# Chapter 4

# Solving of first-order ordinary differential equations

## 4.1 Introduction:

An ordinary differential equation (ODE) is an equation relating a real variable and its derivatives. The ODE is written in the following form:

$$\dot{y} = \frac{dy}{dt} = f(t, y(t)) \qquad (1)$$

Where $y$ is the unknown function, $\dot{y}$ is its derivative, and $t$ is the real variable.

**Exemple 1 :**

$$\dot{y} = \frac{dy}{dt} = y \qquad (2)$$

The solution of this equation can be determined analytically: it is given by:

$$y(t) = ce^{t} \qquad (3)$$

There is an infinity of solutions to this differential equation due to the integration constant c. To determine the appropriate solution to our physical state, we must refer to the data already known, generally specified via two categories of conditions or two categories of problems:

**Initial conditions (Cauchy problem)**

The initial condition gives the value of the function and its derivatives at a given time, called the initial time. In another way, to find the values of the function at other points in the domain, we must know the value of the function y and its derivatives in the initial state.

At an initial value $t_0$, we have

$$y(t_0) = y_0 \qquad (4)$$

We can therefore evaluate the value of c via this initial condition. If we take y(0)=1 at t=0 in the previous example (eq. 3), the value of c will be equal to 1, and the final solution of the differential equation will be given by:

$$y(t) = e^t \quad (6)$$

- **Condition aux limites (Problème aux limites)**

A boundary condition expresses the behavior (gives the values) of a function on the boundary (border) of its area of definition

$$\begin{matrix} y_i = \\ y_f = \end{matrix} \quad (7)$$

In this course, we are only interested in the solution of ODEs in the Cauchy problem (that is, if we know the initial condition).

Consider the following differential equation:

$$\dot{y}(t) = f\big(t, y(t)\big), \; t \in [t_0 \;\; t_f] \quad (8)$$

With the initial condition (Cauchy problem), $y(t_0) = y_0$

The above example (eq. 3) is very easy to solve analytically, but there are many ODEs that cannot be solved analytically. In these cases, numerical analysis offers methods to find an approximate solution.

The Cauchy problem is an evaluation problem; that is to say, from the initial condition, we can evaluate the value of $y_1$ at $t_1 = t_0 + \Delta t$ and evaluate the value of $y_2$ from $y_1$ and so on.

- if $y_{n+1}$ is only function of $t_n$ et $y_n$, we say that we have 'One step diagram'

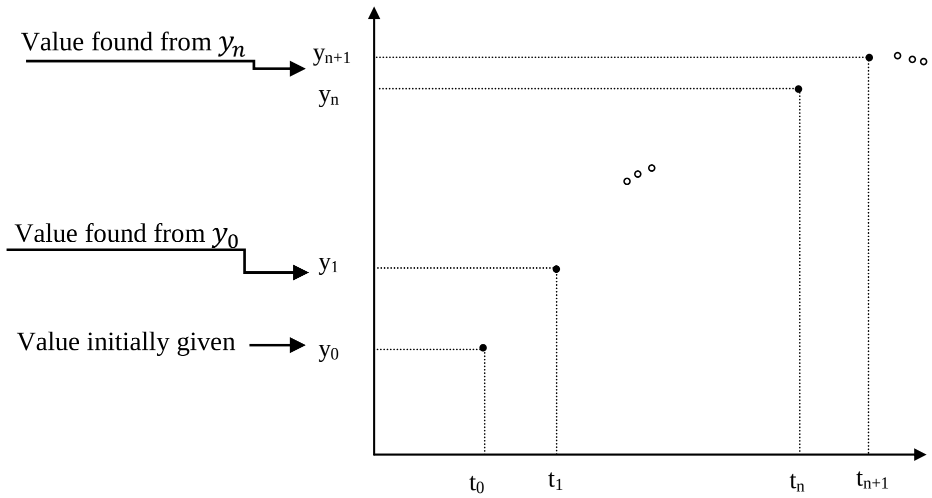



Figure 4.1 : Diagram illustrating the scenario

### 4.2 Uniqueness of the solution

If we assume that, the function f is continuous for the two variables t and y and that f is uniformly Lipschitzian for y, that is to say:

$$\forall t \in [t_0 \quad T], \quad \forall y_1 \; et \; y_2 \in R, \exists L > 0, |f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$$

Therefore, the differential equation admits a unique solution y ∈ $C^1([t_0, T])$.

- **Question:** The question that arises is: what is the method to estimate the value of $y_1$ from $y_0$?
- **Response:** The idea is based on linearization. That is, we assume that the curve between the two points is a straight line.
    - **But**,

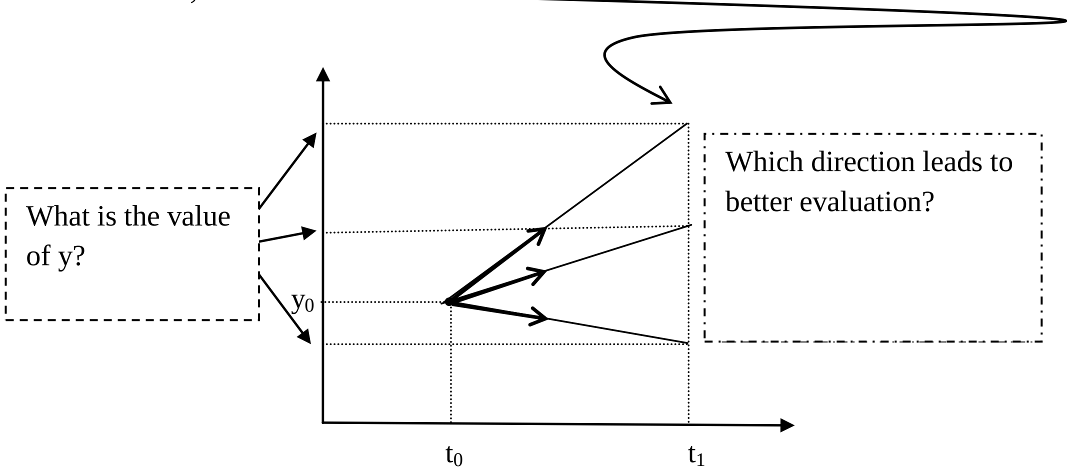



Figure 4.2: Schematic presentation of the problem

- **Reponses:** Taylor expansion responses on this question and gives the direction that better estimates the value of $y_1 \; at \; t_1 = t_0 + \Delta t$

