 + \frac{1}{120}x^4 - \frac{1}{2}\left(-\frac{1}{6}x^2 + \frac{1}{120}x^4\right)^2 + O(x^4) \Rightarrow$$

$$f(x) = -\frac{1}{6}x^2 + \frac{1}{120}x^4 - \frac{1}{72}x^4 + O(x^4)$$

$$= -\frac{1}{6}x^2 + \frac{1}{180}x^4 + O(x^4)$$

7) $f(x) = \frac{1+2x+x^3}{x^3+x^5} = \frac{1}{x^3}\left(\frac{1+2x+x^3}{1+x^2}\right), \quad x_0 = 0, \ n = 2$

$$f(x) = \frac{1}{x^3}(1+2x+x^3)\left(\frac{1}{1+x^2}\right) \Rightarrow$$

$$f(x) = \frac{1}{x^3}(1+2x+x^3)(1-x^2+x^4+O(x^4))$$

$$= \frac{1}{x^3}(1-x^2+x^4+2x-2x^3+2x^5+x^3-x^5+O(x^5))$$

$$= \frac{1}{x^3}(1+2x-x^2-x^3+x^4+x^5+O(x^5))$$

$$= \frac{1}{x^3} + \frac{2}{x^2} - \frac{1}{x} - 1 + x + x^2 + O(x^2).$$

8) $f(x) = \exp(\sqrt{x}), \quad x_0 = 1, \ n = 3.$

Suppose : $y = x - 1 \Rightarrow x = y + 1$ (si $x \to 1 \Rightarrow y \to 0$). Donc

$$e^{\sqrt{x}} = e^{\sqrt{y+1}}$$

On a : $\sqrt{y+1} = 1 + \frac{y}{2} - \frac{1}{8}y^2 + \frac{1}{16}y^3 + O(y^2) \Rightarrow$

$$e^{\sqrt{y+1}} = e^{1 + \frac{1}{2}y - \frac{1}{8}y^2 + \frac{1}{16}y^3 + O(y^2)} = e \cdot e^{\overbrace{\frac{1}{2}y - \frac{1}{8}y^2 + \frac{1}{16}y^3 + O(y^3)}^{z}}$$

(si $y \to 0 \Rightarrow z \to 0$).

$$\Rightarrow e^z = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + O(z^3) \Rightarrow$$

$$e^{\sqrt{y+1}} = e \cdot e^z = e \cdot \left(1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + O(z^3)\right)$$

$$= e\left[1 + \frac{1}{2}y - \frac{1}{8}y^2 + \frac{1}{16}y^3 + \frac{1}{2}\left(\frac{1}{2}y - \frac{1}{8}y^2 + \frac{1}{16}y^3\right)^2 + \frac{1}{6}\left(\frac{1}{2}y - \frac{1}{8}y^2 + \frac{1}{16}y^3\right)^3 + O(y^3)\right]$$

$$= e\left[1 + \frac{1}{2}y - \frac{1}{8}y^2 + \frac{1}{16}y^3 + \frac{1}{8}y^2 - \frac{1}{32}y^3 - \frac{1}{32}y^3 + \frac{1}{48}y^3 + O(y^3)\right]$$

$$= e\left[1+\frac{1}{2}y+\frac{1}{48}y^3+O(y^3)\right] \Rightarrow$$

$$f(x)=e^{\sqrt{x}} = e\left[1+\frac{1}{2}(x-1)+\frac{1}{48}(x-1)^3+O(x-1)^3\right]$$

$$= e+\frac{e}{2}(x-1)+\frac{e}{48}(x-1)^3+O(x-1)^3.$$

9) $f(x)=\dfrac{x^3+2}{x-1}$, $x_0=+\infty$, $n=3$

suppose: $y=\dfrac{1}{x} \Rightarrow x=\dfrac{1}{y}$ (si $x\to+\infty$, $y\to 0$)

$$f(x)=\frac{x^3+2}{x-1}=\frac{\left(\frac{1}{y}\right)^3+2}{\frac{1}{y}-1}=\frac{\frac{1+2y^3}{y^3}}{\frac{1-y}{y}}=\frac{1}{y^2}\left(\frac{1+2y^3}{1-y}\right)$$

