Exercise 6.1

On $R-\left\{\frac{1}{2}\right\}$ we define the operation $(*)$ such as for $x,y\in R-\left\{\frac{1}{2}\right\}: x*y=x+y-2xy$

1- Is $(*)$ an internal composition law?
2- Is it: commutative, associative?
3- What is the identity element for * on $R-\left\{\frac{1}{2}\right\}$?

Solution 6.1

1. $(*)$ is an internal composition law $\Leftrightarrow \forall x,y\in E: x*y\in E$

Let $x,y\in R-\left\{\frac{1}{2}\right\}$, we check if $x*y\in R-\left\{\frac{1}{2}\right\}$?

راينا سابقا ان $P\Rightarrow Q$ تكافئ $\bar{P}\Rightarrow\bar{Q}$ اذن:

$$\left(x,y\in R-\left\{\frac{1}{2}\right\}\Rightarrow x*y\in R-\left\{\frac{1}{2}\right\}\right)\Leftrightarrow\left(x*y=\frac{1}{2}\Rightarrow x=\frac{1}{2}\ or\ y=\frac{1}{2}\right)$$

$$\begin{aligned}
x*y=\frac{1}{2} &\Leftrightarrow x+y-2xy=\frac{1}{2}\\
&\Leftrightarrow x(1-2y)-\frac{1}{2}(1-2y)=0\\
&\Leftrightarrow (1-2y)\left(x-\frac{1}{2}\right)=0\\
&\Leftrightarrow \left(y-\frac{1}{2}\right)\left(x-\frac{1}{2}\right)=0\\
&\Leftrightarrow y=\frac{1}{2}\ ou\ x=\frac{1}{2}.
\end{aligned}$$

So, $(*)$ is an internal composition law.

2. Commutative:

$x,y\in R-\left\{\frac{1}{2}\right\}: x*y=x+y-2xy=y+x-2yx=y*x \Leftrightarrow (*)$ is commutative

Assosiative

$$\begin{aligned}
(x*y)*z &= (x+y-2xy)*z=(x+y-2xy)+z-2(x+y-2xy)z\\
&= x+y+z-2xy-2xz-2yz+4xyz\\
&= x+(y+z-2yz)-2x(y+z-2yz)\\
&= x+(y*z)-2x(y*z)=x*(y*z),
\end{aligned}$$

The operation * is associative

3. The identity element:

Let $e \in R - \left\{\frac{1}{2}\right\}$. $e\ is\ the\ identity\ element\ for * \Leftrightarrow\ \forall x \in E : x * e = e * x = x$

$$x + e - 2xe = e + x - 2ex = x \Leftrightarrow e(1 - 2x) = 0 \Leftrightarrow e = 0$$

So, the identity elemnt for * is 0

Exercise 6.2

Show that $F = \{((0, y, z); y, z \in \mathbb{R}\}$ is a subspace of the vector space R^3

Solution 6.2

To show that $F = \{((0, y, z); y, z \in \mathbb{R}\}$ is a subspace of the vector space R^3 we need to show that:

1- F is non empty $\leftrightarrow$ $the\ vector\ \ is\ in\ F(0_F \in F)$:

$$0_F = (\mathbf{0}, 0,0)\ so\ 0_F \in F$$

Explanation F is the set of vectors with tree composants : triplets (x,y,z)

F هي مجموعة الاشعة (الثلاثيات$((x, y, z)$ ذات الاحداثية الأولى المعدومة : $(0, x, y)$

بما ان احداثية الشعاع المعدوم 0_F الأولى هي $(x = 0)$فان الصفر ينتمي الى المجموعة F ومنه المجموعة Fهي مجموعة غير خالية $F \neq \emptyset$

2- $\forall u, v \in F, \forall \alpha, \beta\ \leftrightarrow\ \alpha u + \beta v \in F$:

We need to prove that $\forall u, v \in F\ and\ \forall \alpha, \beta \mathbb{R}\ the\ vector\ \alpha u + \beta v$ is in F

$For\ \boldsymbol{u}\ and\ \boldsymbol{v}\ two\ vectors\ in\ F\ we\ have : u = (0, y_1, z_1), v = (0, y_2, z_2)$

يجب ان نثبت انه من اجل كل شعاعين v و u في المجموعة F ومن اجل كل عددين حقيقيين β و α : الشعاع $\alpha u + \beta v$ أيضا محتوى في F