The Taylor expansion is given by:

$$y_{n+1} = y_n + h\frac{dy}{dt} + \frac{h^2}{2!}\frac{d^2t}{dt^2} + \frac{h^3}{3!}\frac{d^3t}{dt^3} + \frac{h^4}{4!}\frac{d^4t}{dt^4} + O(h^5) \qquad (9)$$

$$\begin{cases} y_n = y(t_n) \\ h = t_{n+1} - t_n \\ O(h^5): \; the \; error \; that \; can \; be \; made \; at \; this \; degree \end{cases}$$

Using this development, we can deduce the direction we must follow to find the best estimation of $y$ at $t_{(n+1)} = t_n + \Delta t$. We will clarify this idea in the following paragraphs.

## 4. 2 Euler's method

The first-order ordinary differential equation is given by:

$$\dot{y} = \frac{dy}{dt} = f(t, y(t)) \tag{10}$$

Assume that only the first term of the Taylor expansion provides an acceptable evaluation of the value of y, so,

$$y_{n+1} = y_n + h\frac{dy}{dt} \tag{11}$$

Knowing that,

$$\frac{dy}{dt} = f(t, y(t))$$

Example: $\dot{y} = \frac{dy}{dt} = y + t,$

That means $f\big(t_n, y(t_n)\big) = y_n + t_n$

$\frac{dy}{dt}$ : represents the slope of $f$ at the point $(y_n, t_n)$.

First order development of Taylor can be graphically represented in the figure 4.3.

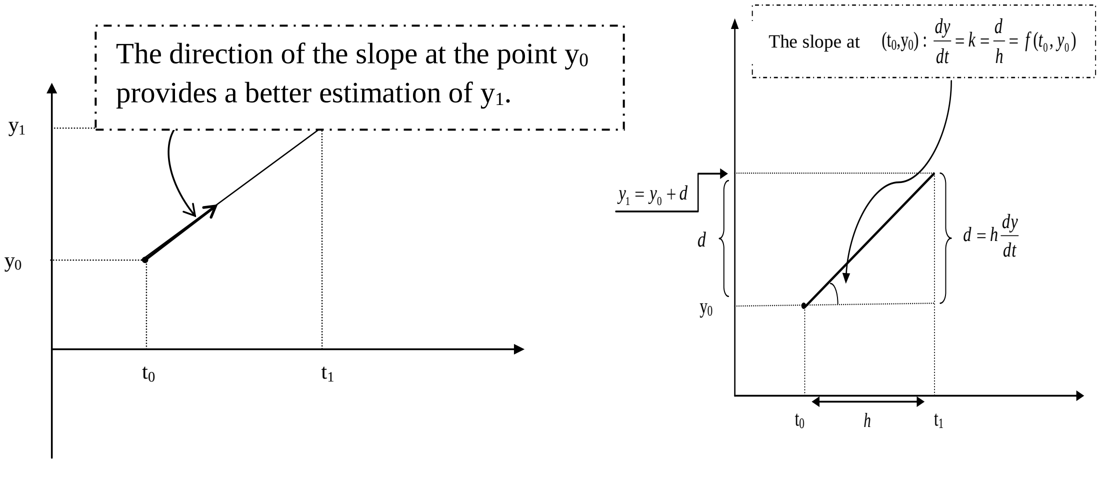



Figure 4.3: schematic representation of Euler idea to evaluate the next solution.

Figure 4.4 represents a graphical explanation of Euler's idea of how to estimate the next value from the known previous value.

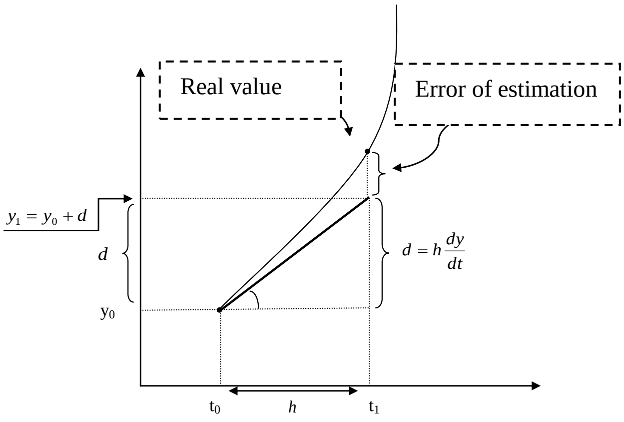



Figure 4. 4: The estimation solution of the second value from the first

With the same analysis, we obtain all ODE values in the study interval.

$$\begin{cases} y_1 = y_0 + hf(t_0, y_0) \\ y_2 = y_1 + hf(t_1, y_1) \\ \quad\quad . \\ \quad\quad . \\ \quad\quad . \\ y_n = y_{n-1} + hf(t_{n-1}, y_{n-1}) \end{cases}$$

Figure 4.5 gives a graphical representation of the approximate solution.

