6 Linear Algebra الجبر الخطي

6.1 introduction

The notion of vector space is a fundamental structure of modern mathematics. This involves identifying the common properties shared by very different sets, such as the set of vectors of the plane, the set of real functions, polynomials, matrices.

6.2 Laws of internal composition, groups

6.2.1 internal composition law قانــون تركيــب داخلــي

Definition 6.1.1 Let E be a non-empty set.

We call " * " law (operation) of internal composition on E the application of $E \times E \ in \ E$

$$* : R \times R \to R$$

$$(x, y) \to x * y$$

$$* \ is \ an \ internal \ composition \ law \ on \ E \Leftrightarrow \forall x, y \in E : x * y \in E$$

القانون الداخلي على مجموعة E هو تطبيق يربط عنصرين من المجموعة بعنصر من نفس المجموعة بمعنى آخر، هو عملية ثنائية تكون فيها المجموعة مستقرة

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Example 6.1.1

- Addition on R is an internal operation because: $\forall x, y \in R \ x + y \in R$
- Addition on ℕ is an internal operation because: $\forall x, y \in \mathbb{N} \ x + y \in \mathbb{N}$
- Multiplication on R is an internal law of composition: $\forall x, y \in R \ x \times y \in R$
- Subtraction on R is an internal law of composition: $\forall x, y \in R \ x \times x \in R$
- Subtraction on ℕ **is NOT** an internal law of composition because : $\exists x, y \in \mathbb{N} \ x \times y \notin \mathbb{N} \ (2 - 4 = -2 \notin \mathbb{N})$

Properties

Let E be a set and "*", "Δ" two laws of internal composition in E, then:

1. **Commutativity تبــادلية**

We say that " * " is commutative in E if and only if

نقول ان العملية * تبادلية ادا وفقط اذا كان

$$\forall x, y \in E \ x * y = y * x$$

2. **Associativity تجميعية**

We say that " " is associative in E if and only if:

نقول ان العملية * تجميعية ادا وفقط اذا كان

$$\forall x, y \in E \ (x * y) * z = x * (y * z)$$

3. **Identity element عنصر حيادي**

The internal law $' * '$has an identity element on E (neutral element), denoted e, if and only if :

نقول ان القانون الداخلي يقبل عنصر محايد نرمز له ب e اذا وفقط اذا كان:

$$\forall x \in E : x * e = e * x = x$$

4. **Inverse element عنصر نظير**

Let $x \in E$, we say that $x' \in E$ is the symmetric of x if and only if:

نقول ان العنصر $x' \in E$ القانون الداخلي هو نظير او عكس العنصر $x \in E$ اذا وفقط اذا كان:

$$x * x' = x' * x = e$$

5. **Distributivity توزيعية**

We say that "* " is distributive with respect to $''\Delta''$ if and only if:

نقول ان العملية * توزيعية بالنسبة لعملية أخرى Δ ادا وفقط اذا كان

$$\forall x, y, z \in E \begin{cases} x * (y \, \Delta \, z) = (x * y) \, \Delta (x * \, z) \\ x \, \Delta \, (y * z) = (x \, \Delta \, y) * (x \, \Delta \, z) \end{cases}$$

Exercise 6.1

On $R - \left\{\frac{1}{2}\right\}$ we define the operation $(*)$ such as for $x, y \in R - \left\{\frac{1}{2}\right\}: x * y = x + y - 2xy$

1- Is $(*)$ an internal composition law.
2- Is it: commutative, associative?
3- What is the identity element by * ?

6.2.2 Groups

Definition 6.2.1 Let E be a set combined with an operation (law) " * ": We say that $(E; *)$ is a group if and only if:

1. **Internal law** the law " * " is **internal** in E.
2. **Associativity** the law " * " is **associative.**
3. **identity element** the law " * " admits an **identity element** (neutral element) in E:

$$\forall x \in E \ \exists x' \in E: x * x' = x' * x = e$$

4. **inverse element** every element of E admits an **inverse element** for the law " * ":

And if moreover " * " is **commutative**, we say that E is an **abelian (or commutative) group**.

