Chapter IV

Elementary Functions

Contents

4 Elementary Functions

4.1 Logarithm function

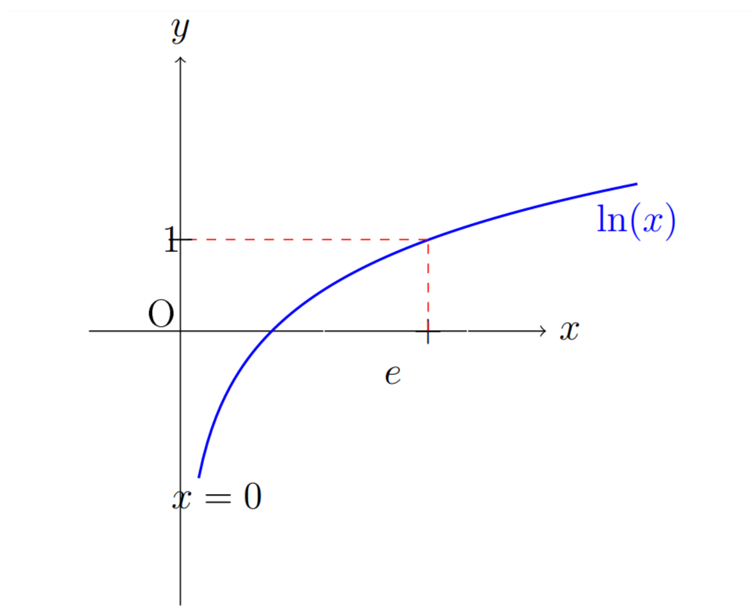
Definition 4.1.1 The natural logarithm function, denoted as $ln(x)$, is a specific type of logarithmic function with a base equal to the mathematical constant "e".

- The natural logarithm $ln(x)$ is the primitive of the reciprocal function $\frac{1}{x}$
- The natural logarithm $ln(x)$

$$ln: \quad \mathbb{R}_+^* \longrightarrow \mathbb{R}$$
$$x \longrightarrow \ln x$$

- ➢ The function $ln\ (x)$ is continuous, strictly increasing and defines a bijection on R^{*+}.
- ➢ $ln\ (x) is\ diferrentiable\ on\ R^{*+}\ with\ \frac{dln(x)}{dx} = \frac{1}{x}$
- ➢ Its reciprocal function is e^x

x	$ln\, x$
$\to 0^+$	$\to -\infty$
1	0
e	1
$\to +\infty$	$\to +\infty$



4.1.1 Properties of logarithms

Let $a > 0$ and $b > 0$ be strictly positive real numbers, and $\alpha \in \mathbb{R}$:

- $ln\ (a.b) = ln\ (a) + ln\ (b)$
- $ln\ \left(\frac{a}{b}\right) = ln\ (a) - ln\ (b)$
- $\ln\left(\frac{1}{a}\right) = -\ln(a)$
- $\ln(a^\alpha) = \alpha \ln(a)$
- $\ln(\sqrt{x}) = \frac{1}{2}\ln x$

4.1.2 Equations and inequalities with logarithms

- $ln\,(a) = ln\,(b) \Leftrightarrow a = b$
- $ln\,(a) \geq ln\,(b) \Leftrightarrow a \geq b$
- $ln\,(a) < ln\,(b) \Leftrightarrow a < b$
- $\ln(a) \leq 0 \Leftrightarrow 0 < a \leq 1$
- $\ln(a) > 0 \Leftrightarrow a > 1.$

