Chapter 5

Direct Methods for Solving Systems of Linear Equations

The solution of large linear systems is of great importance in applied science, and engineering. In general, linear systems can be solved using direct or indirect (iterative) methods.

- Direct methods can give the exact value of the solution after a finite number of operations.

- Indirect (iterative) methods consist of constructing a series of solution vectors x_i from a vector proposed as an initial solution. The sequence of solutions converges to the exact solution x.

Direct methods are often more reliable, but they usually require a computer with a very large memory. These methods are suitable for small systems, but not for large ones.

5. 1 Mathematical review about matrixes

5. 1. 1 Definition

The matrix of IR element is a two-dimensional array composed of m rows and n columns. The set of IR matrices of dimension m, n is denoted $M_{(m,n)}$ and forms a vector space on IR.

Example

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \qquad (1)$$

Matrix A contains three lines and four columns $(m = 3, n = 4)$ we note $A_{(3,2)}$.

- **Square matrix**

A square matrix is a matrix whose number of lines equals the number of columns ($n = m$).

5. 1. 2 Some properties

- **Sum of two matrixes**

Let A and B be two matrixes:

- $A + B = [a_{ij} + b_{ij}]_{m,n}$ $1 \leq i \leq m, \quad 1 \leq j \leq n$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}, \; B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{bmatrix} \Rightarrow$$

$$A + B = \begin{bmatrix} (a_{11} + b_{11}) & (a_{12} + b_{12}) & (a_{13} + b_{13}) & (a_{14} + b_{14}) \\ (a_{21} + b_{21}) & (a_{22} + b_{22}) & (a_{23} + b_{23}) & (a_{24} + b_{24}) \\ (a_{31} + b_{31}) & (a_{32} + b_{32}) & (a_{33} + b_{33}) & (a_{34} + b_{34}) \end{bmatrix} \qquad (2)$$

- **Product of a matrix with un number**

- $\alpha A = [\alpha x_{ij}]_{m,n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \rightarrow \alpha A = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \alpha a_{13} & \alpha a_{14} \\ \alpha a_{21} & \alpha a_{22} & \alpha a_{23} & \alpha a_{24} \\ \alpha a_{31} & \alpha a_{32} & \alpha a_{33} & \alpha a_{34} \end{bmatrix} \qquad (3)$$

- **Transpose**

We call a matrix A^t transpose of the matrix A, the matrix where the rows are inverted with the columns such that

$$a_{ij}^t = a_{ji} \qquad (4).$$

So if the matrix A is of m rows and n columns, the transposed matrix A^t is of n rows and m columns.

$$A_{3,4} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 50 & 60 & 70 & 80 \\ 10 & 20 & 30 & 40 \end{bmatrix} \rightarrow \; A_{4,3}^t = \begin{bmatrix} 1 & 50 & 10 \\ 2 & 60 & 20 \\ 3 & 70 & 30 \\ 4 & 80 & 40 \end{bmatrix} \qquad (5)$$

➢ Symmetric matrix

- A matrix A (necessarily, square matrix) is called symmetric if $A^t = A$ ($a_{i\,j} = a_{j\,i}$)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 8 \end{bmatrix} = A^t \qquad (5.6)$$

➢ Antisymmetric matrix (Skew-symmetric matrix)

An antisymmetric matrix, also known as a skew-symmetric matrix, is a square matrix that satisfies $A^t = -A$ (In other words, if $a_{i\,j} = -a_{j\,i}$)

$$A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix} \Rightarrow A^t = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix} = -A \qquad (5.7)$$

➢ Identity matrix

The identity matrix, or unit matrix, is **a square** matrix with all its elements zero, except the diagonal, which is equal to 1..

$$I = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \ldots & \vdots \\ \vdots & \ddots & 1 & 0 & \vdots \\ \vdots & \cdots & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix} \ldots\ldots\ldots\ldots\ldots$$

➢ Product of two matrices

Let A ∈ $M_{m,n}$ and B ∈ $M_{n,p}$, then the product C ∈ $M_{m,p}$ is given by the formula,

$$c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj} \; 1 \le i \le m, \qquad 1 \le j \le n$$