Example 6.2.1

1. $(Z, +)$ is a commutative group. هي مجموعة تبادلية.
2. The real numbers with respect to addition, which we denote as (R,+) is a group: it has the *identity 0*, any element x has *an inverse −x*, and it satisfies *associativity*.

3. $(N;\ +)$ **is not** a group because the elements of ℕ have no inverse elements on ℕ

4. (ℝ ; ·) **is not** a group: the element $0\ \in\ R$ has no inverse, as there is nothing we can multiply 0 by to get to 1.

5. $(\mathbb{R}^*,\cdot)$ **is** a group! The identity in this group is 1, every element x has an inverse $1/x$ such that $x\ \cdot\ (1/x) = 1$, and this group satisfies *associativity*.

Exemple 6.3 *Soit* $E = \{-1, 1\}$ *, on a :* $(E, \times)$ *est un groupe abélien.*

6.3 Vector space

In this part, (𝕜 ; +; ·) denotes a commutative field حقل شعاعي, in practice $\mathbb{k} = R\ or\ \mathbb{k} = \mathbb{C}$

6.3.1 Vector space structure

Let 𝕜 be a commutative field (generally it is ℝ or ℂ) and let E be a non-empty set provided with an *internal operation* denoted (+):

$$+: E \times E \to E$$

$$(x, y) \to x + y$$

and an *external operation* noted (·):

$$\cdot : \mathbb{k} \times E \to E$$

$$(\lambda, y) \to \lambda \cdot y$$

A vector space $(E, +,\ \cdot)$ is a set E with two operations '+' and '·' satisfying the following properties for all $x, y\ \in\ E$ and $\lambda, \mu \in \mathbb{R}$

$(+)$	1.	'+' in an internal law	$\forall x, y \in E: x + y \in\ E$
	2.	identity element	there exist a zero vector $\exists\ 0_E\ \forall x \in E: 0_E + x = x$
	3.	inverse element	Each vector in E must have an opposite in E.
			$\forall x \in E\quad \exists(-x) \in E: x + (-x) = (-x) * x = 0$

	4. associativity	$\forall x, y, z \in E: (x + y) + z = x + (y + z)$
	5. commutativity	$\forall x, y \in E\ x + y = y + x$
	this means that $(E, +)$ is a commutative group:	
$(\cdot)$	1. The scalar multiple of a vector is a vector: $\forall \lambda \in \mathbb{k}, \forall x \in E: \lambda \cdot x \in E$	
	2. distributivity: $\forall \lambda \in K,\ \forall x, y \in E: \lambda \cdot (x + y) = \lambda \cdot x + \lambda \cdot y$	
	3. $\forall \lambda, \mu \in K, \forall x \in E: (\lambda \cdot \mu) \cdot x = \lambda \cdot (\mu \cdot x)$	
	4. $\forall \lambda, \mu \in K, \forall x \in E: (\lambda + \mu). x = \lambda. x + \mu. x$	
	5. $\forall x \in E, 1_K \cdot x = x$ $(1_{\mathbb{k}} is\ the\ identity\ element\ for\ the\ multiplication\ law\ (\cdot)$	

The elements of the vector space are called ***vectors*** and those of $\mathbb{k}$ **are scalars**.

Examples

- $(\mathbb{R}, +, \cdot) is\ \mathbb{R} - vector\ space$
- The set of all real valued continuous (differentiable or integrable) functions defined on the closed interval I is a real vector space

$$(f + g)(x) = f(x) + g(x)$$
$$(\alpha f)(x) = \alpha f(x)$$

For all $f, g \in E$ and $\alpha \in R$.

Proposition 6.2.1. If E is $K - vector\ space$, then we have the following properties.

1. $\forall x \in E, 0_k. x = 0_E$
2. $\forall x \in E, (-1_{\mathbb{k}}). x = -x$
3. $\forall \lambda \in \mathbb{K}, \lambda \cdot 0_E = 0_E$
4. $\forall \lambda \in \mathbb{K}, \forall x, y \in E, \lambda \cdot (x - y) = \lambda \cdot x - \lambda \cdot y$
5. $\forall \lambda \in \mathbb{K}, \forall x \in E, \lambda \cdot x = 0_E \Leftrightarrow \lambda = 0_K or\ x = 0_E$