Let $\boldsymbol{u} = (\mathbf{0}, \boldsymbol{y_1}, \boldsymbol{z_1}) \in \boldsymbol{F}$, $\boldsymbol{v} = (\mathbf{0}, \boldsymbol{y_2}, \boldsymbol{z_2}) \in \boldsymbol{F}$, and $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}$ Then:

$$\forall u, v \in F, \forall \alpha, \beta\ \leftrightarrow\ \alpha u + \beta v = \alpha(0, y_1, z_1) + (0, y_2, z_2)$$

$$= (\alpha 0, \alpha y_1, \alpha z_1) + (\beta 0, \beta y_2, \beta z_2)$$

$$= (0, \alpha y_1, \alpha z_1) + (0, \beta y_2, \beta z_2)$$

$$= (0, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2)$$

$$\leftrightarrow (0, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2) \in F$$

اذن $\alpha u + \beta v$ ينتمي الى F (الاحداثية الأولى للشعاع $\alpha u + \beta v$ معدومة)

Hence, $F = \{((0, y, z); y, z \in \mathbb{R}\}$ is a vector subspace of the vector space R^3

ومنه نستنتج ان $F = \{((0, y, z); y, z \in \mathbb{R}\}$ هو فضاء شعاعي جزئي من الفضاء الشعاعي R^3

Exercise 6.3

Express $u = (-2,3)$ *in* $\mathbb{R}^2$ *as a linear combination of the vectors* $v_1 = (1,1)$ *and* $v_2 = (1,2)$

Solution 6.3

Let α_1, α_2 be scalars such that: $u = \alpha_1 v_1 + \alpha_2 v_2$

$u = \alpha_1(1,1) + \alpha_2(1,2) = (\alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2)$

$(-2,3) = (\alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2)$

$\begin{cases} -2 = \alpha_1 + \alpha_2 \dots (1) \\ 3 = \alpha_1 + 2\alpha_2 \dots (2) \end{cases} \rightarrow \begin{cases} \alpha_2 = 5 \\ \alpha_1 = -7 \end{cases}$

$$Hence, u = -7v_1 + 5v_2$$

Exercise 6.4

1. Show that the set $V = \{(-1, 0), (2, 1)\}$ is linearly independent.
2. Show that the set V = {(1, 0),(−2, 0)} is linearly dependent.

Solution 6.4

1. $Let\ \alpha_1, \alpha_2 \in R: \alpha_1 (-1, 0) + \alpha_2 (2, 1) = (0, 0)$

$\Rightarrow (-\alpha_1 + 2\alpha_2, \alpha_2) = (0, 0)$
$\Rightarrow -\alpha_1 + 2\alpha_2 = 0\ and\ \alpha_2 = 0$
$\Rightarrow \alpha_1 = \alpha_2 = 0$
$Let\ \alpha_1, \alpha_2 \in R$

2. $\alpha_1 (1, 0) + \alpha_2 (-2, 0) = (0, 0)$

$\Rightarrow (\alpha_1 - 2\alpha_2, 0) = (0, 0)$

$\Rightarrow \alpha_1 - 2\alpha_2 = 0 \Rightarrow \alpha_1 = 2\alpha_2.$

$\Rightarrow \exists \alpha_1 = 2 \neq 0\ and\ \alpha_2 = 1 \neq 0\ such\ that\ 2\,(1,0) + 1(-2,0) = (0,0)$

Exercise 6.5

Consider the set $S = \{(1,1,1), (2,2,0), (3,0,0)\}$

1. Is S a system of generators of the vector space $\mathbb{R}^3$?
2. Write $v = (3,4,2)$ as a linear combination of S

Solution 6.5

1. S is a system of generators of the vector space $\mathbb{R}^3$

$\Leftrightarrow \forall u \in R^3 \ \exists \alpha_1, \alpha_2, \alpha_3 \in R \text{ such that} : u = \alpha_1(1,1,1) + \alpha_2(2,2,0) + \alpha_3(3,0,0)$

$$u \in R^3 \Leftrightarrow u = (x, y, z)$$

$$so, (x, y, z) = \alpha_1(1,1,1) + \alpha_2(2,2,0) + \alpha_3(3,0,0)$$

$$(x, y, z) = (\alpha_1, \alpha_1, \alpha_1) + (2\alpha_2, 2\alpha_2, 0) + (3\alpha_3, 0,0)$$

