Solution de Série N°02

EXO1 : Somme de Riemann

$$\int_a^b f(x)\,dx = \lim_{n\to+\infty}\frac{b-a}{n}\sum_{k=1}^{n} f\left(a+k\frac{b-a}{n}\right)$$

1) $$\int_0^1 x^2\,dx = \lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{k^2}{n^2}$$

$$= \lim_{n\to+\infty}\frac{1}{n^3}\sum_{k=1}^{n}k^2$$

$$= \lim_{n\to+\infty}\frac{1}{n^3}\,\frac{n(n+1)(2n+1)}{6} = \frac{1}{3}$$

2) $$\int_0^1 e^x\,dx = \lim_{n\to+\infty}\frac{1}{n}\underbrace{\sum_{k=1}^{n}\left(e^{\frac{1}{n}}\right)^k}_{\text{Somme de suite géo}}$$

$$= e-1$$

EXO2 :

1) $$\lim_{n\to+\infty} n\sum_{k=0}^{n-1}\frac{1}{k^2+n^2} = \lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{\frac{k^2}{n^2}+1}$$

supposons : $f(x)=\frac{1}{x^2+1}$, $a=0$, $b=1$

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{\frac{k^2}{n^2}+1} = \int_0^1\frac{1}{x^2+1}\,dx = \left[\text{Arctan}\,x\right]_0^1$$

$$= \frac{\pi}{4}$$

2) $$V_n = \prod_{k=1}^{n}\left(1+\frac{k^2}{n^2}\right)^{\frac{1}{n}}$$

$$= e^{\frac{1}{n}\sum_{k=1}^{n}\ln\left(1+\frac{k^2}{n^2}\right)}$$

$$\lim_{n\to+\infty} v_n = e^{\lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\left(1+\frac{k^2}{n^2}\right)}$$

$$\Rightarrow \lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\left(1+\frac{k^2}{n^2}\right) = \int_0^1\ln(1+x^2)\,dx$$

$(f(x)=\ln(1+x^2)$, $a=0$, $b=1)$

et pour utilise l'intégration par partie :

$$\begin{cases}u'=1\\ v=\ln(1+x^2)\end{cases} \Rightarrow \begin{cases}u=x\\ v'=\frac{2x}{1+x^2}\end{cases}$$

$$\int_0^1\ln(1+x^2)\,dx = \ln 2 - 2 + \frac{\pi}{2}$$

alors $$\lim_{n\to+\infty} v_n = e^{\ln 2-2+\frac{\pi}{2}} = 2e^{\frac{\pi}{2}-2}$$

EXO3 :

1) $I_1 = \int \arctan x\,dx$ (intégration par partie I.pp) ($D_f = \mathbb{R}$)

$$\begin{cases}u'=1\\ v=\text{Arctan}\,x\end{cases} \Rightarrow \begin{cases}u=x\\ v'=\frac{1}{1+x^2}\end{cases}$$ Alors

$$I_1 = x\,\text{Arctan}\,x - \int\frac{x}{1+x^2}\,dx$$

$$= x\,\text{Arctan}\,x - \frac{1}{2}\ln|1+x^2| + C$$

2) $I_2 = \int \arcsin x\,dx$, $D_f =]-1,1[$

Comme I.pp

$$I_2 = x\,\text{Arcsin}\,x - \sqrt{1-x^2} + C$$

3) $I_3 = \int \cos x\, e^x\,dx$ (I.p.p 2 fois)

$$I_3 = \frac{1}{2}e^x(\sin x+\cos x)+C$$

4) $I_4 = \int\frac{1}{x\ln x}\,dx$ (changement variable CH.V) $y=\ln x$, $dy=\frac{1}{x}dx$

$$I_4 = \ln|\ln|x|| + C$$

5) $I_5 = \int\frac{x}{\sqrt{x+1}}\,dx$, CH.V

$t = x+1 \Rightarrow dt = dx$.

$$I_5 = \frac{2}{3}(x+1)^{\frac{1}{2}}(x-2)+C$$

6) $$I_6 = \int\frac{1}{3+e^x}\,dx = \int\frac{e^x}{3e^x+1}\,dx$$

CH.V $t=e^x$

$$I_6 = \frac{1}{3}\ln|3e^x+1|+C$$

7) $$I_7 = \int\frac{x+2}{x^2-3x-4}\,dx$$

$$\frac{x+2}{x^2-3x-4} = \frac{x+2}{(x+1)(x-4)} = \frac{a}{x+1}+\frac{b}{x-4}$$

$$= \frac{ax-a4+bx+b}{(x+1)(x-4)} \Rightarrow \begin{cases}a=-\frac{1}{5}\\ b=\frac{6}{5}\end{cases}$$