6.3.2 Subspace of a vector space

Definition Let E be a k- vector space and F a *nonempty subset* of E. we say that F is a *vector subspace* of E if and only if:

$$\forall x, y \in F, \forall \alpha, \beta \in K : \alpha u + \beta v \in F$$

Remark

1. $0_E\ and\ E\ are\ vector\ spaces$
2. $0_E\ is\ a\ vector\ (0_E \in E)\ and\ 0_k is\ a\ scalar\ (0_k \in k)$

Exercice 6.2

Show that $F = \{((0, y, z); y, z \in \mathbb{R}\}$ is a subspace of the vector space R^3

Solution 6.2

To show that $F = \{((0, y, z); y, z \in \mathbb{R}\}$ is a subspace of the vector space R^3 we need to show that:

1- <u>F is non empty</u> $\leftrightarrow$ *the vector is in* $F(0_F \in F)$:

$$0_F = (\mathbf{0}, 0,0) \; so \; 0_F \in F$$

Explanation F is the set of vectors with tree composants : triplets (x,y,z)

F هي مجموعة الاشعة (الثلاثيات(x, y, z)) ذات الاحداثية الأولى المعدومة : $(0, x, y)$

بما ان احداثية الشعاع المعدوم 0_F الأولى هي $(x = 0)$فان الصفر ينتمي الى المجموعة F ومنه المجموعة Fهي مجموعة غير خالية $F \neq \emptyset$

2- <u>$\forall u, v \in F, \forall \alpha, \beta \;\leftrightarrow\; \alpha u + \beta v \in F$:</u>

We need to prove that $\forall u, v \in F$ *and* $\forall \alpha, \beta \in \mathbb{R}$ *the vector* $\alpha u + \beta v$ is in F

For $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *two vectors in F we have* : $u = (0, y_1, z_1), v = (0, y_2, z_2)$

يجب ان نثبت انه من اجل كل شعاعين v و u في المجموعة F ومن اجل كل عددين حقيقيين β و α : الشعاع

$\alpha u + \beta v$ أيضا محتوى في F

Let $\boldsymbol{u} = (\mathbf{0}, \boldsymbol{y_1}, \boldsymbol{z_1}) \in \boldsymbol{F}$, $\boldsymbol{v} = (\mathbf{0}, \boldsymbol{y_2}, \boldsymbol{z_2}) \in \boldsymbol{F}$, and $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}$ Then:

$\forall u, v \in F, \forall \alpha, \beta \;\leftrightarrow\; \alpha u + \beta v = \alpha(0, y_1, z_1) + (0, y_2, z_2)$

$$= (\alpha 0, \alpha y_1, \alpha z_1) + (\beta 0, \beta y_2, \beta z_2)$$

$$= (0, \alpha y_1, \alpha z_1) + (0, \beta y_2, \beta z_2)$$

$$= (0, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2)$$

$$\leftrightarrow (0, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2) \in F$$

اذن $\alpha u + \beta v$ ينتمي الى F (الاحداثية الأولى للشعاع $\alpha u + \beta v$ معدومة)

Hence, $F = \{((0, y, z); y, z \in \mathbb{R}\}$ is a vector subspace of the vector space R^3

ومنه نستنتج ان $F = \{((0, y, z); y, z \in \mathbb{R}\}$ هو فضاء شعاعي جزئي من الفضاء الشعاعي R^3

6.3.3 Sum of two vector subspaces

Definition 6.2.3. Let E be K-vector space, and F_1, F_2 two vector subspaces of E. The set defined as:

$$F_1 + F_2 = (u_1 + u_2 : u_1 \in F_1 \text{ and } u_2 \in F_2$$

is a vector subspace of E, called the sum of F_1 *and* F_2.