4.1.3 Special limits

$$\lim_{+\infty} \ln x = +\infty \qquad \lim_{x\to 0^+} \ln x = -\infty$$

$$\lim_{+\infty} \frac{\ln x}{x} = 0 \qquad \lim_{x\to 0^+} x\, lnx = 0$$

4.2 Exponential function

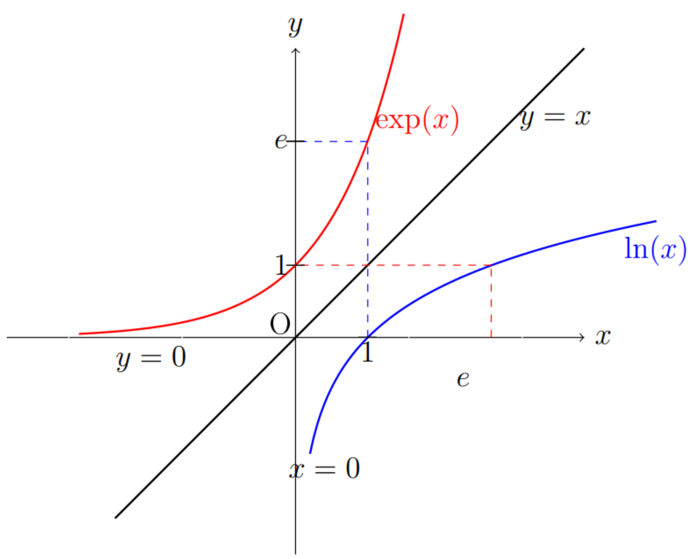
Definition 4.2.1 The exponential is the inverse function of $\ln\,(x)$, it is defined as follows:

$$exp:\ \mathbb{R} \longrightarrow \mathbb{R}_+^*$$
$$x \longrightarrow e^x$$

➢ The function $exp:\ R \rightarrow R_+^*$ is:

- Continuous.
- Strictly increasing.
- Positive.
- Differentiable, with: $\frac{de^x}{dx} = e^x$.

➢ Its reciprocal function is $ln\,(x)$.



4.2.1 Properties of exponentials

Let a, b and n be real numbers:

- $e^a \times e^b = e^{a+b}$
- $\frac{1}{e^a} = e^{-a}$
- $\frac{e^a}{e^b} = e^{a-b}$
- $(e^a)^n = e^{na}$

4.2.2 Equations and inequalities with logarithms

- $e^a = e^b \Leftrightarrow a = b$
- $e^a \geq e^b \Leftrightarrow a \geq b$
- $e^a < e^b \Leftrightarrow a < b$
- $e^a \geq b > 0 \Leftrightarrow a > \ln b$
- $e^a < b \Leftrightarrow a < \ln b \quad b > 0$

4.2.3 Link between exponential function and logarithm

- $\ln(e^a) = a.$
- $e^{\ln(a)} = a$.
- $e^x > 0 \; \forall \, x \in \mathbb{R}.$
- $e^a = b \Leftrightarrow a = \ln(b).$

4.2.4 Special limits

$$\lim_{+\infty} e^x = +\infty \qquad \lim_{x\to -\infty} e^x = 0 \qquad \lim_{x\to +\infty} \frac{e^x}{x} = +\infty$$

$$\lim_{+\infty} \frac{e^x}{x} = 0 \qquad \lim_{x\to+\infty} x\, e^x = +\infty \qquad \lim_{x\to +\infty} \frac{e^x}{x^n} = +\infty$$

Power functions

Definition 4.3.1 A power function is a function of the form $f(x) = ax^b$, where a is a given constant.

For example, $y = x, y = 2x^5, y = \sqrt[3]{x} = x^{\frac{1}{3}}, y = x^{2/3}$ are power functions.

In a power function $f(x) = ax^b$, the base x is a variable, and the exponent b is a constant.

The appearance of the graph of a power function depends on the constant a.

Power function	Degree	Name
$y = a$	0	Constant
$y = a.x$	1	Linear
$y = a.x^2$	2	Quadratic
$y = ax^3$	3	Cubic

- For $n = 0$ the function is constant.
- For n=1 the function is linear. It is increasing if $a > 0$ and decreasing if $a < 0$.
- If $n > 1$, then the overall shape of the graph of $y = ax^n$ is determined by the parity of n (whether n is even or odd) and the sign of a.
  - If <u>n is even</u>, the graph of the function is symmetric with respect to $y - axis$ $(x = 0)$.
  - If <u>n is odd</u>, the graph of the function is symmetric with respect to the origin, (0,0).

4.3 Trigonometric functions

4.3.1 Sine function

$$Sin: \quad \mathbb{R} \rightarrow [-1,1]$$
$$x \rightarrow \sin x$$

- Continuity: $Sin\ x$ is continuous on $\mathbb{R}$.
- Parity: $sin\ x$ is an odd function:

$$\forall x \in \mathbb{R} \sin(-x) = -\sin(x)$$

- $Sin\ x$ is a periodic function, its period is 2π

$$\forall x \in \mathbb{R} : \sin(x + 2\pi) = \sin(x)$$

- $Sin\ x$ is derivable on $\mathbb{R}$ with $(\sin x)' = \cos x$.

