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Marchov chains

- A Markov chain describes a system whose state changes over time.

- The changes are not completely predictable, but rather are governed by probability distributions.

- The future of the system depends only to its present state, and not the path by which the system got to this latter.

I. Marchov chains

Definitions 1

A time-homogeneous Markov chain is a discrete-time stochastic process (X_n,n ≥ 0) with values in a finite or countable set S (the state space) such that:

$P(X_{n+1}=j/X_n=i,X_{n-1}=i_{n-1},X_{n-2}=i_{n-2},\ldots,x_0=i_0)=P(X_{n+1}=j/X_n=i)=P_{ij}$.

for every n ≥ 0 and j, i, i_{n-1} , ... , i_1 , i_o $\in$ S.

I. Marchov chains

The ***transition matrix*** of the chain is the matrix $P = (p_{ij})$ $i,j \in S$ defined as $p_{ij} = P(X_{n+1} = j | X_n = i)$.

It satisfies the following properties:

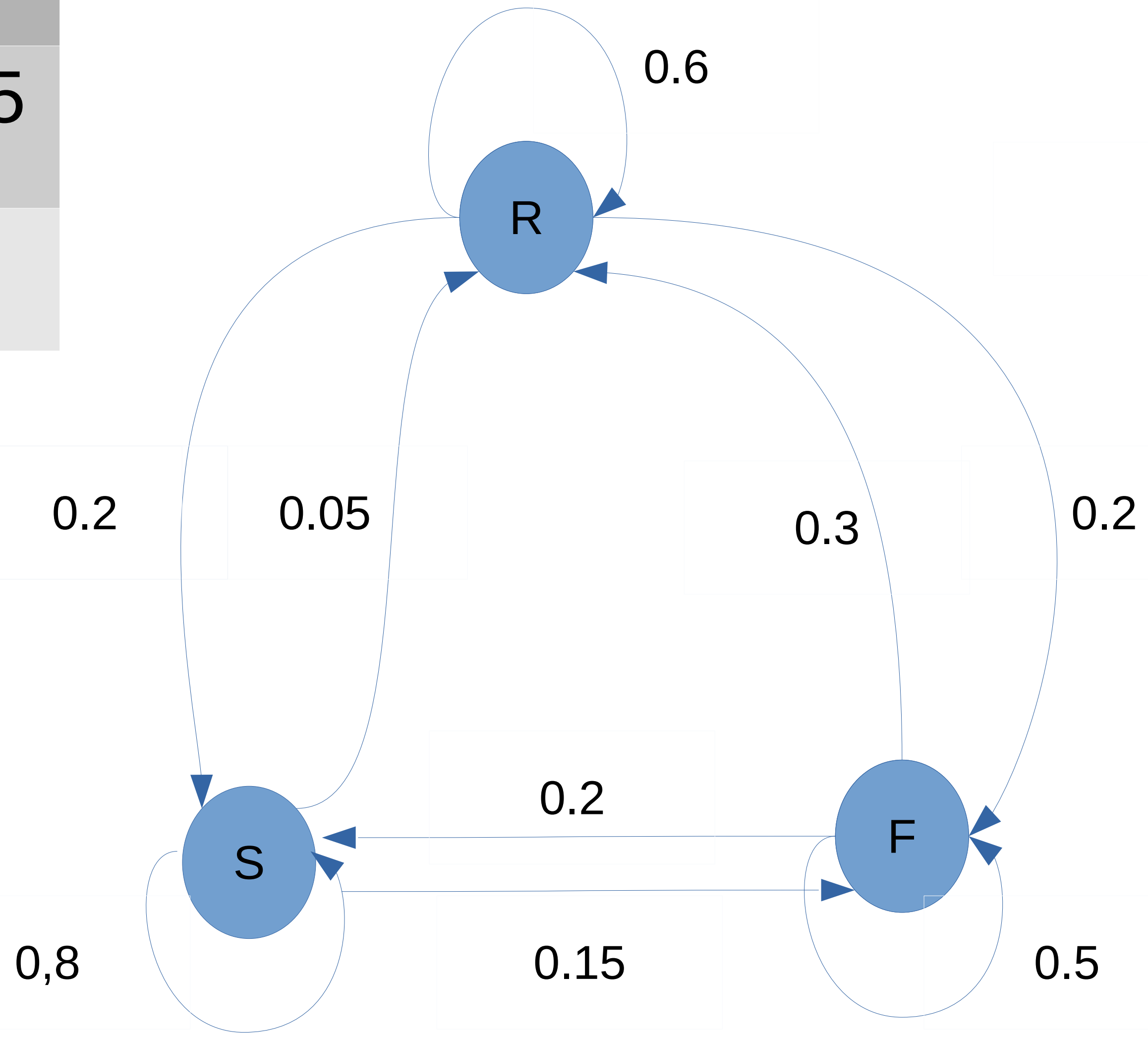
$0 \leq p_{ij} \leq 1$ $\forall i, j \in S$ and $\Sigma_{j \in S} p\ ij = \Sigma_{j \in S} P(X_{n+1} = j | X_n = i) = 1$ $\forall i \in S$