$$=\frac{1}{y^2}(1+2y^3)\left(\frac{1}{1-y}\right)=\frac{1}{y^2}(1+2y^3)(1+y+y^2+y^3+y^4+y^5+O(y^5))$$

$$=\frac{1}{y^2}(1+y+y^2+y^3+y^4+y^5+2y^3+2y^4+2y^5+O(y^5))$$

$$=\frac{1}{y^2}\left[1+y+y^2+3y^3+3y^4+3y^5+O(y^5)\right]$$

$$=\frac{1}{y^2}+\frac{1}{y}+1+3y+3y^2+3y^3+O(y^3)$$

$$=x^2+x+1+\frac{3}{x}+\frac{3}{x^2}+\frac{3}{x^3}+O\left(\left(\frac{1}{x}\right)^3\right).$$

Exercice 09:

$$*\ (g\circ f)(x) = g(f(x)) = 1-\frac{1}{2}\left(x-\frac{x^3}{6}\right)^2+\frac{1}{24}\left(x-\frac{x^3}{6}\right)^3+O(x^4)$$

$$=1-\frac{1}{2}x^2+\frac{1}{6}x^4+\frac{1}{24}x^4+O(x^4)$$

$$=1-\frac{1}{2}x^2+\frac{5}{24}x^4+O(x^4).$$

$$*\ \lim_{x\to 0}\frac{g\circ f(x)-1+\frac{x^2}{2}}{x^4}=\lim_{x\to 0}\frac{\cancel{1}-\cancel{\frac{1}{2}x^2}+\frac{5}{24}x^4+O(x^4)-\cancel{1}+\cancel{\frac{1}{2}x^2}}{x^4}$$

$$=\lim_{x\to 0}\frac{\frac{5}{24}x^4+O(x^4)}{x^4}$$

$$=\lim_{x\to 0}\frac{\cancel{x^4}\left(\frac{5}{24}+\frac{O(x^4)}{x^4}\right)}{\cancel{x^4}}=\frac{5}{24}$$

with $\frac{O(x^4)}{x^4}\to 0$

Exercice 10 :

$$* \lim_{x \to 0} \left(\frac{1}{\ln(1+x)} - \frac{1}{x} \right) = \lim_{x \to 0} \left(\frac{1}{x - \frac{x^2}{2} + o(x^2)} - \frac{1}{x} \right)$$

$$= \lim_{x \to 0} \frac{1}{x} \left(\frac{1}{1 - \frac{x}{2}} - 1 \right) = \lim_{x \to 0} \frac{1}{x} \left(\cancel{1} + \frac{x}{2} + o(x) - \cancel{1} \right)$$

$$= \lim_{x \to 0} \left(\frac{1}{2} + \underbrace{\frac{o(x)}{x}}_{\to 0} \right) = \frac{1}{2}$$

$$* \lim_{x \to 0} \frac{e^{3x} \sin 3x}{Sh(-2x)} = \lim_{x \to 0} \frac{(1 + 3x + o(x))(3x + o(x))}{-2x + o(x)}$$

$$= \lim_{x \to 0} \frac{3x + o(x)}{-2x + o(x)} = \lim \left(\frac{3 + \frac{o(x)}{x} \to 0}{-2 + \frac{o(x)}{x} \to 0} \right)$$

$$= -\frac{3}{2}$$

$$* \lim_{x \to +\infty} \left(1 + \frac{1}{x} \right)^x = \lim_{y \to 0} (1+y)^{\frac{1}{y}} \quad (\text{on pose } y = \tfrac{1}{x}).$$

$$= \lim_{y \to 0} e^{\frac{1}{y} \ln(1+y)} = \lim_{y \to 0} e^{\frac{1}{y}\left(y - \frac{1}{2} y^2 + o(y^2)\right)}$$

$$= \lim_{y \to 0} e^{1 - \frac{1}{2} y + o(y)} = \lim_{y \to 0} \left(e \cdot e^{-\frac{1}{2} y + o(y)} \right)$$

$$= e.$$