Example

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},\; B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \Rightarrow$$

$$AB = \begin{bmatrix} (a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}) & (a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32}) & (a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33}) \\ (a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}) & (a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{33}) & (a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33}) \\ (a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31}) & (a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32}) & (a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33}) \end{bmatrix}$$

Notes:

- $AB \neq BA$
- $ABC = (AB)C = A(BC)$
- $\alpha(AB) = (\alpha A)B = A(\alpha B)$

➢ Invertible matrix

We say that a matrix is invertible if:

- *the matrix is square,*
- *Its determinant is not equal to zero.*

Thus, for a matrix A, there exists a matrix B of the same size where the products AB and BA are equal to the identity matrix.

$$AB = BA = I$$

In this case, matrix B is unique. B is called the inverse matrix of A and is denoted,

$$B = A - 1.$$

$$AA - 1 = A - 1A = I.$$

➢ **Triangular matrix:** there are two types of triangular matrices:

- **Lower triangular matrix**; A is called lower triangular, if $a_{i\,j} = 0$ for $j > i$

$$\mathrm{A_{n,m}} = \begin{bmatrix} \mathrm{a}_{11} & 0 & \cdots & 0 \\ \mathrm{a}_{21} & \mathrm{a}_{22} & 0 & \vdots \\ \vdots & \vdots & \ddots & 0 \\ \mathrm{a}_{n1} & \mathrm{a}_{n2} & \cdots & \mathrm{a}_{nn} \end{bmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

- **Upper triangular matrix**; A is called lower triangular, if $a_{i\,j} = 0$ for $j < i$

$$\mathrm{A_{n,m}} = \begin{bmatrix} \mathrm{a}_{11} & \mathrm{a}_{12} & \cdots & \mathrm{a}_{1n} \\ 0 & \mathrm{a}_{22} & \vdots & \vdots \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \mathrm{a}_{nn} \end{bmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

➢ Positive definite symmetric matrix:

A symmetric matrix is said to be positive definite if it satisfies one of the four equivalent properties (if one of the conditions is verified, the others are necessarily verified).

1. The n main determinants of A (all main minors) are strictly positive.
2. If for any vector $x \in IR^n \neq 0$: $x^tAx > 0$.
3. All eigenvalues of A are strictly positive.
4. There exists a lower triangular matrix L such that, $A = LL^t$.

From the third condition, A is strictly positive if:

$$A=\begin{bmatrix} \underset{I_1}{a_{11}} + a_{12} + \dots\dots\dots\dots\dots\dots\dots a_{1n} \\ a_{21} + \underset{I_2}{a_{22}} + \dots\dots\dots\dots\dots\dots\dots a_{2n} \\ . \\ . \\ . \\ a_{n1} + a_{m2} + \dots\dots\dots\dots\dots\dots\dots \underset{I_n}{a_{nn}} \end{bmatrix} \qquad (5.8)$$

Since every major minor is strictly positive, then, (det $A_{(k)_{1\leq k\leq n}} > 0$)

- $I_1=a_{11}>0$,
- $I_2=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} >0$,

  .

  .

  .
- $I_n=\det A > 0$.

Example

$$A=\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \qquad (5.9)$$

- It is clear that A is symmetric ($A^t=A$),

- **First method:**

1- For A to be strictly positive, all its minors must be strictly positive. A is a (3x3) matrix; we therefore find three main determinants:

2- $I_1 = a_{11} = 1 > 0$

3- $I_2 = \det\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = (1\times 2 - 1\times 1) = 1 > 0$

4- $I_3 = \det\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} = 1\times\det\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - 1\times\det\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} + 0\times\det\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = 2 > 0.$

Therefore, the matrix A is symmetric and strictly positive.

- **Second method:**

$$x^t A x = (x_1\ x_2\ x_3)\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (x_1\ x_2\ x_3)\begin{bmatrix} x_1 + x_2 \\ x_1 + 2x_2 + x_3 \\ x_2 + 2x_3 \end{bmatrix}$$

$$x_1^2 + x_1x_2 + x_1x_2 + 2x_2 + x_2x_3 + x_2x_3 + 2x_3^2 = (x_1 + x_2)^2 + (x_2 + x_3)^2 + 2x_3^2 > 0.$$

Therefore A is symmetric strictly positive.