Note however that for a given $j \in S$, $\Sigma_{i \in S} p_{ij}$ can be anything.

I. Markov chains

The transition graph of the chain is the oriented graph where vertices are states and an arrow from i to j exists if and only if $p_{ij} > 0$, taking value p_{ij} when it exists.

0.6	0.2	0.2
0.05	0.8	0.15
0.3	0.2	0.5



II. Distribution

The distribution of the Markov chain at time $n \geq 0$ is given by:

$\pi_i^{(n)} = P(X_n = i)\ i \in S$

and its initial distribution is given by

$\pi_i^{(0)} = P(X_0 = i)\ i \in S$

For every $n \geq 0$, we have

$\Sigma_{i \in S} \pi_i^{(n)} = 1$

II Distribution

The main question that will retain our attention for the first part of the course:

When does $\pi^{(n)}$ (the distribution at time n) converge at $n \to \infty$ to some limiting distribution π?

III. m-step transition

Definition 2. we define m-step transition probabilities, for $m \geq 1$ and $i, j \in S$,

$$p_{ij}^{(m)} = P(X_{n+m} = j | X_n = i) = P(X_m = j | X_0 = i)$$

III. m-step transition

How to compute these probabilities? Using the Chapman-Kolmogorov equations.

For m = 2, these lead.

$p_{ij}^2 = \Sigma_{k \in S}\, p_{ik}\, p_{kj} = (P \cdot P)_{ij} = (P^2)_{ij}$

Indeed, we check that :

$p_{ij}^{(2)} = P(X_2 = j | X_0 = i) = \Sigma_{k \in S} P(X_2 = j, X_1 = k | X_0 = i)$

$= \Sigma_{k \in S} P(X_2 = j / X_1 = k, X_0 = i) * p(X_1 = k / X_0 = i)$

$= \Sigma_{k \in S} p_{kj} * p_{ik}$

III. m-step transition

For higher values of m and $0 \leq l \leq m$, Chapman-Kolmogorov equations read:

$p_{ij}^{(m)} = \Sigma_{k \in S}\, p_{ik}^{(l)} p_{kj}^{(m-l)} = (P^{(l)} \cdot P^{(m-l)})ij = (P^{m})_{ij}$

$p_{ij}^{(m)} = P[X_m=j/x_0=i] = \Sigma_{k \in S}\, P[X_m=j, X_l=k/x_0=i] =$

$\Sigma_{k \in S}\, P[X_m=j/ X_l=k, x_0=i] * P[X_l=k/ x_0=i]$

$= \Sigma_{k \in S}\, p_{ik}^{(l)} p_{kj}^{(m-l)}.$

III. m-step transition

Example 1

Consider a Markov chain given in previous section.

Determine $P[X_3=F / X_0=S]$ and $P[X_5=R / X_2=F]$.

0.6	0.2	0.2
0.05	0.8	0.15
0.3	0.2	0.5

P

0.43	0.32	0.25
0.115	0.68	0.205
0.34	0.32	0.34

$P^2_{32}=0.32$

0.349	0.392	0.259
0.1645	0.608	0.2275
0.322	0.392	0.268

$P^3_{13}=0.259$

IV Classification of states

Definitions 3.

