

# Solution of series $N^{\circ}4$

**Exercise 1:**

$$1. f(x) = \begin{cases} x^2 \cos \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}, \quad x_0 = 0.$$

we have:

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x^2 \cos \frac{1}{x}}{x - 0} = \lim_{x \rightarrow 0} x \cos \frac{1}{x} = 0$$

because  $\left(-x \leq x \cos \frac{1}{x} \leq x, \text{ and } \lim_{x \rightarrow 0} x = 0\right)$ . So the function  $f$  is differentiable in  $x_0$  and  $f'(0) = 0$ .

$$2. f(x) = \begin{cases} \sin x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}, \quad x_0 = 0. \text{ we have:}$$

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{\sin x \sin \frac{1}{x}}{x - 0} = \lim_{x \rightarrow 0} \frac{\sin x}{x} \sin \frac{1}{x} = \lim_{x \rightarrow 0} \sin \frac{1}{x}, \text{ does not exist.}$$

$\left(\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1\right)$ , therefore  $f$  is not differentiable at  $x_0 = 0$ .

$$3. f(x) = \begin{cases} \exp\left(\frac{1}{x^2 - a^2}\right), & |x| < a \\ 0, & |x| \geq a \end{cases}, \quad |x_0| = a, \quad a \in \mathbb{R}_+.$$

We have

$$f(x) = \begin{cases} \exp\left(\frac{1}{x^2 - a^2}\right), & -a < x < a \\ 0, & x \in ]-\infty, -a] \cup [a, +\infty[ \end{cases}$$

the differentiability of  $f$  in  $x_0 = a$ :  $\lim_{x \rightarrow a^+} \frac{f(x) - f(a)}{x - a} = \lim_{x \rightarrow a^+} \frac{0 - 0}{x - a} = 0 = f'_r(a)$

$$\lim_{x \rightarrow a^-} \frac{f(x) - f(a)}{x - a} = \lim_{x \rightarrow a^-} \frac{\exp\left(\frac{1}{x^2 - a^2}\right) - 0}{x - a} = 0 = f'_l(a)$$

We have  $f'_r(a) = f'_l(a)$ , then  $f$  is differentiable at  $x_0 = a$ , and  $f'(a) = 0$ .

The differentiability of  $f$  in  $x_0 = -a$ :

$$\lim_{x \rightarrow -a^-} \frac{f(x) - f(-a)}{x + a} = \lim_{x \rightarrow -a^-} \frac{0 - 0}{x + a} = 0 = f'_r(-a).$$

$$\lim_{x \rightarrow -a^+} \frac{f(x) - f(-a)}{x + a} = \lim_{x \rightarrow -a^+} \frac{\exp(\frac{1}{x^2 - a^2}) - a}{x + a} = 0 = f'_l(-a).$$

We have:  $f'_r(-a) = f'_l(-a)$ , then  $f$  is differentiable at  $x_0 = -a$ , and  $f'(-a) = 0$ .

**Exercise 2:**

$$f(x) = \begin{cases} ax^2 + bx + 1, & 0 \leq x \leq 1 \\ \sqrt{x}, & x > 1 \end{cases}$$

We determine  $a$  and  $b$  such that  $f$  is differentiable on  $\mathbb{R}_+^*$ , we have  $\sqrt{x}$  is differentiable on  $]0, 1[$ , and  $ax^2 + bx + 1$  is differentiable on  $]1, +\infty[$ , so  $f$  is differentiable on  $]0, 1[ \cup ]1, +\infty[$ .

The differentiability of  $f$  in  $x_0 = 1$ : ( $f(1) = 1$ )

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \sqrt{x} = 1 = f(1)$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} ax^2 + bx + 1 = a + b + 1.$$

$$f \text{ is continuous at } x_0 = 1 \iff \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^-} f(x) = f(1) \implies a + b + 1 = 1 \iff$$

$a = -b$ . Therefore  $f$  is continuous for  $a = -b$ .

$$\lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1^+} \frac{\sqrt{x} - 1}{x - 1} = \lim_{x \rightarrow 1^+} \frac{\sqrt{x} - 1}{(\sqrt{x} - 1)(\sqrt{x} + 1)} = \lim_{x \rightarrow 1^+} \frac{1}{\sqrt{x} + 1} = \frac{1}{2} =$$

$f'_r(1)$ .

$$\lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1^-} \frac{ax^2 + bx + 1 - 1}{x - 1} = \lim_{x \rightarrow 1^-} \frac{ax^2 - ax}{x - 1} = \lim_{x \rightarrow 1^-} \frac{ax(x - 1)}{x - 1} = a =$$

$f'_l(1)$ .