$$(x, y, z) = (\alpha_1 + 2\alpha_2 + 3\alpha_3 , \alpha_1 + 2\alpha_2 , \alpha_1)$$

$$\begin{cases} x = \alpha_1 + 2\alpha_2 + 3\alpha_3 \\ y = \alpha_1 + 2\alpha_2 \\ z = \alpha_1 \end{cases} \Leftrightarrow \begin{cases} \alpha_1 = z \\ \alpha_2 = \dfrac{y - z}{2} \\ \alpha_3 = \dfrac{x - y}{3} \end{cases}$$

Hence, all $u \in R^3 : u = z(1,1,1) + \frac{y-z}{2}(2,2,0) + \frac{x-y}{3}(3,0,0)$

2. $(3,4,2) = 2\,(1,1,1) + \frac{4-2}{2}(2,2,0) + \frac{3-4}{3}(3,0,0) = 2\,(1,1,1) + 1\,(2,2,0) - \frac{1}{3}(3,0,0)$

Exercise 6.6

Consider the set $V = \{(1,0), (1, -1)\}$

نعتبر المجموعة $V = \{(1,0), (1, -1)\}$

Is V a basis of $\mathbb{R}^2$

هل V قاعدة للفضاء $\mathbb{R}^2$

Solution 6.6

$V = \{(1,0), (1, -1)\}$ is a basis of $\mathbb{R}^2 \Leftrightarrow \begin{cases} 1. V \text{ is linearly independent} \\ \text{and} \\ 2. V \text{ is a generator set} \end{cases}$

1. *V is linearly independent means that*:

   $\forall \alpha_1, \alpha_2 \in \mathbb{R}\text{:}\ \alpha_1(1,0) + \alpha_2(1, -1) = 0 \Leftrightarrow \alpha_1 = 0 \text{ and } \alpha_2 = 0$

Let $\alpha_1, \alpha_2 \in \mathbb{R}$

$$\forall \alpha_1, \alpha_2 \in \mathbb{R}: \alpha_1(1,0) + \alpha_2(1, -1) = 0_{R^2} \Leftrightarrow \alpha_1(1,0) + \alpha_2(1, -1) = (0,0)$$

$$\Leftrightarrow (\alpha_1, 0) + \alpha_2(\alpha_2, -\alpha_2) = (0,0)$$

$$\Leftrightarrow (\alpha_1 + \alpha_2, -\alpha_2) = 0$$

$$\Rightarrow \begin{cases} \alpha_1 + \alpha_2 = 0 \\ -\alpha_2 = 0 \end{cases} \Rightarrow \begin{cases} \alpha_1 = 0 \\ \alpha_2 = 0 \end{cases}$$

So $\{(1,0),(1,-1)\}$ are linearly independent.

2. V is a generator set of $\mathbb{R}^2$ means that:

$$\forall\, u(x,y) \in \mathbb{R}^2\ \exists\, \alpha_1, \alpha_2 \in \mathbb{R}\ such\ that: u = \alpha_1(1,0) + \alpha_2(1,-1)$$

Let $\alpha_1, \alpha_2 \in \mathbb{R} : u = \alpha_1(1,0) + \alpha_2(1,-1) \Leftrightarrow (x,y) = (\alpha_1 + \alpha_2, -\alpha_2)$

Exercise 6.7

On R^3, consider the set E defined as: $E = \{(x,y,z) \in R^3 : \boldsymbol{x + y + z = 0}\}$

1. Show that E is a subspace of R^3
2. Find a basis of E

Solution 6.7

1. E is a vector space $\Leftrightarrow \begin{cases} E \neq \emptyset \\ \forall u, v \in E, \forall \alpha, \beta \leftrightarrow \alpha u + \beta v \in E \end{cases}$