4.3.2 Cosine function

$$Cos: \quad \mathbb{R} \rightarrow [-1,1]$$
$$x \rightarrow \cos x$$

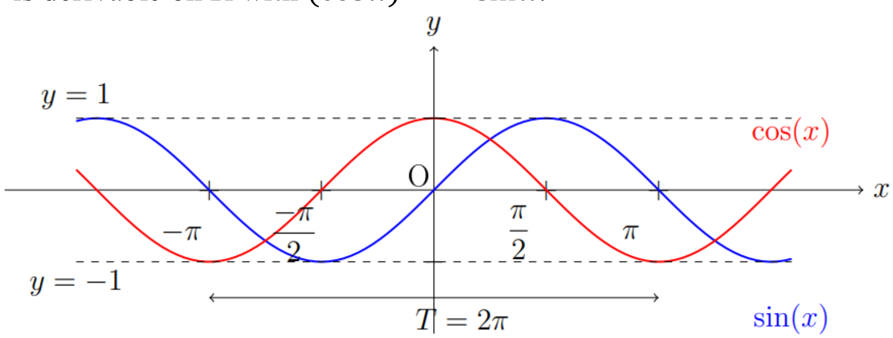
- Continuity: $Cos\ x$ is continuous on $\mathbb{R}$.
- Parity: $cos\ x$ is an even function.

$$\forall x \in \mathbb{R}, \cos(-x) = cos(x)$$

- $Cos\ x$ is a periodic function, its period is 2π.

$$Cos\ (x + 2\pi) = \text{Cos}\, x$$

- $Cos\ x$ is derivable on $\mathbb{R}$ with $(\cos x)' = -\sin x$.



Properties

The sine and cosine functions satisfy the following properties, for all $x \in R$,

- $cos^2(x) + sin^2(x) = 1$

- $cos^2\,(x) = \frac{1}{2}\,(1\, +\, cos\, 2x)$
- $sin^2\,(x)\, =\, \frac{1}{2}\,[1\, -\, cos\,(2x)]$
- $cos\,(2x) =\, cos^2(x) -\, sin^2\,(x)$
- $sin\,(2x)\, =\, 2\, cos\,(x)\, sin\,(x)$

Addition formulas $\forall x, y\, \in\, R$, we have

- $cos\,(x\, +\, y)\, =\, cos\,(x)\, cos\,(y)\, -\, sin\,(x)\, sin\,(y)$
- $cos\,(x\, -\, y)\, =\, cos\,(x)\, cos\,(y)\, +\, sin\,(x)\, sin\,(y)$
- $sin\,(x\, +\, y)\, =\, sin\,(x)\, cos\,(y)\, +\, cos\,(x)\, sin\,(y)$
- $sin\,(x\, -\, y)\, =\, sin\,(x)\, cos\,(y)\, -\, cos\,(x)\, sin\,(y)$

4.3.3 Tangent function

$$tan:\quad \mathbb{R} - \left\{\frac{\pi}{2} + k\pi, k \in \mathbb{Z}\right\} \;\rightarrow\; \mathbb{R}$$

$$x \;\rightarrow\; \tan x = \frac{\sin x}{\cos x}$$

- Continuity: $\tan x$ is continuous on $\mathbb{R} - \left\{\frac{\pi}{2} + k\pi, k \in \mathbb{Z}\right\}$
- Parity: $\tan x$ is an odd function:

$$\forall x \in \mathbb{R} - \left\{\frac{\pi}{2} + k\pi, k \in \mathbb{Z}\right\}, tan\,(-x) = -tan\,(x)$$

- $\tan x$ is a periodic function. Its period is π

$$tan\,(x + \pi) = \tan x$$

- $tan\, x$ is differentiable on $\mathbb{R} - \left\{\frac{\pi}{2} + k\pi, k \in \mathbb{Z}\right\}$.
- $(\tan x)' = 1 + \tan^2 x = \frac{1}{\cos^2 x}$

4.4 Inverse functions of the trigonometric functions

4.4.1 Arcsine function

The function:

$$Sin:\; \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow\; [-1{,}1]$$

$$x \rightarrow \sin x$$

Is bijective (because it is continuous and strictly increasing on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$). Its inverse function is $(sin\, x)^{-1} = arcsine$

$$arcsin: [-1,1] \rightarrow \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

- $Arcsin\ x$ is continuous and differentiable on : $[-1,1]$.
- $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$, $|x| < 1$.
- $(\arcsin x)' > 0$: $Arcsin\ x$ is strictly increasing.