Two states $i, j \in S$ communicate ("$i \longleftrightarrow j$") if $\exists n, m \geq 0$ such that $p_{ij}^{(n)} > 0$ and $p_{ji}^{(m)} > 0$.

A chain is said to be irreducible if all states communicate (a single class).

A state i is said to be absorbing if $p_{ii} = 1$.

IV Classification of states

Definitions 4.

Periodicity. For a state $i \in S$, define $d_i = \gcd(\{n \geq 1 : p_{ii}^n > 0\})$. If $d_i = 1$, we say that state i is aperiodic. Else if $d_i > 1$, we say that state i is periodic with period d_i.

- In a given class, all states have the same period $d_i = d$.

- If there is at least on self-loop in the class ($\exists i \in S$ such as $p_{ii} > 0$), then all states in the class are aperiodic.

$P_{11}^{(0)}=0$

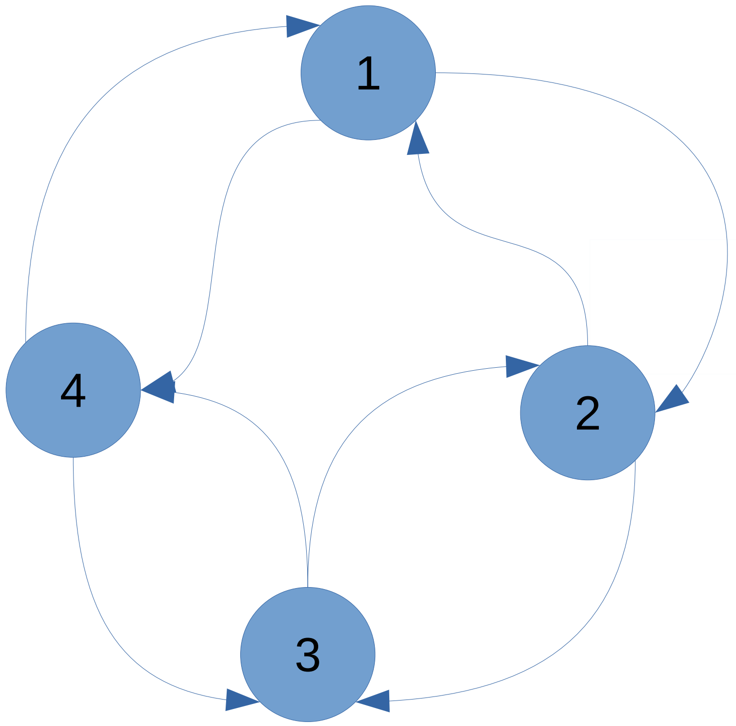
$P_{11}^{(1)}=0$

$P_{11}^{(2)}>0$

$P_{11}^{(3)}=0$

$P_{11}^{(4)}>0$

d1=d=2 so its periodic



IV Classification of states

Definition 5

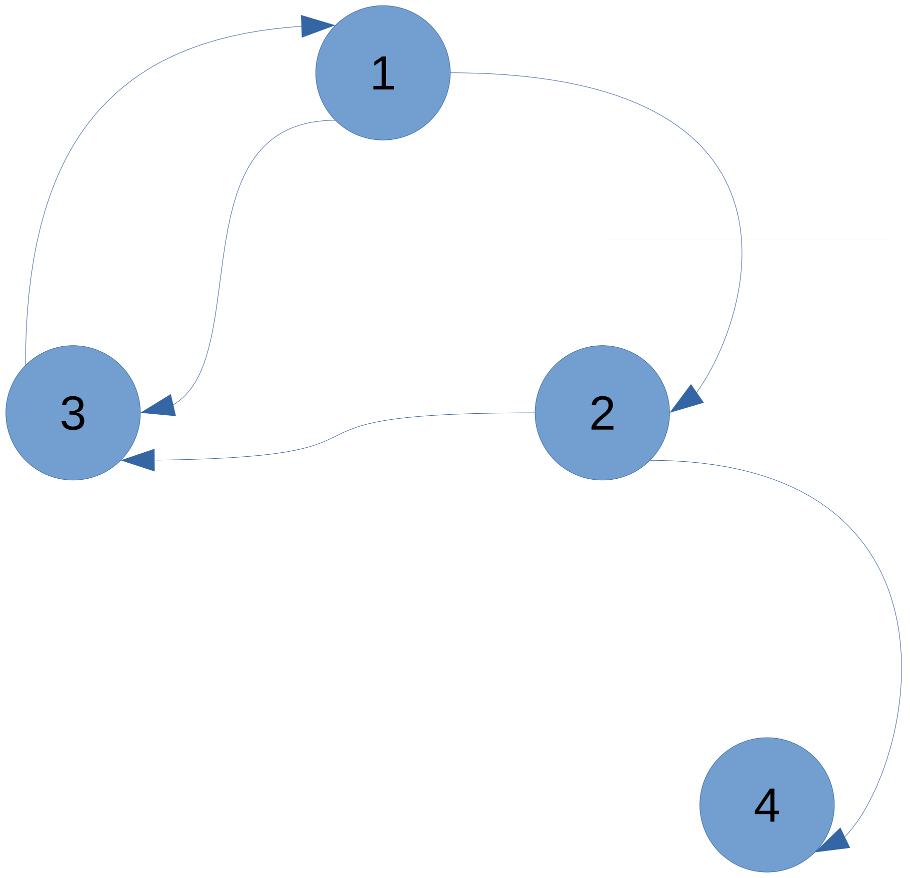
A state $i \in S$ is recurrent if $f_{ii} = P(\exists n \geq 1$ such that $X_n = i \mid X_0 = i) = 1$

(i.e., the probability that the chain returns to state i in finite time is equal to 1).

- A state $i \in S$ is transient if $f_{ii} < 1$.

So a state is recurrent if and only if it is not transient.

Note in particular that it is not necessary that $f_{ii}=0$ for state i to be transient.



states 1, 2 and 3 are transient and 4 is recurrent.

IV Classification of states

- In a given class, either all states are recurrent, or all states are transient.

- In a finite chain, a class is recurrent iff there is no arrow leading out of it. (So a finite irreducible chain is always recurrent.)