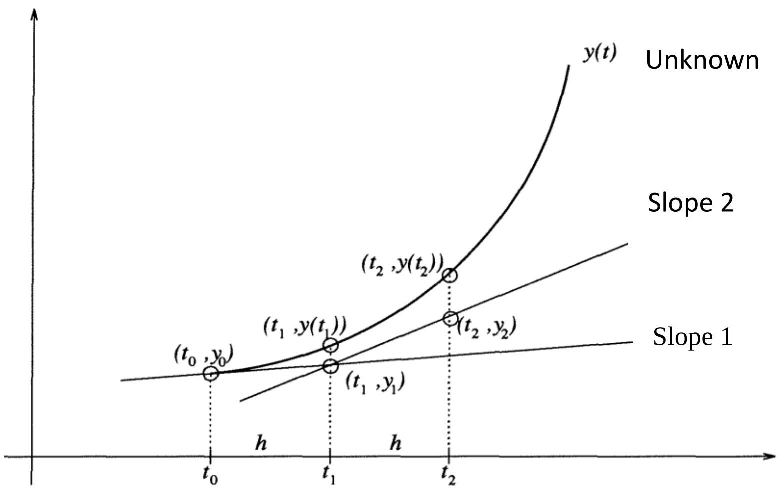



Figure 4. 5: Graphical representation of Euler method

**Notes:**

1. It is very clear that the approximate value of $y_{n+1}$ is better when h is small.
2. We can rewrite Euler's equation as:

$$\begin{cases} y_{n+1} = y_n + hak \\ k = f(t_n, y_n) \\ a = 1 \end{cases}$$

**Example 4. 1**

Taking the following differential equation

$$\begin{cases} \dfrac{dy}{dt} = y \\ y(0) = 1 \end{cases}$$

The exact solution (analytical solution) of this equation is given by,

$$y = e^t$$

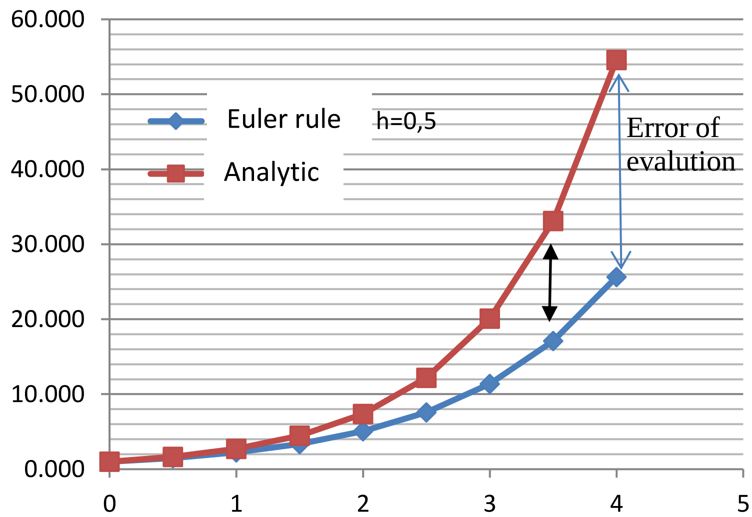
We want to find numerically the values of y (the solution to the differential equation) in the interval $t \in [0, 4]$ with Euler's method in two cases; h=0.5 and h=0.1.

$$y_{i+1} = y_i + hf(t_i, y(t_i))$$

$$\begin{cases} y_1 = y_0 + hf(t_0, y(t_0)) = 1 + 0.5 \times 1 = 1.50 \\ y_2 = y_1 + hf(t_1, y(t_1)) = 1.5 + 0.5 \times 1.5 = 2.50 \\ \quad . \\ \quad . \\ \quad . \\ y_n = y_{n-1} + hf(t_{n-1}, y(t_{n-1})) = 17.11 + 0.5 * 17.11 = 54.60 \end{cases}$$

De cette manière on va remplir le tableau suivant ;

| h=0,5 | | | $y = e^t$ |
|---|---|---|---|
| n | Time | Euler Method h=0,5 | Analytic |
| 0 | 0 | 1,000 | 1,000 |
| 1 | 0,5 | 1,500 | 1,649 |
| 2 | 1 | 2,250 | 2,718 |
| 3 | 1,5 | 3,375 | 4,482 |
| 4 | 2 | 5,063 | 7,389 |
| 5 | 2,5 | 7,594 | 12,182 |
| 6 | 3 | 11,391 | 20,086 |
| 7 | 3,5 | 17,086 | 33,115 |
| 8 | 4 | 25,629 | 54,598 |






*S.AOULMIT*

| h=0,1 | | | |
|---|---|---|---|
| n | xn | yn | y=ex(x) |
| 0 | 0 | 1,000 | 1,000 |
| 1 | 0,1 | 1,100 | 1,105 |
| 2 | 0,2 | 1,210 | 1,221 |
| 3 | 0,3 | 1,331 | 1,350 |
| 4 | 0,4 | 1,464 | 1,492 |
| 5 | 0,5 | 1,611 | 1,649 |
| 6 | 0,6 | 1,772 | 1,822 |
| 7 | 0,7 | 1,949 | 2,014 |
| 8 | 0,8 | 2,144 | 2,226 |
| 9 | 0,9 | 2,358 | 2,460 |
| 10 | 1 | 2,594 | 2,718 |
| 11 | 1,1 | 2,853 | 3,004 |
| 12 | 1,2 | 3,138 | 3,320 |
| 13 | 1,3 | 3,452 | 3,669 |
| 14 | 1,4 | 3,797 | 4,055 |
| 15 | 1,5 | 4,177 | 4,482 |
| 16 | 1,6 | 4,595 | 4,953 |
| 17 | 1,7 | 5,054 | 5,474 |
| 18 | 1,8 | 5,560 | 6,050 |
| 19 | 1,9 | 6,116 | 6,686 |
| 20 | 2 | 6,727 | 7,389 |

