CHAPTER 6

Elementary Functions

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Contents

1

Elementary Functions

In our calculus course, we are going to deal mostly with **elementary functions**. They are

- Power functions $\left(x^2,\ \sqrt{x},\ x^{\frac{1}{3}}, \cdots\right)$,
- Exponential functions $(2^x,\ e^x,\ \pi^x, \cdots)$,
- Logarithmic functions $(\ln x,\ \log_2 x, \cdots)$,
- Trigonometric functions $(\sin x,\ \cos x,\ \tan x, \cdots)$,
- Inverse trigonometric functions $(\arcsin x,\ \arccos x,\ \arctan x, \cdots)$,
- Hyperbolic functions $(chx,\ shx,\ thx, \cdots)$,

and their sums, differences, products, quotients, and compositions. For example

$$f(x) = \frac{\arcsin \sqrt{x^2 - 3}}{\ln(x^4 + 3) - \tan e^{\cos x}} \text{ is an elementary function.}$$

1.1 Power functions

1.1.1 *Review of exponents*

We start at the beginning. For a number a and a positive integer n,

$$a^n = \underbrace{a.a.a.\cdots.a}_{n\ times}.$$

1.1.2 *Basic laws of exponents*

$$a^1 = a, \quad (ab)^n = a^n b^n, \quad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n},$$
$$a^m a^n = a^{m+n}, \quad \frac{a^m}{a^n} = a^{m-n}, \quad (a^m)^n = a^{mn}.$$

1.1.3 *Definition of power functions*

Definition 1.1.1. *A power function is a function of the form $f(x) = x^a$, where a is a given constant. For example, $y = x$, $y = x^4$, $y = x^{\frac{2}{3}}$ are power functions.*
In a power function $f(x) = x^a$, the base x is a variable, and the exponent a is a constant.
The appearance of the graph of a power function depends on the constant a .

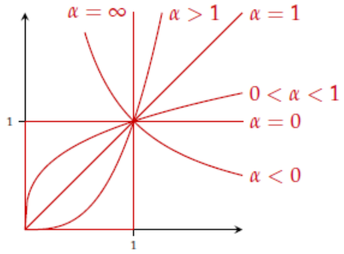

Figure 1.1: Power function with real exponents.

Definition 1.1.2. *(**Power functions** $y = x^n$)*
If n is an integer greater than 1, then the overall shape of the graph of $y = x^n$ is determined by the parity of n (whether n is even or odd).

- *If n is even, then the graph has a shape similar to the parabola* $y = x^2$.

- *If n is odd, then the graph has a shape similar to the cubic parabola* $y = x^3$.

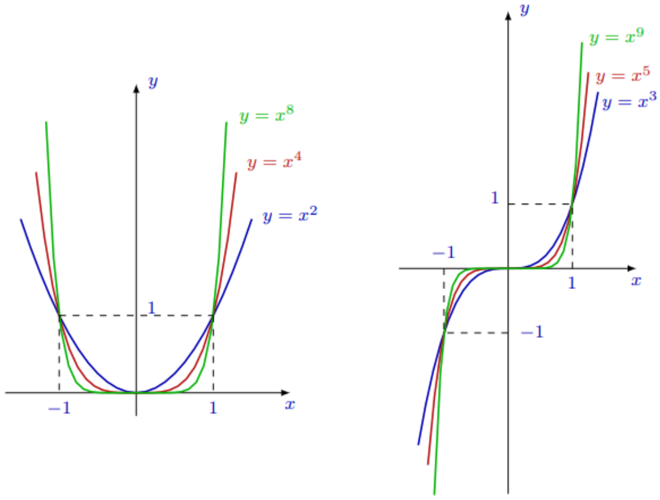

Figure 1.2: Power function with integer exponents.

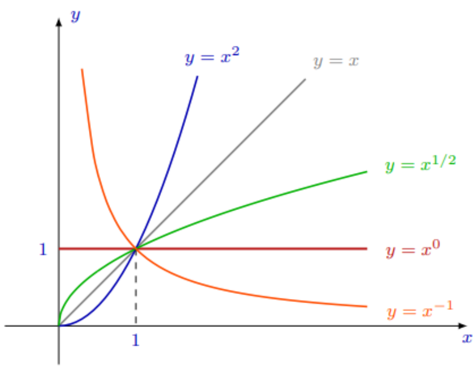

Figure 1.3: The graphs of $y = x^n$ for some rational n and $x > 0$.

1.2 Logarithm and Exponential Functions

1.2.1 *Logarithm*

Definition 1.2.1. *There exists a unique function, denoted* $\ln :]0, +\infty[\longrightarrow \mathbb{R}$ *such that:*

$$\ln'(x) = \frac{1}{x}, \qquad \textit{and} \qquad \ln(1) = 0.$$

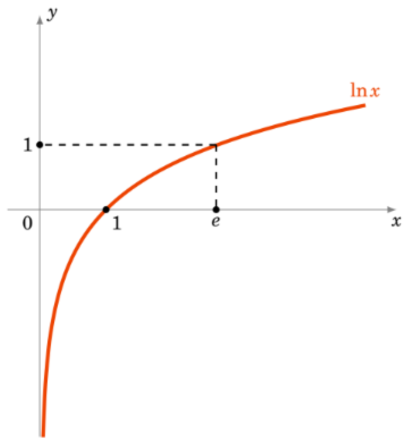

Figure 1.4: Logarithm function

Proposition 1.2.1. *The logarithm function satisfies the following properties: (for all* $a,\ b > 0$ *):*

1. $\ln(a \times b) = \ln a + \ln b$,
2. $\ln(\frac{1}{a}) = -\ln a$,
3. $\ln(a^n) = n \ln a$, *for all* $n \in \mathbb{N}$,
4. $\ln$ *is a continuous, strictly increasing function and defines a bijection of* $]0, +\infty[$ *on* $\mathbb{R}$.
5. $\lim\limits_{x \to +\infty} \frac{\ln(x)}{x} = 0,\ \ \lim\limits_{x \to 0^+} \frac{\ln(x+1)}{x} = 1,\ \ \lim\limits_{x \to +\infty} \frac{\ln(x)}{x^p} = 0,\ \ p \in \mathbb{R}_+^*$.

