Solution:

Exo 1: (10 pts)

a) show that: $\forall x \in \mathbb{R} \setminus \mathbb{Z}$: $E(x) + E(-x) = -1$.

we have, $\forall x \in \mathbb{R}\setminus\mathbb{Z}$: $E(x) \leq x < E(x)+1 \ldots (1)$ 0,25

multiply (1) by (-1) we get:

0,25

$$-E(x) - 1 < -x \leq -E(x) \ldots (2).$$

On the other hand we have: $E(-x) \leq -x < E(-x)+1 \ldots (3)$ 0,25

From (2) and (3) we obtain:

$-E(x) - 1 = E(-x)$ 0,25 so: $E(-x) + E(x) = -1$.

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b) $\forall x, y \in \mathbb{R}$, if $x \leq y$ then $E(x) \leq E(y)$.

we have: $E(x) \leq x \ldots (1)$ 0,25

$E(y) \leq y \ldots (2)$

$(1) - (2) \Leftrightarrow E(x) - E(y) \leq x - y$, 0,25 since $x \leq y$ so

$x - y \leq 0$ 0,25 then $E(x) - E(y) \leq 0 \Rightarrow$ 0,25 $E(x) \leq E(y)$

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c) A and B are bounded, then:

$\inf(A) \leq A \leq \sup(A)$ and 0,1 $\inf(B) \leq B \leq \sup(B)$.

we have $A \cap B \subset A$ and $A \cap B \subset B$ so:

$$\begin{cases} \sup(A\cap B) \leq \sup(A) \\ \inf(A \cap B) \geq \inf(A) \end{cases} \text{ and } 0,1 \quad \begin{cases} \sup(A\cap B) \leq \sup(B) \\ \inf(A\cap B) \geq \inf(B). \end{cases}$$

then, $\sup(A\cap B) \leq \min(\sup(A), \sup(B))$ 0,1

$\inf(A\cap B) \geq \max(\inf(A), \inf(B))$.

d) Assume that $\exists x \in \mathbb{Q}$ such that $x^2 = 2$, 0,1

$x \in \mathbb{Q} \Rightarrow \exists p, q \in \mathbb{Z}^*$ such that $x = \frac{p}{q}$,

0,1

thus we assume that p and q are coprime.

we have: $x^2 = 2 \Leftrightarrow \frac{p^2}{q^2} = 2 \Leftrightarrow p^2 = 2q^2$

then p^2 is even, therefore: p is also 0,1 even i.e:

$\exists p' \in \mathbb{Z}$ such that: $p = 2p' \Rightarrow p^2 = 4p'^2$,

0,1

hence $q^2 = 2p'^2$ so q^2 is even and so:

q is also even. (contradiction).

then $x^2 = 2$ doesn't admit a solution in $\mathbb{Q}$.

2/ $A = [2, 4[\cup]0, 1]$.

Assume that: $A_1 = [2, 4[$ 0,25 and 0,25 $A_2 =]0, 1]$.

A_1, A_2 are bounded and:

$\inf(A_1) = \min(A_1) = 2$ 0,25 $\sup(A_1) = 4$, $\max(A)$ doesn't exist.

$\inf(A_2) = 0$, $0 \notin A_2$ ($\min(A_2)$ doesn't exist),

0,25

$\sup(A_2) = \max(A_2) = 1$.

so: • $\sup(A) = \max(\sup(A_1), \sup(A_2)) = \max(4, 1) = 4$,

0,25

since $4 \notin A$ then $\max(A)$ does not exist.

• $\inf(A) = \min(\inf(A_1), \inf(A_2)) = \min(0, 1) = 0$,

0,25

since $0 \notin A$ then $\min(A)$ does not exist.

3/ a/ $|z+1| = |z-i|$

$\Leftrightarrow |x+iy+1| = |x+iy-i| \Leftrightarrow |(x+1)+iy| = |x+i(y-1)|$

$\Leftrightarrow \sqrt{(x+1)^2+y^2} = \sqrt{x^2+(y-1)^2}$ 0,25

$\Leftrightarrow (x+1)^2+y^2 = x^2+(y-1)^2$

$\Leftrightarrow 2x = -2y \Rightarrow \boxed{-x = y}$ 0,25

Then: $Z = \{x \in \mathbb{R} \mid z = x(1-i)\}$

b/ $|z+1| = 1 \Leftrightarrow |x+iy+1| = 1$

$\Leftrightarrow \sqrt{(x+1)^2+y^2} = 1$ 0,25

$\Leftrightarrow (x+1)^2+y^2 = 1$

(the circle of radius 1, centred on $(-1,0)$). 0,25

4/ $x = \ln\left(\tan\left(\frac{y}{2}+\frac{\pi}{4}\right)\right)$

$$\operatorname{ch} x = \frac{1}{2}(e^{x}+e^{-x}) = \frac{1}{2}\left(e^{\ln\left(\tan\left(\frac{y}{2}+\frac{\pi}{4}\right)\right)} + \frac{1}{e^{\ln\left(\tan\left(\frac{y}{2}+\frac{\pi}{4}\right)\right)}}\right)$$