| h=0,1 | | | |
|---|---|---|---|
| n | xn | yn | y=ex(x) |
| 21 | 2,1 | 7,400 | 8,166 |
| 22 | 2,2 | 8,140 | 9,025 |
| 23 | 2,3 | 8,954 | 9,974 |
| 24 | 2,4 | 9,850 | 11,023 |
| 25 | 2,5 | 10,835 | 12,182 |
| 26 | 2,6 | 11,918 | 13,464 |
| 27 | 2,7 | 13,110 | 14,880 |
| 28 | 2,8 | 14,421 | 16,445 |
| 29 | 2,9 | 15,863 | 18,174 |
| 30 | 3 | 17,449 | 20,086 |
| 31 | 3,1 | 19,194 | 22,198 |
| 32 | 3,2 | 21,114 | 24,533 |
| 33 | 3,3 | 23,225 | 27,113 |
| 34 | 3,4 | 25,548 | 29,964 |
| 35 | 3,5 | 28,102 | 33,115 |
| 36 | 3,6 | 30,913 | 36,598 |
| 37 | 3,7 | 34,004 | 40,447 |
| 38 | 3,8 | 37,404 | 44,701 |
| 39 | 3,9 | 41,145 | 49,402 |
| 40 | 4 | 45,259 | 54,598 |

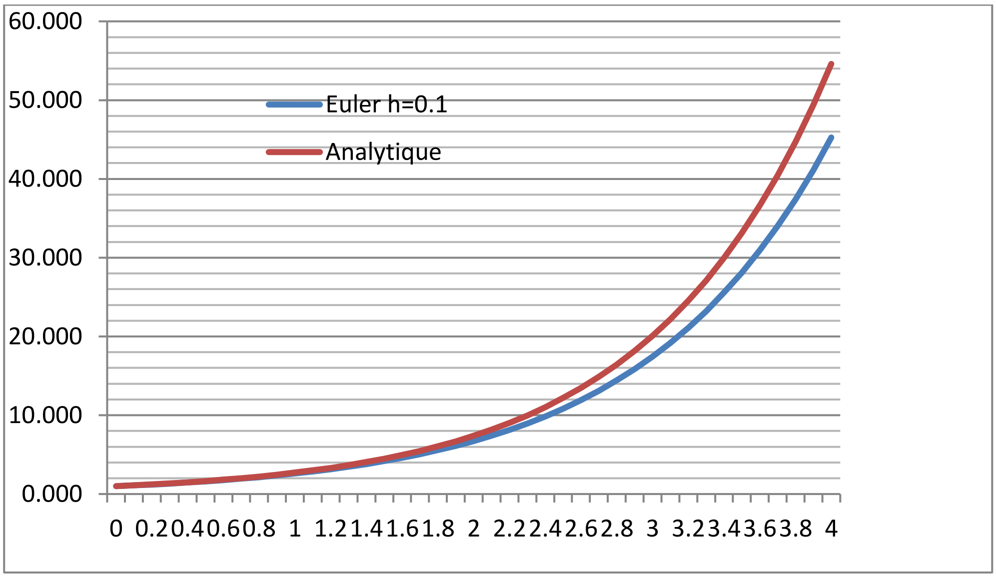



Figure 4.6 : Numerical and analytical solutions of the differential equation 2.

The two figures below give a good illustration of the effect of the step h on Euler's method.

### 4.3 Runge-Kutta Methods

To find the direction that we must follow to evaluate the value of $y_{n+1}$ from $y_n$, Runge-Kutta methods take into consideration higher orders in the Taylor expansion.

$$y_{n+1} = y_n + h\frac{dy}{dt} + \frac{h^2}{2!}\frac{d^2y_n}{dt^2} + \frac{h^3}{3!}\frac{d^3y_n}{dt^3} + \frac{h^4}{4!}\frac{d^4y_n}{dt^4} + O(h^5)$$

We can write:

$$y_{n+1} = y_n + hf\big(t_n, y(t_n)\big) + +\frac{h^2}{2!}\frac{df\big(t_n, y(t_n)\big)}{dt} + \frac{h^3}{3!}\frac{d^2f\big(t_n, y(t_n)\big)}{dt^2} + \frac{h^4}{4!}\frac{d^3f\big(t_n, y(t_n)\big)}{dt^3}$$

$$y_{n+1} = y_n + h\left(f\big(t_n, y(t_n)\big) + +\frac{h}{2!}\frac{df\big(t_n, y(t_n)\big)}{dt} + \frac{h^2}{3!}\frac{d^2f\big(t_n, y(t_n)\big)}{dt^2} + \frac{h^3}{4!}\frac{d^3f\big(t_n, y(t_n)\big)}{dt^3}\right)$$

$$y_{n+1} = y_n + h\emptyset\big(t_n, y(t_n)\big)$$

Where ;

$$\emptyset\big(t_n, y(t_n)\big) = \left(f\big(t_n, y(t_n)\big) + +\frac{h}{2!}\frac{df\big(t_n, y(t_n)\big)}{dt} + \frac{h^2}{3!}\frac{d^2f\big(t_n, y(t_n)\big)}{dt^2} + \frac{h^3}{4!}\frac{d^3f\big(t_n, y(t_n)\big)}{dt^3}\right)$$

represents the average slope (i.e., the average of the directions to be followed in order to get the best estimate of $y_{n+1}$ from the $y_n$ value). According to the order of Taylor expansion, we classify the families of Runge-Kutta methods. The most commonly used classes are Runge-Kutta of order 2 (RK2) and Runge-Kutta of order 4 (RK4).