$0_{R^3} = (0,0,0) \in E\ because\ 0 + 0 + 0 = 0\ so\ E \neq \emptyset$

$let\ u, v \in E\ and\ \alpha, \beta \in R$

$we\ have\ u \in E \leftrightarrow u = (x,y,z)\ such\ that\ x + y + z = 0$

$and, \quad v \in E \leftrightarrow (x', y', z')\ such\ that: x' + y' + z' = 0$

$$so, \alpha u + \beta v = \alpha(x,y,z) + \beta(x',y',z')$$
$$= (\alpha x, \alpha y, \alpha z) + (\beta x', \beta y', \beta z')$$
$$= (\alpha x + \beta x', \alpha y + \beta y', \alpha z + \beta z')$$

$$\alpha u + \beta v = \left(\underbrace{\alpha x + \beta x'}_{\text{المركبة الاولى}}, \underbrace{\alpha y + \beta y'}_{\text{المركبة الثانية}}, \underbrace{\alpha z + \beta z'}_{\text{المركبة الثالثة}} \right)$$

هل $\alpha u + \beta v$ ينتمي الى E. E تضم الاشعة الثلاثية التي مجموع مركباتها معدومة.

اذا, اذا كان $\alpha u + \beta v$ ينتمي فان مجموع مركباته يساوي الصفر:

$$(\alpha x + \beta x') + (\alpha y + \beta y') + (\alpha z + \beta z') = (\alpha x + \alpha y + \alpha z) + (\beta x' + \beta y' + \beta z')$$
$$= \alpha \underbrace{(x + y + z)}_{0} + \beta \left(\underbrace{x' + y' + z'}_{0} \right) = 0$$

اذن $\alpha u + \beta v \in E$

ومنه E هو فضاء شعاعي جزئي

2. A basis of E البحث عن قاعدة ل

$$E = \{(x,y,z) \in R^3 : \boldsymbol{x+y+z=0}\} = \{(x,y,z) \in R^3 : \boldsymbol{z = -x-y}\}$$
$$= \{(x,y,-x-y)\ ; x,y \in R\} = \{(x,0,-x) + (0,y,-y)\ ; x,y \in R\}$$
$$= \left\{x\underbrace{(1,0,-1)}_{v_1} + y\ \underbrace{(0,1,-1)}_{v_2} ; x,y \in R\right\}$$

So $(1,0,-1) and\ (0,1,-1)$ is a generating set of E. we need to show that they are linearly independent.

$(1,0,-1) and\ (0,1,-1)$ هي مجموعة مولدة. لتكون قاعدة يجب ان تكون مستقلة خطيا فيما بينها

Let $\alpha_1, \alpha_2 \in R$: $\alpha_1\ (1,0,-1) + \alpha_2(0,1,-1) = 0_E$

$$\Rightarrow (\alpha_1, 0, -\alpha_1) + (0, \alpha_2, -\alpha_2) = (0,0,0)$$

$$\Rightarrow (\alpha_1, \alpha_2, -\alpha_2 - \alpha_2) = (0,0,0)$$

$\Rightarrow \begin{cases} \alpha_1 = 0 \\ \alpha_2 = 0 \end{cases} \Rightarrow (1,0,-1) and\ (0,1,-1)\ are$ linearly independent

$$so\ (1,0,-1) and\ (0,1,-1)\ is\ a\ basis\ of\ E$$

Extra Exercise

Consider the set $R_2\ [x]$ defined as follow:

$$R_2\ [x] = \{P\ (x) = a + bx + cx^2\ ;\ a,b,c \in R\ \}$$

$R_2\ [x]$ is the set of polynomials of degree ≤ 2 with real coefficients.

Show that $V = \{p_1\ (x) = 1, p_2\ (x) = x, p_3\ (x) = x^2\}$ is a basis of $R_2[x]$.

Solution

a. Let α, β, γ ∈ R, then:

$$\forall x \in R : \alpha_1 p_1\ (x) + \alpha_2 p_2\ (x) + \alpha_3 p_3\ (x) = 0 \Leftrightarrow \forall x \in R : \alpha + \beta x + \gamma x^2 = 0.$$

which gives $\alpha = \beta = \gamma = 0, so$ $\{1, x, x^2\}$ ***is linearly independent.***

b. Let $P \in R_2\ [x]$, then there exists $a, b, c \in R$, such that:

$$\forall x \in R : P\ (x) = a + bx + cx^2 = ap_1\ (x) + bp_2\ (x) + cp_3\ (x),$$
$$P = ap_1 + bp_2 + cp_3$$

So V is a ***generating system of*** $\boldsymbol{R_2}$

Then, V is a basis of R_2