Ex(03)

Soit $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 4 & 1 & -2 \\ 2 & 1 & 2 & -1 \\ 1 & 2 & 1 & 0 \end{pmatrix}$

1) $\underline{Sp_A(\lambda)}$?

$$P_A(\lambda) = (1-\lambda) \begin{vmatrix} 4-\lambda & 1 & -2 \\ 1 & 2-\lambda & -1 \\ 2 & 1 & -\lambda \end{vmatrix} \xrightarrow{C_1 \leftarrow C_1 + C_3} (1-\lambda) \begin{vmatrix} 2-\lambda & 1 & -2 \\ 0 & 2-\lambda & -1 \\ 2-\lambda & 1 & -\lambda \end{vmatrix}$$

$$P_A(\lambda) = (1-\lambda)(2-\lambda) \begin{vmatrix} 1 & 1 & -2 \\ 0 & 2-\lambda & -1 \\ 1 & 1 & -\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)(2-\lambda)(2-\lambda).$$

Alors $Sp_A(\lambda) = \{ \overset{S}{1}, \overset{T}{2} \}$

2) $\underline{\text{Jordanisation}}$: 1) $E_2 = \left\{ (x,y,z,t) \in \mathbb{R}^4 \,/\, (A-2I)\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

Soit $\begin{cases} -x = 0 \\ -x + 2y + z - 2t = 0 \\ 2x + y - t = 0 \\ x + 2y + z - 2t = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = t \\ z = 0 \end{cases}$; $\boxed{E_2 = \langle (0,1,0,1) \rangle}$

Complétons la base :

* $(A-2I)\underbrace{\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}}_{\omega_1} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} -x = 0 \\ -x + 2y + z - 2t = 1 \\ 2x + y - t = 0 \\ x + 2y + z - 2t = 1 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = t \\ z = 1 \end{cases}$

Alors $\boxed{\omega_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}}$

* $(A-2I)\,\omega_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} x = 0 \\ -x + 2y + z - 2t = 1 \\ 2x + y - t = 1 \\ x + 2y + z - 2t = 1 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = t + 1 \\ z = -1 \end{cases}$

Alors $\boxed{\omega_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}}$

(2) $E_1 = \left\{ (x,y,z,t) \in \mathbb{R}^4 \,/\, (A-I)\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$.

Soit $\begin{cases} 0x = 0 \\ -x + 3y + z - 2t = 0 \\ 2x + y + z - t = 0 \\ x + 2y + z - t = 0 \end{cases} \Rightarrow \begin{cases} x - 3y - z + 2t = 0 \\ 7y + 3z - 5t = 0 \\ 5y + 2z - 3t = 0 \end{cases} \Rightarrow \begin{cases} x - 3y - z + 2t = 0 \\ 7y + 3z - 5t = 0 \\ z + 4t = 0 \end{cases}$

$\Rightarrow \begin{cases} x = -t \\ y = -t \\ z = 4t \end{cases}$ $\boxed{E_1 = \langle (-1,-1,4,1) \rangle}$

Donc : $P = \begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ 4 & 0 & 1 & -1 \end{pmatrix}$ et $J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

Ex(24) Soit $A_k = \begin{pmatrix} 0 & k & -k \\ -1 & k+1 & -k \\ 0 & 0 & 1 \end{pmatrix}$

1) $P_A(\lambda)$? $P_{A_k}(\lambda) = \begin{vmatrix} -\lambda & k & -k \\ -1 & k+1-\lambda & -k \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} -\lambda & k \\ -1 & k+1-\lambda \end{vmatrix}$

$P_{A_k}(\lambda) = (1-\lambda)\left(\lambda^2 - (k+1)\lambda + k\right) = (1-\lambda)^2(k-\lambda)$.

2) Si $k=1$ Alors $P_{A_1}(\lambda) = (1-\lambda)^3$.

Si A_1 est diagonalisable Alors $A_1 = PDP^{-1} = P\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}P^{-1}$
$= PP^{-1} = I_3$

Mais $A_1 \neq I_3$ contradiction.

Donc A_1 n'est pas diagonalisable.

3) Trigonalisation. $E_1 = \left\{ (x,y,z) \in \mathbb{R}^3 \,\middle|\, (A_1 - I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$.

Soit $\begin{cases} -x+y-z=0 \\ -x+y-z=0 \\ 0=0 \end{cases} \Rightarrow x = y - z$; $E_1 = \langle (1,1,0), (-1,0,1) \rangle$

Complétons la base de E_1 : $(A_1 - I)\,\omega = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Soit : $\begin{cases} -x+y-z=1 \\ -x+y-z=1 \end{cases} \Rightarrow x = y - z - 1$; $\omega = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$

Donc : $P = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ et $J = \begin{pmatrix} \boxed{1} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

3) A_1^n ? $A_1^n = (PJP^{-1})^n = PJ^nP^{-1}$

$J^n = (D+N)^n = \sum_{k=0}^{n} C_k^n D^k N^{n-k}$ $\left(N \cdot N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right)$

Donc $J^n = \sum_{0}^{1} C_k^n D^{n-k} N^{k} = C_0^n D^n N^0 + C_1^n D^{n-1} N = D^n + nD^{n-1}N$

$J^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$

Donc : $A_1^n = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} n+1 & -1 & n \\ 0 & 1 & 0 \\ -n & n & 1-n \end{pmatrix}$