Example 2

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$$

$$x^t A x = (x_1 \quad x_2)\begin{pmatrix} 1 & 2 \\ 2 & 8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (x_1 + 2x_2)^2 + 4{x_2}^2 > 0,$$

Therefore A is symmetric strictly positive.

Note: To check if a large matrix is strictly positive, we apply Gaussian scaling. That is to say, make the matrix upper triangular and check that all the diagonal elements are strictly positive.

5.3 Systems of equations

We call the system of m equations with n unknowns $x_{1,}x_{2,}$………………., $x_{n,}$ the family of equations.

$$\begin{cases} a_{11}x_1 + a_{12}\,x_2 + \dots\dots\dots\dots\dots a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}\,x_2 + \dots\dots\dots\dots\dots a_{2n}x_n = b_2 \\ . \\ . \\ . \\ a_{m1}x_1 + a_{m2}\,x_2 + \dots\dots\dots\dots\dots a_{mn}x_n = b_m \end{cases} \qquad \dots\dots\dots\dots\dots\dots\dots$$

The system cannot be solvable if only the number of equations equals the number of unknowns, i.e n=m**,**

$$\begin{cases} a_{11}x_1 + a_{12}\,x_2 + \dots\dots\dots\dots\dots a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}\,x_2 + \dots\dots\dots\dots\dots a_{2n}x_n = b_2 \\ . \\ . \\ . \\ a_{n1}x_1 + a_{m2}\,x_2 + \dots\dots\dots\dots\dots a_{nn}x_n = b_n \end{cases}$$

The system can be expressed using the format, $A\,X = B$. Where:

- A is a square matrix given by the elements $\left(a_{i\,j} \;\; 1 \le i \le n, 1 \le j \le n \;\right)$
- X is the column matrix of unknowns, and
- B is the vector representing the second member of the system.

$$A = \begin{bmatrix} a_{11} + a_{12} + \dots\dots\dots\dots\dots a_{1n} \\ a_{21} + a_{22} + \dots\dots\dots\dots\dots a_{2n} \\ . \\ . \\ . \\ a_{n1} + a_{m2} + \dots\dots\dots\dots\dots a_{nn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix} et\, b = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{bmatrix} \dots\dots\dots\dots\dots$$

5.4 Solving a system of equations

There are several direct methods for solving systems of equations. In this course, three methods will be studied: Gauss method, Cholesky method, and LU factorization method (Crout factorization and Doolittle factorization).

5.3.1 Gauss method:

The Gauss method, or Gauss elimination, as it is also called the Gauss pivot method, consists of transforming the system $AX = b$ into a triangular system using an algorithm called the Gaussian elimination algorithm..

The $AX = B$ system can be transformed into an upper triangular system or a lower triangular system. Generally, the Gaussian method transforms the system into a lower triangular system. The upper triangular system is written in the form,

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots\cdots\cdots + a_{1\,n}x_n & = b_1 \\ \qquad\quad a_{22}x_2 + \cdots\cdots\cdots + a_{2\,n}\,x_n & = b_2 \\ \qquad\qquad\qquad . & \\ \qquad\qquad\qquad . & \\ \qquad\qquad\qquad . & \\ \cdots\cdots\cdots \qquad\qquad a_{n\,n}\,x_n = b_n & \end{cases} \qquad \ldots\ldots\ldots\ldots\ldots\ldots 10$$

Note: *The elements* a_{ij} *and* b_i *are not the same as the elements given in the first system*.

At this point, we can find x_n from the last row, then we go back to find x_{n-1} from the line before the last one, and so on until we get x_1.

To transform a non-triangular system into a triangular system, the following transformations can be used:

- Altering lines.