$$I_7 = \int \frac{x+2}{x^2-3x-4}dx = -\frac{1}{5}\int \frac{1}{x+1}dx + \frac{6}{5}\int \frac{1}{x-4}dx$$

$$= -\frac{1}{5}\ln|x+1| + \frac{6}{5}\ln|x-4| + C$$

8) $$I_8 = \int \frac{1}{(x+2)(x^2+2x+5)}dx$$

$$\frac{1}{(x+2)(x^2+2x+5)}dx = \frac{a}{x+2} + \frac{bx+c}{x^2+2x+5}$$

$$\Rightarrow \begin{cases} a = \frac{1}{5} \\ b = -\frac{1}{5} \end{cases}, \quad c = 0$$

$$I_8 = \frac{1}{5}\int \frac{1}{x+2}dx - \frac{1}{5}\underbrace{\int \frac{x}{x^2+2x+5}dx}_{I'_8}$$

$$= \frac{1}{5}\ln|x+2| - \frac{1}{5} I'_8$$

$$I'_8 = \frac{1}{2}\int \frac{2x+2-2}{x^2+2x+5}dx$$

$$= \frac{1}{2}\int \frac{2x+2}{x^2+2x+5}dx - \int \frac{1}{x^2+2x+5}dx$$

$$= \frac{1}{2}\ln|x^2+2x+5| - \frac{1}{4}\underbrace{\int \frac{1}{\left(\frac{x+1}{2}\right)^2+1}dx}_{I''_8}$$

on pose : $t = \frac{x+1}{2} \Rightarrow dt = \frac{1}{2}dx$

$$I''_8 = \int \frac{2}{t^2+1}dt = 2\,\mathrm{Arctan}\, t + C$$

$$= 2\,\mathrm{Arctan}\left(\frac{x+1}{2}\right). \text{ Alors}$$

$$I_8 = \frac{1}{5}\ln|x+2| - \frac{1}{5}\Big[\frac{1}{2}\ln(x^2+2x+5)$$
$$- \frac{1}{2}\mathrm{Arctan}\left(\frac{x+1}{2}\right)\Big] + C.$$

$$= \frac{1}{5}\ln|x+2| - \frac{1}{10}\ln(x^2+2x+5)$$
$$+ \frac{1}{10}\mathrm{Arctan}\left(\frac{x+1}{2}\right) + C$$

EXO 5:

1) $$I_1 = \int \cos^{2017} x \sin x\, dx$$

CH.V. $t = \cos x$

$$I_1 = -\frac{1}{2018}\cos^{2018} x + C$$

2) $$I_2 = \int \sin^2 x \cos^2 x\, dx$$

en utilisant $\sin^2 x = \frac{1-\cos 2x}{2}$
$\cos^2 x = \frac{1+\cos 2x}{2}$ $\Rightarrow$

$$I_2 = \int \frac{1}{4}(1-\cos 2x)(1+\cos 2x)\,dx.$$

$$= \frac{1}{4}\int (1-\cos^2 2x)\,dx.$$

$$= \frac{1}{4}x - \frac{1}{4}\int \left(\frac{1+\cos 4x}{2}\right)dx$$

$$= \frac{1}{4}x - \frac{1}{8}x - \frac{1}{8}\int \cos 4x\, dx.$$

$$= \frac{1}{8}x - \frac{1}{32}\sin 4x + C$$

3) $$I_3 = \int \frac{1}{\sin x}dx$$

CH.V : $t = \tan\frac{x}{2} \Rightarrow \sin x = \frac{2t}{t^2+1}$

$\cos x = \frac{1-t^2}{1+t^2}$, $dx = \frac{2dt}{1+t^2}$

$$I_3 = \ln\left|\tan\frac{x}{2}\right| + C.$$

4) $$I_4 = \int \frac{1}{\cos x}dx. \quad \text{CH.V.}$$

$t = \tan\frac{x}{2} \Rightarrow \cos x = \frac{1-t^2}{1+t^2}$, $dx = \frac{2dt}{1+t^2}$

$$I_4 = \ln\left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right| + C$$

5) $$I_5 = \int \frac{\sin x}{\sin x + \cos x}dx$$

6) $$I_6 = \int \frac{\cos x}{\sin x + \cos x}dx.$$

$$I_5 + I_6 \Rightarrow \int \frac{\sin x + \cos x}{\sin x + \cos x}dx = x. \quad (*)$$

$$I_6 - I_5 \Rightarrow \int \frac{\cos x - \sin x}{\sin x + \cos x}dx.$$

$$= \ln(\sin x + \cos x) \quad (**)$$

$(*) + (**) \Rightarrow$

$$I_6 = \frac{1}{2}\left(x + \ln(\sin x + \cos x)\right) + C$$

(*) - (**) $\Rightarrow$

$I_5 = \frac{1}{2}\left(x - \ln(\cos x + \sin x)\right) + c$

Exo 5:

$I_1 = \int_{-1}^{1} (\text{Arc}\cos x)^2 \, dx$ (I.p.p 2 fois)

$\begin{cases} U = (\text{Arc}\cos x)^2 \\ V' = 1 \end{cases} \Rightarrow \begin{cases} U' = -\frac{2\,\text{Arc}\cos x}{\sqrt{1-x^2}} \\ V = x \end{cases}$

$I_1 = x(\text{Arc}\cos x)^2\Big]_{-1}^{1} + \int_{-1}^{1} \frac{2x\,\text{Arc}\cos x}{\sqrt{1-x^2}}$

$\begin{cases} l = \text{Arc}\cos x \\ h' = \frac{2x}{\sqrt{1-x^2}} \end{cases} \Rightarrow \begin{cases} l' = -\frac{1}{\sqrt{1-x^2}} \\ h = -2\sqrt{1-x^2} \end{cases}$

$I_1 = \pi^2 + \left(-2\sqrt{1-x^2}\,\text{Arc}\cos x\right)\Big]_{-1}^{1} - 2x\Big]_{-1}^{1}$

$= \pi^2 - 4$

2) $I_2 = \int_0^{\frac{\pi}{2}} x \sin x \, dx$ (I.p.p)

$\begin{cases} u = x \\ v' = \sin x \end{cases} \Rightarrow \begin{cases} u' = 1 \\ v = -\cos x \end{cases}$

$I_2 = -x\cos x\Big]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x \, dx$

$= \sin x\Big]_0^{\frac{\pi}{2}} = 1$

3) $I_3 = \int_0^1 \frac{1}{(1+x^2)^2} \, dx$

en utilisant la formule:

$J_{k+1} = \frac{x}{2k(1+x^2)^k} + \frac{2k-1}{2k} J_k$

où: $J_k = \int \frac{1}{(1+x^2)^k} \, dx$

$J_1 = \int \frac{1}{1+x^2} = \text{Arc}\tan x$. Alors

pour $k = 1$:

$J_2 = \int_0^1 \frac{1}{(1+x^2)^2} \, dx = \frac{x}{2(1+x^2)}\Big]_0^1 + \frac{1}{2} J_1\Big]_0^1 \Rightarrow$

$I_3 = \int_0^1 \frac{1}{(1+x^2)^2} \, dx = \frac{1}{4} + \frac{1}{2}\text{Arc}\tan x\Big]_0^1$

$= \frac{1}{4} + \frac{1}{2}\cdot\frac{\pi}{4} = \frac{1}{4} + \frac{\pi}{8}$

4) $I_4 = \int_0^1 \frac{3x+1}{(1+x)^2} \, dx$

CH.V: $t = x + 1 \Rightarrow x = t - 1$

$I_4 = \int_1^2 \frac{3(t-1)+1}{t^2} \, dt$

$= \int_1^2 \frac{3t}{t^2} \, dt - \int_1^2 \frac{2}{t^2} \, dt$

$= 3\ln t\Big]_1^2 + \frac{2}{t}\Big]_1^2$

$= 3\ln 2 + 2\left(\frac{1}{2} - 1\right)$

$= 3\ln 2 - 1$

5) $I_5 = \int_1^2 x^2 \ln x \, dx$ I.p.p.

6) $I_6 = \int_{-1}^{1} \frac{1}{x^2 + 4x + 7} \, dx$

$= \int_{-1}^{1} \frac{1}{(x+2)^2 + 3} \, dx$

$= \frac{1}{3}\int_{-1}^{1} \frac{1}{\left(\frac{x+2}{\sqrt{3}}\right)^2 + 1} \, dx$

CH.V: $t = \frac{x+2}{\sqrt{3}}$

$I_6 = \frac{\sqrt{3}}{3} \int_{\frac{\sqrt{3}}{3}}^{\sqrt{3}} \frac{dt}{t^2+1} = \frac{\sqrt{3}}{3}\text{Arc}\tan t\Big]_{\frac{\sqrt{3}}{3}}^{\sqrt{3}}$

$= \frac{\sqrt{3}}{3}\left[\text{Arc}\tan\sqrt{3} - \text{Arc}\tan\frac{\sqrt{3}}{3}\right]$