The mathematical demonstration is very difficult, so we only give the general relationship:

$$\begin{cases} y_{n+1} = y_n + h\emptyset\big(t_n, y(t_n)\big) \\ \emptyset\big(t_n, y(t_n)\big) = \sum_{i}^{s} a_i k_i \end{cases}$$

- s : order of Runge-Kutta method,
- $k_i$ the slope at given points in the interval $[t_n, t_{n+1}]$
- $\sum_i^s a_i = 1$

- $$\begin{cases} k_1 = f(t_n, y_n), \\ k_2 = f(t_n + b_2 h, y_n + h c_{21} k_1) \\ k_3 = f(t_n + b_3 h, y_n + h(c_{31} k_1 + c_{32} k_2) \\ k_4 = f(t_n + b_4 h, y_n + h(c_{41} k_1 + c_{42} k_2 + c_{43} k_3) \\ . \\ . \\ . \\ k_s = f(t_n + b_s h, y_n + h(c_{s1} k_1 + c_{s2} k_2 + ... c_{ss-1} k_{s-1}) \end{cases} \quad \text{where} \begin{cases} b_i : 1....s \\ c_{ij} 1 \leq j \prec i \leq s \end{cases}$$

The coefficients $a_i, b_i \, et \, c_{ij}$ are arranged in the Butcher table:

| | | | | | |
|---|---|---|---|---|---|
| 0 | | | | | |
| $b_2$ | $c_{21}$ | | | | |
| $b_3$ | $c_{31}$ | $c_{32}$ | | | |
| $b_4$ | $c_{41}$ | $c_{42}$ | $c_{43}$ | | |
| $b_s$ | $c_{s1}$ | $c_{s2}$ | | $c_{ss-1}$ | |
| | $a_1$ | $a_2$ | | | $a_s$ |

$$\begin{cases} \sum_{i}^{s} a_i = 1 \\ \sum_{j=1}^{i-1} c_{ij} = bi \end{cases}$$

**4. 3. 1 Runge-Kutta Methods of d'ordre 1 (RK1)**

$$\begin{cases} a_1 = a = 1 \\ k_1 = k = f(t_n, y_n) \\ y_{n+1} = y_n + hf(t_n, y_n) \end{cases}$$

It is very clear that RK1 corresponds to the Euler method that we studied in the previous paragraph. In another way, we can consider that Euler's method is a special case of Range-Kutta.

### 4. 3. 2 Runge-Kutta Methods of d'ordre 2 (RK2)

$$\begin{cases} y_{n+1} = y_n + h(a_1 k_1 + a_2 k_2) \\ k_1 = f(t_n, y_n), \\ k_2 = f(t_n + b_2 h, y_n + h c_{21} k_1) \end{cases}$$

Depending on the values of $a_1$, $a_2$, $b_2$, and $c_{21}$ we distinguish two types of Runge-Kutta of order 2:

$$\begin{array}{c|cc} 0 & & \\ b_2 & c_{21} & \\ \hline & a_1 & a_2 \end{array} \qquad \begin{array}{c|cc} 0 & & \\ 1 & 1 & \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array} \qquad \begin{array}{c|cc} 0 & & \\ \frac{1}{2} & \frac{1}{2} & \\ \hline & 0 & 1 \end{array}$$

$$\sum_{i=1}^{2} a_i = 1$$

$$\sum_{j=1}^{i-1} c_{ij} = b_i$$

(a) Heun

(b) Mid-point

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- **Demonstration**

  Taking into account only the second term of Taylor's expansion.

$$y_{n+1} = y_n + h\frac{dy}{dt} + \frac{h^2}{2!}\frac{d^2y}{dt^2}, \qquad h = \Delta t,\ y_n = y_n(t_n)$$

$$y_{n+1} = y_n + hf_n(t_n, y_n) + \frac{h^2}{2}\frac{df_n(t_n, y_n)}{dt} \ldots\ldots\ldots\ldots$$

$$y_{n+1} = y_n + hf_n(t_n, y_n) + \frac{h^2}{2}\left[\frac{df_n(t_n, y_n)}{dt} + \frac{df_n(t_n, y_n)}{dy}.\frac{dy_n(t_n, y_n)}{dt}\right] \ldots\ldots\ldots\ldots$$

$$y_{n+1} = y_n + hf_n(t_n, y_n) + \frac{h^2}{2}\left[f_{tn}(t_n, y_n) + f_{yn}(t_n, y_n).f_n(t_n, y_n)\right] \ldots\ldots\ldots$$

Where $f_{tn}(t_n, y_n)$ et $f_{yn}(t_n, y_n)$ are the derivatives of $f_n(t_n, y_n)$ with respect to $t$ and to $y$ respectively.

$$y_{n+1} = y_n + hf_n(t_n, y_n) + \frac{h^2}{2} f_{tn}(t_n, y_n) + \frac{h^2}{2} f_{yn}(t_n, y_n).f_n(t_n, y_n) \ldots\ldots\ldots\ldots \text{(*)}$$

Equation 1 gives the final formula for the Taylor expansion.