Example

$$\begin{cases} x_2 + 4x_3 = -3 \\ 2x_1 - 3x_2 + x_3 = 2 \\ 2x_1 - x_2 + 3x_3 = 1 \end{cases} \Rightarrow \begin{cases} 2x_1 - 3x_2 + x_3 = 2 \\ x_2 + 4x_3 = -3 \\ 2x_1 - x_2 + 3x_3 = 1 \end{cases}$$

- Multiplying an equation by a non-zero constant

Example

$$\begin{cases} 2x_1 - 3x_2 + x_3 = 2 \\ (x_2 + 4x_3) = (-3) \\ 2x_1 - x_2 + 3x_3 = 1 \end{cases} \Rightarrow \begin{cases} 2x_1 - 3x_2 + x_3 = 2 \\ -2(x_2 + 4x_3) = -2(-3) \\ 2x_1 - x_2 + 3x_3 = 1 \end{cases}$$

- An equation can be replaced by another one by adding or subtracting a certain number of times from another equation.

Example

$$\begin{array}{cccccc} 2x_1 & -3x_2 & x_3 & = & 2 \\ & -2x_2 & -8x_3 & = & 6 \\ 2x_1 & -x_2 & +3x_3 & = & 1 \end{array} \xrightarrow{L_3 = L_3 - 1\times L_1} \begin{array}{cccccc} 2x_1 & -3x_2 & x_3 & = & 2 \\ & -2x_2 & -8x_3 & = & 6 \\ & +2x_2 & +2x_3 & = & -1 \end{array}$$

❖ **Gauss method procedure**

- Initially, we group A and b into a single matrix,

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} \times \begin{vmatrix} x_1 \\ \vdots \\ x_n \end{vmatrix} = \begin{vmatrix} b_1 \\ \vdots \\ b_n \end{vmatrix} \rightarrow \begin{vmatrix} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & b_n \end{vmatrix}$$

- Transformation of matrix A into an upper triangular matrix

Step 1: We put $A = A^{(1)}$ $and\ b = b^{(1)}$

$$[A{:}b]^1 = \begin{bmatrix} a_{11}^{(1)} & \dots & a_{1\,n}^{(1)} & b_1^{(1)} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n\,1}^{(1)} & \dots & a_{n\,n}^{(1)} & b_n^{(1)} \end{bmatrix}$$

- Choose the first equation such that ${a_{11}}^{(1)} \neq 0$.
- We carry out the following operations :

L1 is maintained $\Leftrightarrow \begin{cases} L_1^{(2)} = L_1^{(1)} \\ L_i^{(2)} = L_i^{(1)} - \frac{{a_{i1}}^{(1)}}{{a_{11}}^{(1)}} L_1^{(1)}\ ; i = 2 \dots n \end{cases}$

We then obtain:

$$[A{:}b]^2 = \begin{bmatrix} a_{11}^{(2)} & a_{12}^{(2)} & \dots & b_1^{(2)} \\ 0 & {a_{22}}^{(2)} & \dots & b_2^{(2)} \\ 0 & \vdots & & \vdots \\ 0 & \vdots & & \vdots \\ 0 & a_{n\,2}^{(2)} & \dots & b_n^{(2)} \end{bmatrix}.$$

Step 2:

- Choose the second equation such that ${a_{22}}^{(2)} \neq 0$.

$$\begin{cases} L_1^{(3)} = L_1^{(1)} \\ L_2^{(3)} = L_2^{(2)} \\ L_i^{(3)} = L_i^{(2)} - \dfrac{a_{i2}^{(2)}}{a_{22}^{(2)}} L_i^{(2)} \quad i = 3 \dots n \end{cases}$$

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$$\begin{cases} L_k^{(k+1)} = L_k^{(k)} \\ L_i^{(k+1)} = L_i^{(k)} - \dfrac{a_{ik}^{(k)}}{a_{kk}^{(k)}} L_k^{(k)} \quad i = k-1 \dots n \end{cases}$$

At last, we get a triangular system that can be used to calculate x_n, then x_{n-1}... (resolution by ascent).