0,1

$$= \frac{1}{2}\left[\tan\left(\frac{y}{2}+\frac{\pi}{4}\right) + \frac{1}{\tan\left(\frac{y}{2}+\frac{\pi}{4}\right)}\right]$$

$$= \frac{1}{2}\left[\frac{\sin\left(\frac{y}{2}+\frac{\pi}{4}\right)}{\cos\left(\frac{y}{2}+\frac{\pi}{4}\right)} + \frac{\cos\left(\frac{y}{2}+\frac{\pi}{4}\right)}{\sin\left(\frac{y}{2}+\frac{\pi}{4}\right)}\right]$$ 0,1

$$= \frac{1}{2}\left[\frac{1}{\cos\left(\frac{y}{2}+\frac{\pi}{4}\right)\cdot\sin\left(\frac{y}{2}+\frac{\pi}{4}\right)}\right] = \frac{1}{\sin\left(2\left(\frac{y}{2}+\frac{\pi}{4}\right)\right)}$$ 0,1

$$= \frac{1}{\sin\left(y+\frac{\pi}{2}\right)} = \frac{1}{\cos y}$$ 0,1

Exo 2: (5 pts)

1/ show that $(u_n)_{n\in\mathbb{N}}$, and $(v_n)_{n\in\mathbb{N}}$ are adjacent:

$$u_{n+1} - u_n = \sum_{k=1}^{n+1} \frac{1}{k^2} - \sum_{k=1}^{n} \frac{1}{k^2} = \frac{1}{(n+1)^2} > 0$$

0,1

So $(u_n)_{n\in\mathbb{N}}$ is increasing sequence.

$$v_{n+1} - v_n = u_{n+1} + \frac{2}{n+2} - u_n - \frac{2}{n+1}$$

$$= \frac{1}{(n+1)^2} + \frac{2}{n+2} - \frac{2}{n+1} = \frac{(n+2) + 2(n+1)^2 - 2(n+1)(n+2)}{(n+1)^2(n+2)}$$

0,1

$$= \frac{-n}{(n+1)^2(n+2)} < 0$$

So $(v_n)_{n\in\mathbb{N}}$ is decreasing sequence.

0,1

$$\lim_{n\to+\infty} u_n - v_n = \lim_{n\to+\infty} \cancel{u_n} - \cancel{u_n} - \frac{2}{n+1} = \lim_{n\to+\infty}\left(\frac{-2}{n+1}\right) = 0$$

Then $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are adjacent sequences.

2/ $x_n = \dfrac{n^2-1}{n^2}$ is a Cauchy sequence if:

$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n, m \geq N : |x_m - x_n| < \varepsilon$ 0,1

$$|x_m - x_n| = \left|\frac{m^2-1}{m^2} - \frac{n^2-1}{n^2}\right| = \left|\frac{m^2-n^2}{m^2n^2}\right| = \left|\frac{1}{m^2} - \frac{1}{n^2}\right|$$ 0,1

we have $\left|\frac{1}{m^2} - \frac{1}{n^2}\right| \leq \frac{1}{m^2} + \frac{1}{n^2} \leq \frac{2}{N^2}$ 0,1

So $(x_n)_{n\in\mathbb{N}}$ is cauchy sequence iff: $\frac{2}{N^2} < \varepsilon \Rightarrow N > \sqrt{\frac{2}{\varepsilon}}$

So $\exists N > \sqrt{\frac{2}{\varepsilon}}$ i.e. $N = \left[\sqrt{\frac{2}{\varepsilon}}\right] + 1$ such that $|x_m - x_n| < \varepsilon$. 0,1

3/ $\frac{n-1}{n+1} \longrightarrow 1$

we must show that: $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}$

$(n \geq N) \Rightarrow \left| \frac{n-1}{n+1} - 1 \right| < \varepsilon$. 01

Let $\varepsilon > 0$, $\left| \frac{n-1}{n+1} - 1 \right| < \varepsilon \iff \left| \frac{-2}{n+1} \right| < \varepsilon$

$\iff \frac{2}{n+1} < \varepsilon$

$\iff n > \frac{2}{\varepsilon} - 1$ 01

Just choose: $N = E\left(\frac{2}{\varepsilon} - 1\right) + 1$.

So $\forall \varepsilon > 0, \exists N = E\left(\frac{2}{\varepsilon} - 1\right) + 1 \in \mathbb{N}, \forall n \in \mathbb{N}$: 01

$\left| \frac{n-1}{n+1} - 1 \right| < \varepsilon$.

Exo 3: (5 pts)

1/ The function g is continuous on $]0,1[$, because it is the sum and the ratio of continuous functions on $]0,1[$. 01

$* \lim_{x \overset{>}{\to} 0} g(x) = \lim_{x \overset{>}{\to} 0} \frac{f(x)}{x} - \frac{f(x)-1}{x-1} = f'(0) - 1$ 01

$= -1 = g(0)$. 01

$* \lim_{x \overset{<}{\to} 1} g(x) = \lim_{x \overset{<}{\to} 1} \frac{f(x)}{x} - \frac{f(x)-1}{x-1}$ 01

$= 1 - f'(1) = 1 = g(1)$. 01

So g is continuous on $[0,1]$. 01

2/ g is continuous on $[0,1]$, differentiable on $]0,1[$, and $g(0).g(1) < 0$, then according to the intermediate value theorem, there exists a real α in $]0,1[$ such that: 0,1

$g(\alpha) = 0$.

$$g(\alpha) = 0 \iff \frac{f(\alpha)}{\alpha} - \frac{f(\alpha) - 1}{\alpha - 1} = 0$$

0,1

$$\iff \frac{\alpha - f(\alpha)}{\alpha(\alpha-1)} = 0 \iff f(\alpha) = \alpha .$$

3/ f is continuous on $[0,1]$, differentiable on $]0,1[$ then according to the Mean ~~value~~ theorem, we have: 0,1

$\exists \beta \in]0,1[$ such that: $\dfrac{f(1) - f(0)}{1-0} = f'(\beta)$.

0,1

$\iff \exists \beta \in]0,1[$ such that: $f'(\beta) = 1$

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