We will find the Taylor expansion, but this time using the Runge-Kutta equation of order 2, given by:

$$y_{n+1} = y_n + ah_1 f(t_n, y_n) + a_2 hf(t_n + b_2 h, y_n + hc_{21} f(t_n, y_n)) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..$$

Then,

$$y_{n+1} = y_n + ah_1 f(t_n, y_n) + a_2 h\lfloor f(t_n + y_n) + b_2 hf_{tn}(t_n, y_n) + hc_{21} f_{yn}(t_n, y_n) f_n(t_n, y_n)\rfloor$$
…………

$$y_{n+1} = y_n + (a_1 + a_2)hf(t_n, y_n) + a_2 b_2 hf_{tn}(t_n, y_n) + a_2 c_{21} hf_{yn}(t_n, y_n) f_n(t_n, y_n) \ldots\ldots\ldots\ldots(**)$$

Comparing (*) and (**) we find,

$$\begin{cases} a_1 + a_2 = 1 \\ a_2 b_2 = \dfrac{1}{2} \\ a_2 c_{21} = \dfrac{1}{2} \end{cases} \quad \ldots\ldots\ldots\ldots\ldots$$

- ✓ $a_1 = a_2 = \frac{1}{2} \Rightarrow b_2 = 1, c_{21} = 1 \ldots\ldots$ case of Heun,
- ✓ $a_1 = 0,\ a_2 = 1 \Rightarrow b_2 = \frac{1}{2}, c_{21} = \frac{1}{2} \ldots$ case of Mid-point.

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### 4. 3. 2. 1 Heun's method (modified Euler)

$$\begin{cases} a_1 = a_2 = \dfrac{1}{2} \\ c_{21} = b_2 = 1 \end{cases} \qquad y_{n+1} = y_n + \frac{h}{2}(k_1 + k_2) \qquad \begin{cases} k_1 = f(t_n, y_n), \\ k_2 = f(t_n + h, y_n + hk_1) \end{cases}$$

- **Graphical Representation:**

Figure 4.4 shows the graphic representation of Heun's method, where:

$k_1$ : The slope of y at the point ($t_n$,$y_n$),

$k_2$ : The slope of $y_{n+1}$ at the point ($t_{n+1}$, $y_{n}$+$hk_1$).

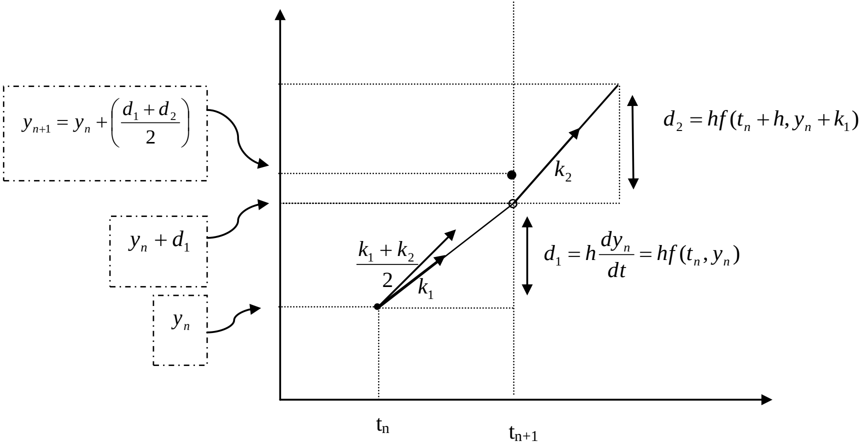



Figure 4. Graphical representation of Heun's method

## 4. 3. 2. 2 Midpoint Method

$$\begin{cases} a_1 = 0, \quad a_2 = 1 \\ c_{21} = b_2 = \dfrac{1}{2} \end{cases}$$

$$\begin{cases} k_1 = f(t_n, y_n), \\ k_2 = f(t_n + \dfrac{h}{2}, y_n + \dfrac{h}{2}k_1) \end{cases}$$