Example

We use the Gauss method to solve the following system;

$$\begin{cases} x_1 + 3x_2 + 3x_3 = 0 \\ 2x_1 + 2x_2 + 2x3 = 2 \\ 3x_1 + 2x_2 + 6x_3 = 11 \end{cases}$$

Grouping A and b into a single matrix

$$\mathrm{A}^{(1)} = \begin{bmatrix} 1 & 3 & 3 & \vdots & 0 \\ 2 & 2 & 2 & \vdots & 2 \\ 3 & 2 & 6 & \vdots & 11 \end{bmatrix}$$

Step 1 : elimination of x_1

$$L_2^{(2)} \longrightarrow L_2^{(1)} - 2L_1^{(1)}$$

$$L_3^{(2)} \longrightarrow L_3^{(1)} - 3L_1^{(1)}$$

$$\mathrm{A}^{(2)} = \begin{bmatrix} 1 & 3 & 3 & \vdots & 0 \\ 0 & -4 & -4 & \vdots & 2 \\ 0 & -7 & -3 & \vdots & 11 \end{bmatrix}$$

Etape 2 : elimination of x_2 :

$$L_3^{(3)} \longrightarrow L_3^{(2)} - (-7)L_2^{(2)}$$

$$A^{(3)} = \begin{bmatrix} 1 & 3 & 3 & \vdots & 0 \\ 0 & -4 & -4 & \vdots & 2 \\ 0 & 0 & 4 & \vdots & 15/2 \end{bmatrix}$$

Finding x_n by ascent resolution

$$4x_3 = 15/2 \Rightarrow x_3 = 15/8$$

$$-x_2 - x_3 = \frac{1}{2} \Rightarrow x_2 = -19/8$$

$$x_1 + 3x_2 + 3x_3 = 0 \Rightarrow x_1 = 3/2$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3/2 \\ -19/8 \\ 15/8 \end{pmatrix}$$

5.3.2 Cholesky method

Cholesky method can only be used if the matrix A is symmetric positive definite. (see paragraph 5.2.2)

Let us take the following system $Ax = b$.

If A is a positive definite symmetric matrix, then A can be decomposed into the form $A = LL^t$. Where, L is a lower triangular matrix.

$$Ax = b \quad \rightarrow \quad LL^t x = b \quad \rightarrow \quad L(L^t x) = b \; \ldots..$$

Since L and L^t are triangular matrixes, we can transform the system into two systems that are easy to solve. We put, $L^t x = Y$, then we find the vector Y from the equation,

$$LY = b \; \ldots\ldots\ldots..$$

Then we find the vector x from the equation,

$$L^t x = Y \; \ldots\ldots\ldots\ldots$$

Construction of the lower triangular matrix $L = (l_{i\,j})$. We have $A = L\,L^t$ where $A = (a_{\,ij})$.

$$a_{\,ij} = \sum_{k=1}^{n} \ell_{i\,k}\,\ell_{j\,k} \qquad j \leq i$$

Then : $a_{11} = \ell_{11}^{\;2} \Rightarrow \ell_{11} = \sqrt{a_{11}}$, and $a_{i1} = \ell_{i1}\ell_{11} \Rightarrow \ell_{i1} = \frac{a_{i1}}{\ell_{11}}$ $i = 2, \ldots n$

The construction of the matrix L is done column by column,

$$\ell_{k\,k} = \sqrt{a_{kk} - \sum_{j=1}^{\ell-1} \ell_{k\,j}^2}$$

And: $a_{ik} = \sum_{j=1}^{k} \ell_{ij}\ell_{kj} = \ell_{ik}\ell_{kk} + \sum_{j=1}^{k-1} \ell_{ij}\, \ell_{kj}$

Then: $\ell_{ik} = a_{ik} - \sum_{j=1}^{k-1} \ell_{ij}\, \ell_{kj})/\ell_{kk}$

Example 3

Let us consider the system $Ax = b$, where A = $\begin{bmatrix} 9 & 3 & 15 \\ 3 & 5 & 7 \\ 15 & 7 & 42 \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ 5 \\ 15 \end{bmatrix}$.

- Verifying that A is symmetric definite positive.

- $A=A^t$ is symmetric

- is A definite positive?