4.4.2 Arccosine function

The function

$$Cos: \quad [0,\pi] \ \rightarrow \ [-1,1]$$

$$x \ \rightarrow \cos x$$

is bijective (because it is continuous and strictly decreasing on $[0,\pi]$. It admits an inverse function $(\cos x)^{-1} = \arccos x$

$$\arccos: \ [-1,1] \rightarrow \ \ [0,\pi]$$

$$x \ \ \rightarrow \ \arccos x$$

- $\arccos x$ is continuous and strictly decreasing on $[-1,1]$
- $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$. $|x| < 1$

4.4.3 Arctangentn function

The function

$$tan: \quad \left]-\frac{\pi}{2},\frac{\pi}{2}\right[\ \rightarrow \ \ \mathbb{R}$$

$$x \ \ \rightarrow \ \tan x = \frac{\sin x}{\cos x}$$

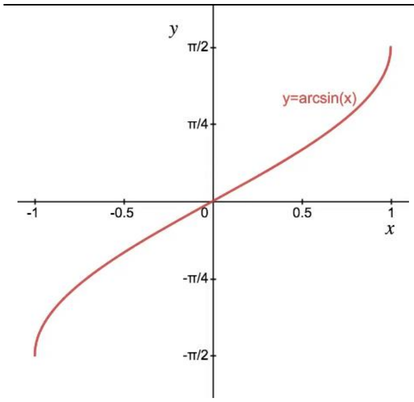
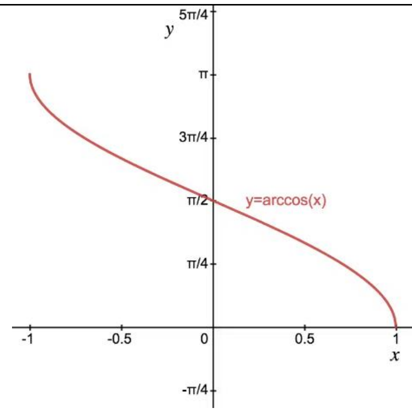
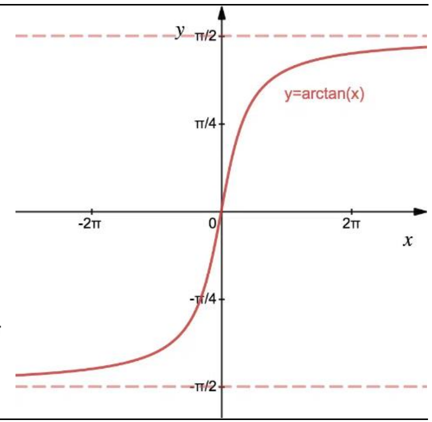
$$\forall\, x \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[: (\tan x)' = \frac{1}{\cos^2 x} > 0$$

So $\tan x$ is bijective on $\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$, so it admits an inverse function on this domain $(\tan x)^{-1} = \arctan x$

$$\arctan: \ \mathbb{R} \rightarrow \ \ \left]-\frac{\pi}{2},\frac{\pi}{2}\right[$$

$$x \ \ \rightarrow \arctan x$$

- $\arctan x$ is continuous and strictly increasing on $\mathbb{R}$
- $(\arctan)'(x) = \frac{1}{1+x^2}$

$\arcsin x$	$\arccos x$	$\arctan x$
		

4.5 Hyperbolic functions

4.5.1 Hyperbolic sine function

The hyperbolic sine function is defined by:

$$sinh\colon \mathbb{R} \rightarrow \ \mathbb{R}\ \mathrm{x} \rightarrow \sinh x = \frac{e^{x} - e^{-x}}{2}$$

4.5.2 Hyperbolic cosine function

The hyperbolic sine function is defined by:

$$cosh\colon \mathbb{R} \rightarrow \ [1, +\infty[$$

$$x \rightarrow cosh\, x = \frac{e^{x} + e^{-x}}{2}$$

4.5.3 Hyperbolic tangent function

The hyperbolic sine function is defined by:

$$cosh\colon \mathbb{R} \rightarrow \ ]-1, +1[$$

$$x \rightarrow tanh\, x = \frac{\sinh x}{\cosh x} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

4.6 Inverse functions of hyperbolic functions

4.6.1 Hyperbolic sine argument function

4.6.2 Hyperbolic cosine argument function

4.6.3 Hyperbolic tangent argument function