$$y_{n+1} = y_n + hk_2$$

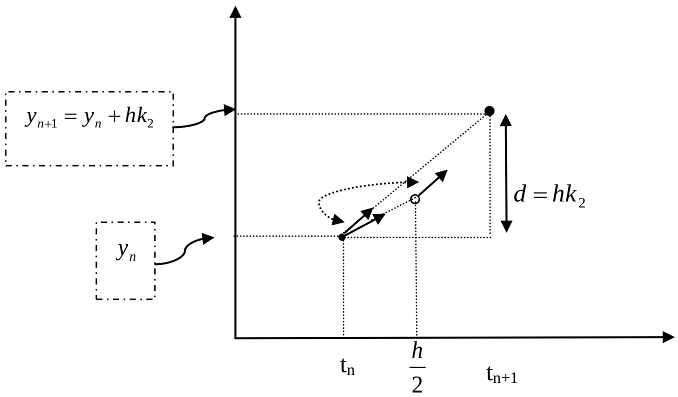



Figure 4. . Graphical representation of Midpoint

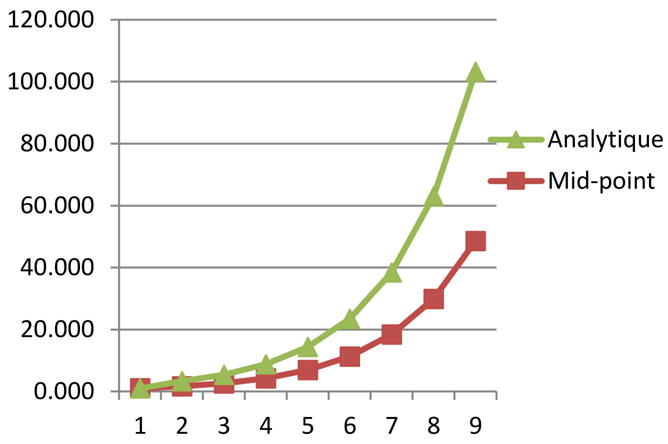
**Example 4.3**

Let’s take the previous example with h=0.5

$$\begin{cases} \dfrac{dy}{dt} = y \\ y(0) = 1 \end{cases}$$

| h=0,5 | | | |
|---|---|---|---|
| n | $t_n$ | Mid-point | Analytic |
| 1 | 0,000 | 1,000 | 0,000 |
| 2 | 0,500 | 1,625 | 1,649 |
| 3 | 1,000 | 2,641 | 2,718 |
| 4 | 1,500 | 4,291 | 4,482 |
| 5 | 2,000 | 6,973 | 7,389 |
| 6 | 2,500 | 11,331 | 12,182 |
| 7 | 3,000 | 18,413 | 20,086 |
| 8 | 3,500 | 29,921 | 33,115 |
| 9 | 4,000 | 48,621 | 54,598 |

a)- Values of y found with Mid-point method (h=0.5)




b)- Graphical comparison of analytic to Mid-point numerical results.

### 4.3.3.Runge-Kutta d'ordre 4 (RK4)

In this class of methods, we take into consideration the 4th degree of Taylor expansion.

| | | | | |
|---|---|---|---|---|
| 0 | | | | |
| $b_2$ | $c_{21}$ | | | |
| $b_3$ | $c_{31}$ | $c_{32}$ | | |
| $b_4$ | $c_{41}$ | $c_{42}$ | $c_{43}$ | |
| | $a_1$ | $a_2$ | $a_3$ | $a_4$ |

$$\begin{cases} \sum_{i}^{s} a_i = 1 \\ \sum_{j=1}^{i-1} c_{ij} = bi \end{cases}$$

Depending on the conditions that must be fulfilled by the constants $a_i, b_i,$ and $c_i$, we can find several rank 4 methods (RK4 family). The most stable method that gives good results is when the constants are as given in the following Butcher table,

| | | | | |
|---|---|---|---|---|
| $0$ | | | | |
| $\frac{1}{2}$ | $\frac{1}{2}$ | | | |
| $\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | | |
| $1$ | $0$ | $0$ | $1$ | |
| | $\frac{1}{6}$ | $\frac{2}{6}$ | $\frac{2}{6}$ | $\frac{1}{6}$ |

$$\begin{cases} y_{n+1} = y_n + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) \\ k_1 = f(t_n, y_n), \\ k_2 = f(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1) \\ k_3 = f(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_2) \\ k_4 = f(t_n + h, y_n + hk_3) \end{cases}$$

The figure below gives a graphical illustration of RK4 method.

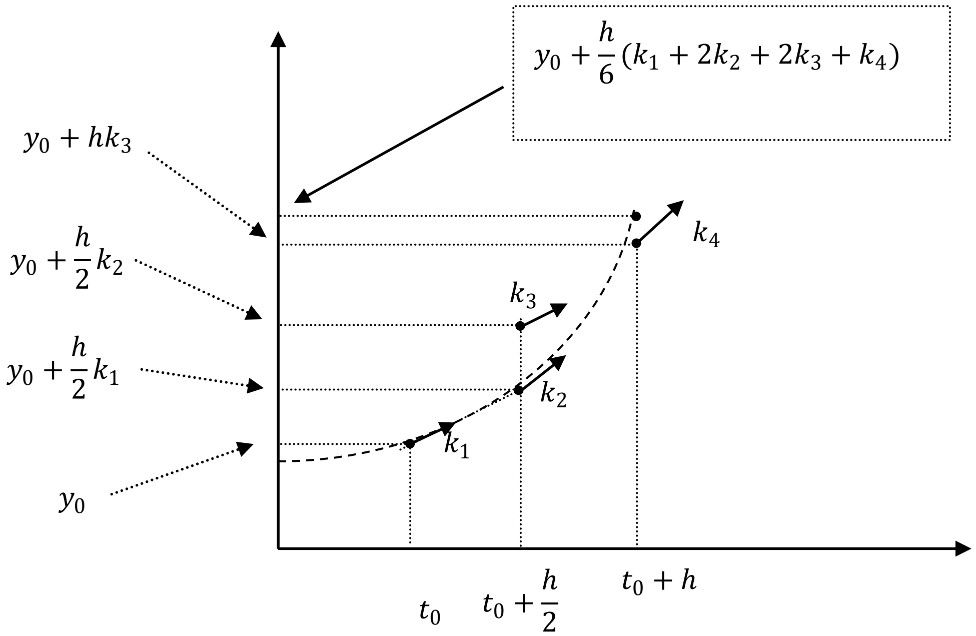



Figure : Graphical representation of the Runge-Kutta 4 (RK4) method

**Example 4**

a- Use the Runge-Kutta method of order 4 to calculate the first three iterations with h=0.1 to solve the differential equation.

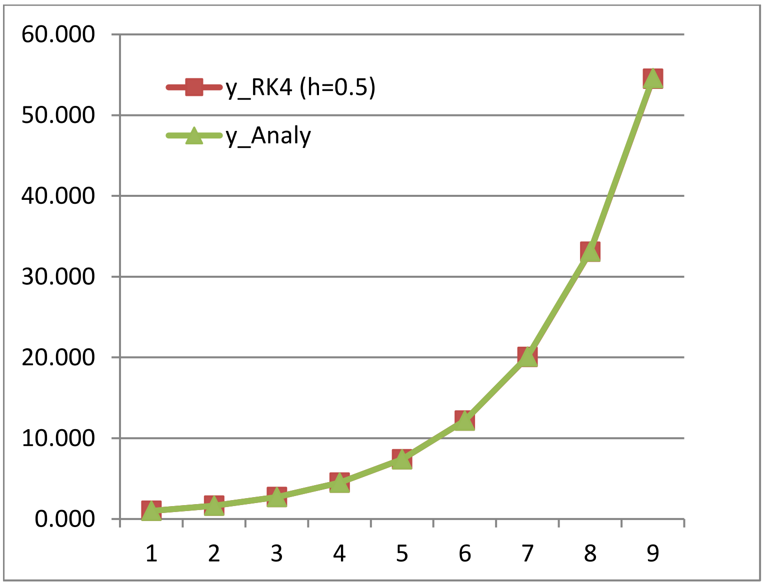
$$\begin{cases} \frac{dy}{dt} = y \\ y(0) = 1 \end{cases}$$