$$A=\begin{bmatrix} 9 & 3 & 15 \\ 3 & 5 & 7 \\ 15 & 7 & 42 \end{bmatrix}$$

$I_1 = 9 > 0$

$I_2 = det \begin{vmatrix} 9 & 3 \\ 3 & 5 \end{vmatrix} = 45 - 9 = 36 > 0$

$$\begin{aligned} I_3 = \det(A) &= 9\begin{vmatrix} 5 & 7 \\ 7 & 42 \end{vmatrix} - 3\begin{vmatrix} 3 & 7 \\ 15 & 42 \end{vmatrix} + 15\begin{vmatrix} 3 & 5 \\ 15 & 7 \end{vmatrix} \\ &= 9(161) - 3(21) + 15(54) \\ &= 1449 - 63 - 810 = 576 > 0 \end{aligned}$$

$$A = L\,L^t = \begin{bmatrix} \ell_{11} & 0 & 0 \\ \ell_{21} & \ell_{22} & 0 \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix}\begin{bmatrix} \ell_{11} & \ell_{21} & \ell_{31} \\ 0 & \ell_{22} & \ell_{32} \\ 0 & 0 & \ell_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \ell_{11}^2 & \ell_{11}\ell_{21} & \ell_{11}\ell_{31} \\ \ell_{21}\ell_{11} & \ell_{21}^{\;2} + \ell_{22}^{\;2} & \ell_{21}\ell_{31} + \ell_{22}\ell_{32} \\ \ell_{31}\ell_{11} & \ell_{31}\ell_{21} + \ell_{32}\ell_{22} & \ell_{31}^2 + \ell_{32}^2 + \ell_{33}^2 \end{bmatrix} = \begin{bmatrix} 9 & 3 & 15 \\ 3 & 5 & 7 \\ 15 & 7 & 42 \end{bmatrix}$$

1st column

$\ell_{11}^2 = 9 \Rightarrow \ell_{11} = 3$

$\ell_{21}\ell_{11} = 3 \Rightarrow \ell_{21} = 1$

$\ell_{31}\ell_{11} = 15 \Rightarrow \ell_{31} = 5$

2nd column

$\ell_{21}^2 + \ell_{22}^2 = 5 \Rightarrow \ell_{22} = 2$

$\ell_{31}\ell_{21} + \ell_{32}\ell_{22} = 7 \Rightarrow \ell_{32} = 1$

3rd column

$$\ell_{31}^2 + \ell_{32}^2 + \ell_{33}^2 = 42 \quad \ell_{33} = 4$$

$$\Rightarrow L = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ 5 & 1 & 4 \end{bmatrix} \to L^t = \begin{bmatrix} 3 & 1 & 5 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

$$LY = b \Leftrightarrow \begin{bmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ 5 & 1 & 4 \end{bmatrix} . \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 15 \end{bmatrix} \Rightarrow y = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$\begin{cases} LY = b \\ L^t x = y \end{cases} \qquad L^t x = y \leftrightarrow \begin{bmatrix} 3 & 1 & 5 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{3}{4} \\ \frac{3}{4} \\ \frac{1}{2} \end{bmatrix}$$

5.4 Factorization method LU (Crout and Doolittle).

This method consists of factoring matrix A into two triangular matrices; a lower triangular ***L*** (L comes from Lower) and the other upper triangular ***U*** (U comes from upper), provided that one of the two matrices has all diagonal elements equal to unity.

- If the elements of the diagonal of L are equal to unity ($l_{ii} = 1$), the method is called a decomposition method ***LU*** of ***Doolittle***,
- • If the elements of the diagonal of U are equal to unity ($u_{ii} = 1$), the method is called a decomposition method ***LU*** of ***Crout***.

The system is given by,

$$AX = b.$$

A will be written in the form

$$A = LU,$$

so the system will be written as

$$LUX = b$$

$$L(UX) = b$$

We put

$$UX = Y.$$

We solve the system in two steps:

1. Since U is upper triangular, we can easily find the vector Y using,

$$LY = b.$$

Thus, since L is lower triangular, we easily find the vector x using,

$$UX = Y$$

So, the system $AX = b$ is decomposed into two triangular systems easy to solve.

$$Ax = b :\to LUx = b :\to L(Ux) = b :\to \begin{cases} Ux = y ....1 \\ Ly = b........2 \end{cases} ....$$

The system with the upper triangular matrix is solved by direct (downward) substitution, and the one with the lower triangular matrix is solved by reverse (upward) substitution.

5.4.1 Determination of matrixes L and U

Theorem 1

A necessary and sufficient condition for A to be decomposable into a product LU is that all its fundamental minors are different from zero.

Theorem 2

If A is invertible and decomposable into a product LU, then this decomposition is unique.