6.3.4 Linear combination, linear dependence, independence

- **Linear combinations**

Let E be a K-vector space. We say that the vector u is a linear combination of the vectors $v_1, v_2, .., v_n$ of E if $\exists\ \alpha_1, \alpha_2, .., \alpha_n \in K : u = \alpha_1 v_1 + \alpha_2 v_2 + .. + \alpha_n v_n$

Exercise 6.3

Express $u = (-2,3)$ *in* $\mathbb{R}^2$ *as a linear combination of the vectors* $v_1 = (1,1)$ *and* $v_2 = (1,2)$

Solution 6.3

Let α_1, α_2 be scalars such that: $u = \alpha_1 v_1 + \alpha_2 v_2$

$u = \alpha_1(1,1) + \alpha_2(1,2) = (\alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2)$

$(-2,3) = (\alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2)$

$$\begin{cases} -2 = \alpha_1 + \alpha_2 \ldots (1) \\ 3 = \alpha_1 + 2\alpha_2 \ldots (2) \end{cases} \rightarrow \begin{cases} \alpha_2 = 5 \\ \alpha_1 = -7 \end{cases}$$

$Hence, u = -7v_1 + 5v_2$

- **Linear independence and linear dependence**

Definition (Linear independence)

Let $V = \{v_1, v_2, .., v_n\}$. We say that S is <u>linearly independent</u> if for all $\alpha_1, \alpha_2, \ldots, \alpha_n \in K$:

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0_E \Rightarrow \alpha_1 = \alpha_2 = \ldots = \alpha_n = 0_K.$$

Definition (Linear dependence)

The set $V = \{v_1, v_2, .., v_n\}$ is linearly dependent if there are scalars $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ *<u>not all zero</u>* for which $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0_E$

That is to say:

$for\ \alpha_1, \alpha_2, .., \alpha_n \in K : \exists \alpha_i \neq 0, i \in \{1, .., n\}: \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0_E$

Exercise 6.4

1. Show that the set $V = \{(-1, 0), (2, 1)\}$ is linearly independent.
2. Show that the set V = {(1, 0),(−2, 0)} is linearly dependent.

Solution 6.4

1. $Let\ \alpha_1, \alpha_2 \in R: \alpha_1(-1,0) + \alpha_2(2,1) = (0,0)$

$\Rightarrow (-\alpha_1 + 2\alpha_2, \alpha_2) = (0,0)$
$\Rightarrow -\alpha_1 + 2\alpha_2 = 0\ and\ \alpha_2 = 0$
$\Rightarrow \alpha_1 = \alpha_2 = 0$
$Let\ \alpha_1, \alpha_2 \in R$

2. $\alpha_1(1,0) + \alpha_2(-2,0) = (0,0)$

$\Rightarrow (\alpha_1 - 2\alpha_2, 0) = (0,0)$

$\Rightarrow \alpha_1 - 2\alpha_2 = 0 \Rightarrow \alpha_1 = 2\alpha_2.$

$$\Rightarrow \exists \alpha_1 = 2 \neq 0\ and\ \alpha_2 = 1 \neq 0\ such\ that\ 2(1,0) + 1(-2,0) = (0,0)$$

6.3.5 Generating sets and basis of a vector space مولدة وقاعـــدة الفضاء الشعـــاعـــي

Definition Let E be a vector space, a finite set of vectors $V = \{v_1, v_2, .., v_n\} \subset E$ is called ***a generator set*** of E if every vector $u \in E$ can be expressed as a linear combination of vectors of V

$$\forall u \in E, \exists \alpha_1, \alpha_2, \dots, \alpha_n \in E : u = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$$

And we say E is generated by V.

المجموعة المولدة هي التي نستطيع كتابة جميع عناصر المجموعة بدلالتها يعني انه يمكن كتابة أي شعاع من اشعة المجموعة E بواسطة تركيبة خطية من الاشعة المولدة

Example $(0,1)\ and\ (1,0)$ is a generating set of $R^2 = \{(x,y); x,y \in R\}$

$$u \in R^2 \Leftrightarrow u = (x,y)$$

Definition. Let E be a vector space over R, the finite set $V = \{v_1, v_2, .., v_n\} \subset E$ is called a *basis of E* if:

1. V is linearly independent.
2. V is a generator of E.

Example $(0,1)\ and\ (1,0)$ is a base of $R^2 = \{(x,y); x,y \in R\}$ because $(0,1)\ and\ (1,0)$ are linearly independent.

$$R^2 = \{(x,y); x,y \in R\} = \underbrace{\{x(1,0) + y(0,1); x,y \in R\}}_{\text{هكذا نستخرج القاعدة}}$$

Exercise 6.5

Consider the set $S = \{(1,1,1), (2,2,0), (3,0,0)\}$ نعتبر المجموعة التالية

1. Is S a system of generators of the vector space $\mathbb{R}^3$?

هل المجموعة مولدة للفضاء الشعاعي $\mathbb{R}^3$

2. Write $v = (3,4,2)$ as a linear combination of S

اكتب الشعاع $v = (3,4,2)$ في شكل تركيبة خطية من اشعة S

Proposition

If S is a basis, every vector can be written as a linear combination of its elements in a unique way.