| n | h=0.1 | |
|---|---|---|
| 1 | $k_1$=0.2442 | $y_1$=**1.221** |
| | $k_2$=0.2687 | |
| | $k_3$=0.2711 | |
| | $k_4$=0.2985 | |
| 2 | $k1$=0.2442 | $y_2$=**1.49**2 |
| | $k2$=0.2687 | |
| | $k3$=0.2711 | |
| | $k4$=0.2985 | |
| 3 | $k1$=0.2442 | $y_3$=1.822 |
| | $k2$=0.2687 | |
| | $k3$=0.2711 | |
| | $k4$=0.2985 | |

a- Use the Runge-Kutta method of order 4 to calculate the nine iterations with h=0.5 for the differential equation

$$\begin{cases} \frac{dy}{dt} = y \\ y(0) = 1 \end{cases}$$

| h=0,5 | | | |
|---|---|---|---|
| n | t | y_RK4 | y_Analy |
| 1 | 0,000 | 1,000 | 1,000 |
| 2 | 0,500 | 1,648 | 1,649 |
| 3 | 1,000 | 2,717 | 2,718 |
| 4 | 1,500 | 4,479 | 4,482 |
| 5 | 2,000 | 7,384 | 7,389 |
| 6 | 2,500 | 12,172 | 12,182 |
| 7 | 3,000 | 20,065 | 20,086 |
| 8 | 3,500 | 33,076 | 33,115 |
| 9 | 4,000 | 54,523 | 54,598 |



In the following table and graph, we give the values and curves for the four previous methods (Euler, Mid-point, Heun, and RK4) as well as the analytical result.

| n | t | y_euler | y_mid | y_heun | y_RK4 | y_Analy |
|---|---|---|---|---|---|---|
| 1 | 0,000 | 1,000 | 1,000 | 1,000 | 1,000 | 1,000 |
| 2 | 0,500 | 1,500 | 1,625 | 1,625 | 1,648 | 1,649 |
| 3 | 1,000 | 2,250 | 2,641 | 2,641 | 2,717 | 2,718 |
| 4 | 1,500 | 3,375 | 4,291 | 4,291 | 4,479 | 4,482 |
| 5 | 2,000 | 5,063 | 6,973 | 6,973 | 7,384 | 7,389 |
| 6 | 2,500 | 7,594 | 11,331 | 11,331 | 12,172 | 12,182 |
| 7 | 3,000 | 11,391 | 18,413 | 18,413 | 20,065 | 20,086 |
| 8 | 3,500 | 17,086 | 29,921 | 29,921 | 33,076 | 33,115 |
| 9 | 4,000 | 25,629 | 48,621 | 48,621 | 54,523 | 54,598 |

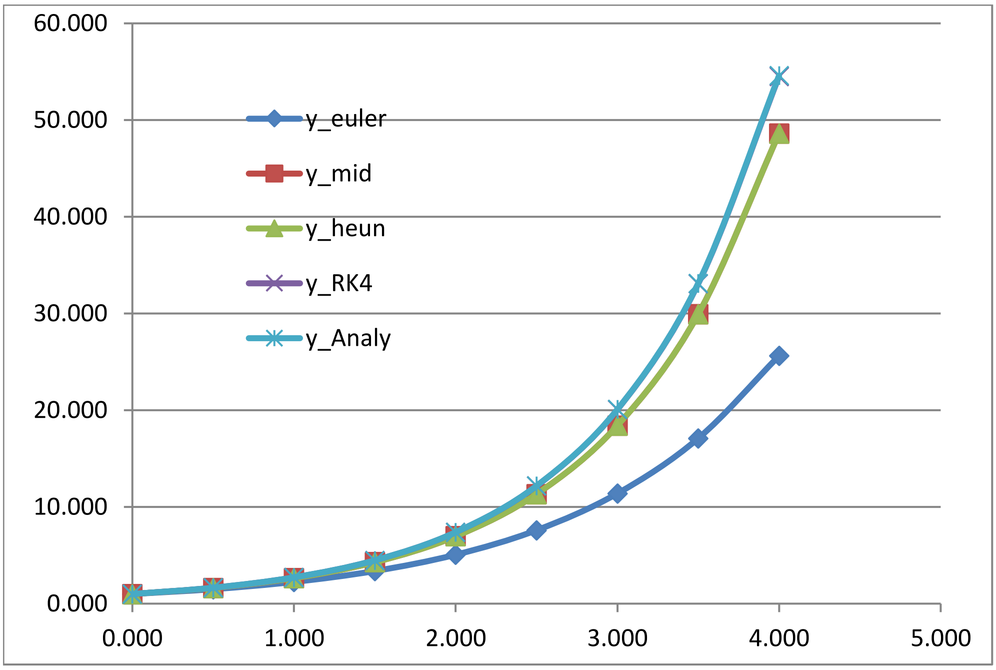



Figure : Graphic representation of the four numerical methods and the analytical curve.