- **Factorization algorithm A = L.U (DOOLITTLE Version)**

To determine the elements $l_{ij} (\forall i > j)$ of the matrix L and the elements $u_{ij} (\forall j \leq i)$ of the matrix U, we use the following version of the factorization algorithm:

$$\begin{cases} l_{ii} = 1 \; \forall i \\ l_{ij} = \dfrac{\left[a_{ij} - \sum\limits_{k=1}^{j-1} l_{ik} u_{kj} \right]}{u_{jj}} \quad \forall i > j \\ u_{ij} = a_{ij} - \sum\limits_{k=1}^{i-1} l_{ik} u_{kj} \quad \forall i \leq j \end{cases}$$

- **Factorization algorithm A = L.U (Crout Version)**

To determine the elements of matrix L and matrix U, we use the following version of the factorization algorithm:

$$\begin{cases} u_{ii} = 1 \; \forall i \\ l_{ij} = a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj} \quad \forall i \leq j \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ u_{ij} = \dfrac{\left[a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj} \right]}{l_{ii}} \quad \forall i < j \end{cases}$$

$$L=\begin{bmatrix} l_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ l_{n1} & \cdots & l_{nn} \end{bmatrix} \qquad U=\begin{bmatrix} 1 & u_{12} & \cdots & u_{1n} \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & 1 \end{bmatrix}$$

the elements of each matrix are given by:

$l_{ki} = a_{ki} - \sum_{j=1}^{i-1} l_{kj}\, u_{ji}$

Avec : $i=2,3,\ldots\ldots\ldots,n$ et $k=i,i+1,\ldots\ldots\ldots,n$

$$u_{ik} = \frac{\left(a_{ik} - \sum_{j=1}^{i-1} l_{ij}\, u_{jk}\right)}{l_{ii}}$$

Exemple :

We take the system where; A= $\begin{bmatrix} 2 & 1 & -2 \\ 4 & 5 & -3 \\ -2 & 5 & 3 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 6 \\ 6 \end{bmatrix}$

Using the factorization of Crout to solve the system AX=b. Then,

L= $\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$ and U= $\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$ such as A=LU

- We identify the first column of A and the first column of LU, This allows to obtain the first column of L:

$$\begin{bmatrix} 2 & 0 & 0 \\ 4 & l_{22} & 0 \\ -2 & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -2 \\ 4 & 5 & -3 \\ -2 & 5 & 3 \end{bmatrix}$$

– We identify the first line of A with the first line of LU, this allows to obtain the first line of U :

$$\begin{bmatrix} 2 & 0 & 0 \\ 4 & l_{22} & 0 \\ -2 & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & -1 \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -2 \\ 4 & 5 & -3 \\ -2 & 5 & 3 \end{bmatrix}$$

– We identify the second column of A with the second column of LU, This allows us to obtain the second column of L:

$$\begin{bmatrix} 2 & 0 & 0 \\ 4 & 3 & 0 \\ -2 & 6 & l_{33} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & -1 \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -2 \\ 4 & 5 & -3 \\ -2 & 5 & 3 \end{bmatrix}$$

– We identify the second line of A with the second line of LU, This allows to obtain the second line of U :

$$\begin{bmatrix} 2 & 0 & 0 \\ 4 & 3 & 0 \\ -2 & 6 & l_{33} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & -1 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -2 \\ 4 & 5 & -3 \\ -2 & 5 & 3 \end{bmatrix}$$

- We identify the third column of A with the third column of LU, This allows us to obtain the third column of L:

$$\begin{bmatrix} 2 & 0 & 0 \\ 4 & 3 & 0 \\ -2 & 6 & -1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & -1 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -2 \\ 4 & 5 & -3 \\ -2 & 5 & 3 \end{bmatrix}$$

Then we replace in,

$\begin{cases} LY = b \\ UX = Y \end{cases}$ on obtient

$$\begin{bmatrix} 2 & 0 & 0 \\ 4 & 3 & 0 \\ -2 & 6 & -1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 4/3 \\ 1 \end{bmatrix}$$ Et

$$\begin{bmatrix} 1 & \frac{1}{2} & -1 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{4}{3} \\ 1 \end{bmatrix}$$

Finely we get the vector x:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$