V Stationary distribution

As a reminder, the distribution of the Markov chain $(X_n , n \geq 0)$ at time n is given by:

$\pi_j^{(n)} = P(X_n = j), \ j \in S$:

Let us compute for $j \in S$:

$\pi_j^{(n+1)} = P(X_{n+1} = j) = \Sigma_{i \in S} P(X_{n+1} = j, X_n = i)$

$= \Sigma_{i \in S} P(X_{n+1} = j / X_n = i) \ P(X_n = i) = \Sigma_{i \in S} \pi_i^{(n)} p_{ij}$

V Stationary distribution

In vector notation (considering $\pi^{(n)}$, $\pi^{(n+1)}$ as row vectors), this reads: $\pi^{(n+1)} = \pi^{(n)} P$

which further implies that $\pi^{(n)} = \pi^{(0)} P^n$. This motivates also the following definition:

Let $(X_n , n \geq 0)$ be a Markov chain with transition matrix P

A probability distribution $\pi = (\pi_i , i \in S)$ is said to be a stationary distribution of the chain X if:

$\pi_j=\Sigma_{i \in S}\pi_i p_{ij}$ $\forall j \in S$ i.e. $\pi = \pi P$.

VI Ergodic Markov chain

Definition 6

A Markov chain is said to be ergodic if it is irreducible, aperiodic and recurrent.

Theorem (Ergodic theorem). Let X be an ergodic Markov chain. Then it admits a unique limiting and stationary distribution π, i.e., $\forall \pi^{(0)}$, $\lim_{n\to\infty} \pi^{(n)} = \pi$ and $\pi = \pi P$.

VII Stationary distribution

If π is stationary then $\pi P^n = \pi P\, P^{n-1} = \pi P^{n-1} = ... = \pi$.

• In particular, if the initial distribution $\pi^{(0)} = \pi$, then $\forall n \geq 0$, $\pi^{(n)} = \pi^{0)} P^n = \pi P^n = \pi$.

Remarks

we can see than π is a (left-)eigenvector of P .

Stationary distribution

The stationary distribution of the Markov chain previously mentioned is

$\pi=[\frac{1}{4}, \frac{1}{2}, \frac{1}{4}]$.