$f$  is differentiable on  $x_0 = 1 \iff f'_r(1) = f'_l(1) \Rightarrow a = \frac{1}{2}$ , and  $b = -a = -\frac{1}{2}$ . So  $f$  is differentiable on  $x_0 = 1$  for  $a = \frac{1}{2}$ , and  $b = -\frac{1}{2}$ .

Calculate:  $f'(x)$ :

$$f'(x) = \begin{cases} \frac{1}{2\sqrt{x}} & : 0 < x \leq 1, \\ x - \frac{1}{2} & : x > 1. \end{cases}$$

**Exercise 3:**

1. Calculate derivatives:

- $y_1(x) = \sqrt{\ln x + 1} + \ln(\sqrt{x} + 1) \implies y'_1 = \frac{1}{2x\sqrt{\ln x + 1}} + \frac{1}{2(x + \sqrt{x})}$ .
- $y_2(x) = \frac{\sqrt{\cos x}}{1 - e^x} \implies y'_2 = \frac{-\sin x + \sin x e^{-x} - \sqrt{\cos x} e^{-x}}{(1 - e^{-x})^2}$ .
- $y_3(x) = e^{\cos \sqrt{x}} \implies y'_3 = \frac{-1}{2\sqrt{x}} \sin(\sqrt{x}) e^{\cos \sqrt{x}}$ .

2. Calculate  $n - th$  derivatives:

- $y_1(x) = \ln(1 + x)$

$$y'_1(x) = \frac{1}{1 + x}$$

$$y''_1(x) = \frac{-1}{(1 + x)^2}$$

$$y_1^{(3)}(x) = \frac{2}{(1 + x)^3}$$

$$y_1^{(4)}(x) = -\frac{2 \times 3}{(1 + x)^4}$$

$$y_1^{(5)}(x) = \frac{2 \times 3 \times 4}{(1 + x)^5}$$

$$y_1^{(6)}(x) = -\frac{2 \times 3 \times 4 \times 5}{(1 + x)^6}.$$

⋮

$$y_1^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$$

- $y_2(x) = \frac{1+x}{1-x}$

$$y'_2 = \frac{2}{(1-x)^2}$$

$$y''_2 = \frac{2 \times 2}{(1-x)^3}$$

$$y_2^{(3)} = \frac{2 \times 2 \times 3}{(1-x)^4}$$

$$y_2^{(4)} = \frac{2 \times 2 \times 3 \times 4}{(1-x)^5}$$

⋮

$$y_2^{(n)} = \frac{2n!}{(1-x)^{n+1}}$$

- $y_3(x) = (x+1)^3 e^{-x}$

Assume that  $g(x) = (x+1)^3$ , and  $f(x) = e^{-x}$ , so

$$g'(x) = 3(x+1)^2, \quad g''(x) = 6(x+1), \quad g^{(3)}(x) = 6, \quad g^{(n)}(x) = 0, \quad \forall n \geq 4$$

$$f'(x) = -e^{-x}, \quad f''(x) = e^{-x}, \quad f^{(n)}(x) = (-1)^n e^{-x}, \quad \forall n \in \mathbb{N}$$

then,

$$\begin{aligned} (y_3)^{(n)} &= \sum_{k=0}^3 C_n^k (e^{-x})^{(n-k)} (1+x)^{(k)} \\ &= \sum_{k=0}^3 C_n^k (-1)^{n-k} e^{-x} \cdot \frac{3!}{(3-k)!} (1+x)^{(3-k)} \end{aligned}$$

- $y_4(x) = x^2 \sin 3x$ , according to Leibniz

$$\begin{aligned} f^{(n)} &= \sum_{k=0}^n C_n^k (\sin 3x)^{(n-k)} (x^2)^{(k)} \\ &= 3^n x^2 \sin \left( 3x + \frac{n\pi}{2} \right) + 2xn3^{n-1} \sin \left( 3x + \frac{(n-1)\pi}{2} \right) + n(n-1)3^{n-2} \sin \left( 3x + \frac{(n-2)\pi}{2} \right) \end{aligned}$$

#### Exercise 4:

Determine the extrema:

$x_0$  is extremum  $\iff f'(x_0) = 0$  and  $f''(x_0) \neq 0$ .