Exercise 6.6

Consider the set $V = \{(1,0),(1,-1)\}$

Is V a basis of $\mathbb{R}^2$

Solution 6.6

$$V = \{(1,0),(1,-1)\} \text{ is a basis of } \mathbb{R}^2 \Leftrightarrow \begin{cases} 1. V \text{ is linearly independent} \\ \quad\quad and \\ 2. V \text{ is a generator set} \end{cases}$$

1. V *is linearly independent means that*:

$\forall \alpha_1, \alpha_2 \in \mathbb{R}$: $\alpha_1(1,0) + \alpha_2(1,-1) = 0 \Leftrightarrow \alpha_1 = 0 \text{ and } \alpha_2 = 0$

Let $\alpha_1, \alpha_2 \in \mathbb{R}$

$$\forall \alpha_1, \alpha_2 \in \mathbb{R}: \alpha_1(1,0) + \alpha_2(1,-1) = 0_{R^2} \Leftrightarrow \alpha_1(1,0) + \alpha_2(1,-1) = (0,0)$$

$$\Leftrightarrow (\alpha_1, 0) + \alpha_2(\alpha_2, -\alpha_2) = (0,0)$$

$$\Leftrightarrow (\alpha_1 + \alpha_2, -\alpha_2) = 0$$

$$\Rightarrow \begin{cases} \alpha_1 + \alpha_2 = 0 \\ \quad -\alpha_2 = 0 \end{cases} \Rightarrow \begin{cases} \alpha_1 = 0 \\ \alpha_2 = 0 \end{cases}$$

So $\{(1,0),(1,-1)\}$ are linearly independent.

2. V is a generator set of $\mathbb{R}^2$ means that:

$$\forall\, u(x,y) \in \mathbb{R}^2\ \exists\ \alpha_1, \alpha_2 \in \mathbb{R} \text{ such that}: u = \alpha_1(1,0) + \alpha_2(1,-1)$$

Let $\alpha_1, \alpha_2 \in \mathbb{R} : u = \alpha_1(1,0) + \alpha_2(1,-1) \Leftrightarrow (x,y) = (\alpha_1 + \alpha_2, -\alpha_2)$

6.3.6 Dimension of a vector space

Definition Let $V = \{v_1, v_2, .., v_n\} \subset E$ a ***basis*** of the vector space E

The number of elements of the basis V: n is called ***dimension of the vector space E*** and denoted $dim\ E = n$.

Remark

- If $V' = \{v_1', v_2', \dots,\ v_m'\}$ is another basis of E then $dim\ V' =\ dim\ V \Rightarrow m =\ n$
- A basis of a vector space E is the smallest generating set of E.
- A generator has at least n vectors $(dim\ E = n)$

Example

- $\dim \mathbb{R} = 1$
- $\dim \mathbb{R}^2 = 2 : \{(1,0), (0,1)\}\ is\ an\ example\ of\ a\ basis\ of\ R^2$
- $\dim \mathbb{R}^3 = 3: \{(1,0,0), (0,1,0), (0,0,1)\}\ is\ an\ example\ of\ a\ basis\ of\ R^3$

Theorem

Let E be a vector space ***of dimension n***, then:

- If $\{v_1, v_2, \dots, v_n\}\ is\ a\ basis\ of\ E \Leftrightarrow$

  $$\{v_1, v_2, \dots, v_n\}\ is\ a\ generator\ and\ is\ linearely\ independent$$

- If $\{v_1, v_2, \dots, v_p\}$ are p vectors in E with $p > n$, then $\{v_1, v_2, \dots, v_p\}$ can not be linearly independent. Furthermore, if $\{v_1, v_2, \dots, v_p\}$ is a generating set, then there exist n vectors of p $\{v_1, v_2, \dots, v_n\}$ that forms a basis of E
- If F is a subspace of E then $\dim F \le \dim E$. furthermore, If $\dim F = \dim E$ then $F = E$

Exercise 6.7

On R^3, consider the set E defined as: $E = \{(x, y, z) \in R^3 : \boldsymbol{x + y + z = 0}\}$

1. Show that E is a subspace of R^3
2. Find a basis of E

Solution 6.7

1. E is a vector space $\Leftrightarrow \begin{cases} E \neq \emptyset \\ \forall u, v \in E, \forall \alpha, \beta \ \leftrightarrow\ \alpha u + \beta v \in E \end{cases}$

$0_{R^3} = (0,0,0) \in E\ because\ 0 + 0 + 0 = 0\ so\ E \neq \emptyset$

$let\ u, v \in E\ and\ \alpha, \beta \in R$

$we\ have\ u \in E\ \leftrightarrow u = (x, y, z)\ such\ that\ \ x + y + z = 0$

$and, \quad v \in E\ \leftrightarrow (x', y', z')\ such\ that: x' + y' + z' = 0$

$so, \alpha u + \beta v = \alpha(x, y, z) + \beta(x', y', z')$

$$= (\alpha x, \alpha y, \alpha z) + (\beta x', \beta y', \beta z')$$

$$= (\alpha x+\beta x', \alpha y+\beta y', \alpha z+\beta z')$$

$$\alpha u+\beta v = \left(\underbrace{\alpha x+\beta x'}_{\text{المركبة الاولى}}, \underbrace{\alpha y+\beta y'}_{\text{المركبة الثانية}}, \underbrace{\alpha z+\beta z'}_{\text{المركبة الثالثة}} \right)$$

هل $\alpha u+\beta v$ ينتمي الى E. E تضم الاشعة الثلاثية التي مجموع مركباتها معدومة.

اذا, اذا كان $\alpha u+\beta v$ ينتمي فان مجموع مركباته يساوي الصفر:

$$(\alpha x+\beta x')+(\alpha y+\beta y')+(\alpha z+\beta z') = (\alpha x+\alpha y+\alpha z)+(\beta x'+\beta y'+\beta z')$$

$$= \alpha \underbrace{(x+y+z)}_{0} + \beta \left(\underbrace{x'+y'+z'}_{0} \right) = 0$$

اذن $\alpha u+\beta v \in E$

ومنه E هو فضاء شعاعي جزئي

2. A basis of E البحث عن قاعدة ل

$$E = \{(x,y,z) \in R^3 : \boldsymbol{x+y+z=0}\} = \{(x,y,z) \in R^3 : \boldsymbol{z=-x-y}\}$$
$$= \{(x,y,-x-y)\,;x,y \in R\} = \{(x,0,-x)+(0,y,-y)\,;x,y \in R\}$$
$$= \left\{ x \underbrace{(1,0,-1)}_{v_1} + y \underbrace{(0,1,-1)}_{v_2} ; x,y \in R \right\}$$

So $(1,0,-1) and\ (0,1,-1)$ is a generating set of E. we need to show that they are linearly independent.

$(1,0,-1) and\ (0,1,-1)$ هي مجموعة مولدة. لتكون قاعدة يجب ان تكون مستقلة خطيا فيما بينها

Let $\alpha_1, \alpha_2 \in R$: $\alpha_1\,(1,0,-1)+\alpha_2(0,1,-1) = 0_E$

$$\Rightarrow (\alpha_1, 0, -\alpha_1)+(0,\alpha_2,-\alpha_2) = (0,0,0)$$

$$\Rightarrow (\alpha_1, \alpha_2, -\alpha_2-\alpha_2) = (0,0,0)$$

$\Rightarrow \begin{cases} \alpha_1 = 0 \\ \alpha_2 = 0 \end{cases} \Rightarrow (1,0,-1) and\ (0,1,-1)\ are$ linearly independent

$$so\ (1,0,-1) and\ (0,1,-1)\ is\ a\ basis\ of\ E$$

6.4 Linear Maps التطبيقات الخطّية

Definition Let E and F be two R −vector spaces. A linear map f from E into F is a relation that assigns to each vector u in E a unique vector $f(u)$ in F:

$$f: E \to F$$

$$x \to f(x)$$

Such that:

a. $\forall u_1, u_2 \in E : f(u_1 + u_2) = f(u_1) + f(u_2)$.
b. $\forall u \in E, \forall \alpha \in \mathbb{R} : f(\alpha u) = \alpha f(u)$.