1.  $f(x) = \sin x^2$ , on  $[0, \pi]$ ,  $f'(x) = 2x \cos x^2$  the critical points are:

$$f'(x) = 0 \iff 2x \cos x^2 = 0 \iff \begin{cases} x = 0 \\ \cos x^2 = 0 \end{cases}$$

$$\text{therefore, } \begin{cases} x = 0 \\ x^2 = \frac{\pi}{2} + k\pi \end{cases} \iff \begin{cases} x = 0 \\ x = \sqrt{\frac{\pi}{2} + k\pi} \end{cases}$$

$f''(x) = 2 \cos x^2 - 4x^2 \sin x^2$ , so:

- Four  $x = 0$ ,  $f''(0) = 2 > 0$ , then 0 is an extremum (minimum).
- Four  $x = \sqrt{\frac{\pi}{2} + k\pi}$ ,  $f''(\sqrt{\frac{\pi}{2} + k\pi}) = -4 \left( \frac{\pi}{2} + k\pi \right) \sin \left( \frac{\pi}{2} + k\pi \right) \neq 0$ .
  - if  $k$  is even:  $\sin \left( \frac{\pi}{2} + k\pi \right) = 1$ , and  $f''(\sqrt{\frac{\pi}{2} + k\pi}) = -4 \left( \frac{\pi}{2} + k\pi \right) < 0$ , so  $\sqrt{\frac{\pi}{2} + k\pi}$  is an extremum (maximum).

– if  $k$  is odd  $\sin\left(\frac{\pi}{2} + k\pi\right) = -1$ , and  $f''\left(\sqrt{\sin\frac{\pi}{2} + k\pi}\right) = 4\left(\sqrt{\frac{\pi}{2} + k\pi}\right) > 0$ , so  $\sqrt{\frac{\pi}{2} + k\pi}$  is an extremum (minimum).

2.  $g(x) = x^4 - x^3 + 1$ , on  $\mathbb{R}$

$g'(x) = 4x^3 - 3x^2$ , the critical points are:

$$g'(x) = 0 \iff 4x^3 - 3x^2 = 0 \iff x^2(4x - 3) = 0 \iff \begin{cases} x = 0 \\ x = \frac{3}{4} \end{cases}$$

$$g''(x) = 12x^2 - 6x$$

- For  $x = \frac{3}{4}$ ,  $g''\left(\frac{3}{4}\right) = \frac{9}{4} > 0$ , so  $\frac{3}{4}$  is an extremum (minimum).
- For  $x = 0$ ,  $g''(0) = 0 \implies f^{(3)}(x) = 24x - 6 \implies f^{(3)}(0) \neq 0$ , so 0 is not an extremum.

**Exercise 5:**

1. (a)  $f(x) = \sin^2 x$ , on  $[0, \pi]$

we have  $f$  is continuous on  $\mathbb{R}$ , so it is continuous on  $[0, \pi]$ , and differentiable on  $]0, \pi[$ .

$f(0) = 0$ , and  $f(\pi) = 0 \implies f(0) = f(\pi)$ , so we can apply Rolle's theorem on  $f$ .

(b) the same for  $g(x) = \frac{\sin x}{2x}$ , on  $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ .

2. We show that  $x < \frac{y-x}{\ln y - \ln x} < y$ .  $\forall x, y \in \mathbb{R}_+^*$ , we apply the Mean value theorem on the function  $f(t) = \ln t$  on the interval  $[x, y]$  such that  $0 < x < y$ .

$f(t) = \ln t$  is continuous on  $[x, y]$ , and differentiable on  $]x, y[$ , then according to the Mean value theorem:

$\exists x \in ]x, y[$ :  $f'(c) = \frac{f(y) - f(x)}{y - x}$  so  $\frac{1}{c} = \frac{\ln y - \ln x}{y - x} \implies c = \frac{y - x}{\ln y - \ln x}$ , and  $c \in ]x, y[$ , then  $x < \frac{y - x}{\ln y - \ln x} < y$ .

**Exercise 6:**

1.  $\lim_{x \rightarrow 0} \frac{1 - \cos x}{e^x - 1} = \frac{0}{0}$   
 $\lim_{x \rightarrow 0} \frac{1 - \cos x}{e^x - 1} = \lim_{x \rightarrow 0} \frac{(1 - \cos x)'}{(e^x - 1)'} = \lim_{x \rightarrow 0} \frac{\sin x}{e^x} = 0.$
2.  $\lim_{x \rightarrow \pi} \frac{\sin x}{x^2 - \pi^2} = \frac{0}{0}$   
 $\lim_{x \rightarrow \pi} \frac{\sin x}{x^2 - \pi^2} = \lim_{x \rightarrow \pi} \frac{(\sin x)'}{(x^2 - \pi^2)'} = \lim_{x \rightarrow \pi} \frac{\cos x}{2x} = \frac{-1}{2\pi}.$