Or, $\forall u_1, u_2 \in E, \forall \alpha, \beta \in \mathbb{R} : f(\alpha x_1 + \beta u_2) = \alpha f(u_1) + \beta f(u_2)$.

Example

The map $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined as $f(x, y) = (x^2, x + y, 1)$ is not linear.

We can easily find vectors $u_1, u_2 \in E$ for which the condition:

$\forall u_1, u_2 \in E, \forall \alpha, \beta \in \mathbb{R} : f(\alpha x_1 + \beta u_2) = \alpha f(u_1) + \beta f(u_2)$. is false.

We take $u_1 = (1,0) and\ u_2 = (0,0)$

We have: $f(u_1 + u_2) = f\big((1,0) + (0,0)\big) = f(1,0) = (1,1,1)$

On the other hand, we have: $f(u_1) = f(1,0) = (1,1,1)\ and\ f(u_2) = f(0,0) = (0,0,1)$

$$so\ f(1,0) + f(0,0) = (1,1,1) + (0,0,1) = (1,1,2)$$

$$(1,1,1) \neq (1,1,2)$$

$$Hence, f((1,0) + (0,0))\ f(1,0) + f(0,0)$$

So f is not linear map.

Exercise 6.8

$f : R^2 \to R^3$ defined as $f(x, y) = (3x - y, 0, 2y)$ is linear map.

a. $Let\ u_1(x_1, y_1), u_2(x_2, y_2) \in \mathbb{R}^2$:

$$f\big((x_1, y_1) + (x_2, y_2)\big) = f(x_1 + x_2, y_1 + y_2)$$
$$= (3(x_1 + x_2) - (y_1 + y_2), 0, 2(y_1 + y_2))$$
$$= (3x_1 - y_1, 0, 2y_1) + (3x_2 - y_2, 0, 2y_2)$$
$$= f(x_1, y_1) + f(x_2, y_2)$$
$$\Rightarrow f\big((x_1, y_1) + (x_2, y_2)\big) = f(x_1, y_1) + f(x_2, y_2).$$

b. Let $\alpha \in \mathbb{R}\ f and\ u(x, y) \in \mathbb{R}$

$$f(\alpha.u) = f(\alpha.(x, y)) = f(\alpha x, \alpha y) = (3(\alpha.x) - (\alpha y), 0, 2(\alpha y))$$
$$= \alpha.(3x - y, 0, 2y) = \alpha f(x, y)$$

$$from\ a\ and\ b\ f\ is\ a\ linear\ map$$

Properties

Here are some simple properties of linear maps $f : E \to F$

- $f(0_E) = 0_W$ $\left(example: if\ f\ R^2 \to R^3\ we\ have\ (0,0) = (0,0,0)\right)$
- $f(-x) = -f(x)$
- If V is a subspace of E , then $f(V)$ is a subspace of F.
- If W is a subspace of F, then $f^{-1}(W)$ is a subspace of E.
- The composite map of two linear maps is a linear map.

6.4.1 Kernel, image, and rank of a linear map

Definition Let E and F be two R −vector spaces and let f be a linear map from $E\ into\ F$

- The set $f(E)$ is called the image of the linear map f and is denoted Imf.

نسمي المجموعة $f(E)$ صورة التطبيق الخطي وتضم $f(u)$ بحيث $u \in E$:

that is to say: $Imf = \{f(u) : u \in E\}$.

- The set of all $u \in E$ such that $f(u) = 0_F$ is called the kernel of f and is denoted $ker\ f$.

العناصر u التي صورها معدومة: $f(u) = 0_E$ تسمى نواة التطبيق الخطي

that is to say: $Kerf = \{u \in E : f(u) = 0_F\}$.

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Properties. Let $f : E \to F$ be a linear map.

- The kernel of f: $Kerf$ is a subspace of E.
- The image of f: Imf is a subspace of F.

To be continued …