Remark 1.2.1. $\ln(x)$ *is called the natural logarithm or also the neperian logarithm. It is characterized by* $\ln(e) = 1$. *We define the logarithm to base a by*

$$\log_a = \frac{\ln x}{\ln a}.$$

So that $\log_a(a) = 1$.

1.2.2 *Exponential*

Definition 1.2.2. *The reciprocal bijection of* $\ln :]0, +\infty[\longrightarrow \mathbb{R}$ *is called the exponential function, denoted* $\exp : \mathbb{R} \longrightarrow]0, +\infty[$.

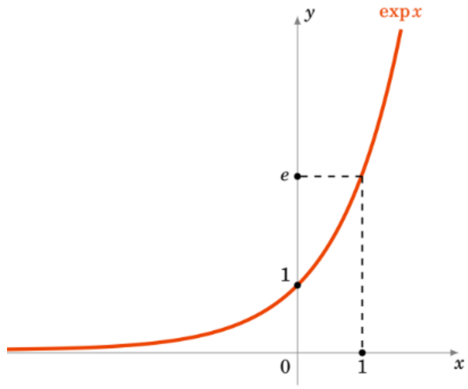

Figure 1.5: Exponential function

For $x \in \mathbb{R}$, e^x *is also noted for* $\exp x$.

Proposition 1.2.2. *The exponential function satisfies the following properties:*

1. $\exp(\ln x) = x, \ \forall x > 0$, *and* $\ln(\exp x) = x, \ \forall x \in \mathbb{R}$,

2. $\exp(a + b) = \exp(a) \times \exp(b)$,

3. $\exp(nx) = (\exp x)^n$,

4. $\exp : \mathbb{R} \longrightarrow]0, +\infty[$ *is a continuous, strictly increasing function satisfying* $\lim_{x\to-\infty} \exp x = 0$, *and* $\lim_{x\to+\infty} \exp x = +\infty$,

5. *The exponential function is differentiable and* $\exp' x = \exp x$, *for all* $x \in \mathbb{R}$

6. $\lim_{x\to+\infty} \frac{e^x}{x^a} = +\infty, \quad \lim_{x\to+\infty} \frac{\exp x}{x} = +\infty, \quad \lim_{x\to+\infty} \frac{x^a}{(\ln x)^b} = +\infty, \quad \lim_{x\to 0^+} x^a(\ln(x))^b = 0.$

1.3 Trigonometric Functions

1.3.1 *Sine function*

Definition 1.3.1. *The sine function* $y = \sin x$ *is defined as follows*

$$\begin{aligned} \sin : \quad \mathbb{R} &\longrightarrow [-1, 1] \\ x &\longrightarrow \sin x. \end{aligned}$$

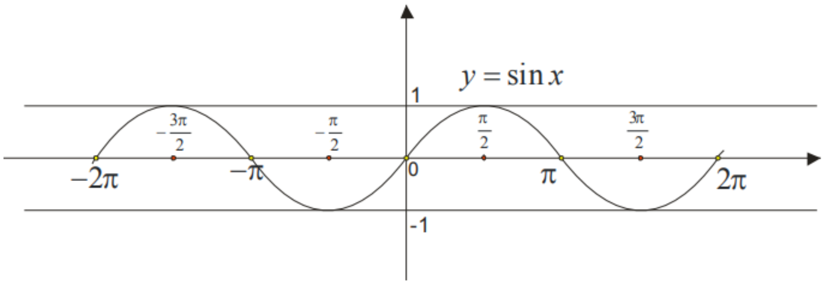

Figure 1.6: Sine function

1.3.2 *Cosine function*

Definition 1.3.2. *The cosine function* $y = \cos x$ *is defined as follows*

$$\begin{aligned} \cos : \quad \mathbb{R} &\longrightarrow [-1, 1] \\ x &\longrightarrow \cos x. \end{aligned}$$

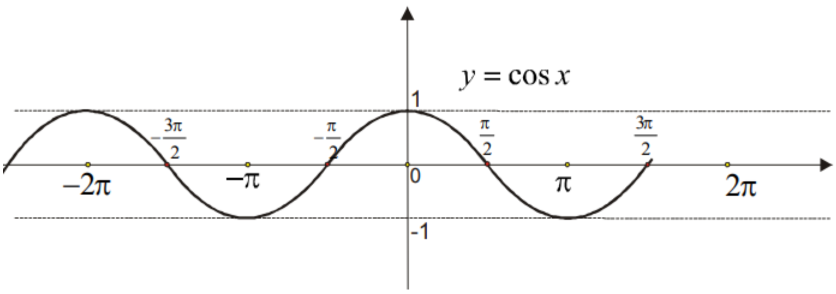

Figure 1.7: Cosine function

1.3.3 *Tangent function*

Definition 1.3.3. *The tangent function $y = \tan x$ is defined as follows*

$$\begin{array}{llll} \tan: & \mathbb{R}|\left\{\frac{\pi}{2}+k\pi\right\} & \longrightarrow \mathbb{R} & \\ & x & \longrightarrow \tan x = \dfrac{\sin x}{\cos x} & , k \in \mathbb{Z} \end{array}$$

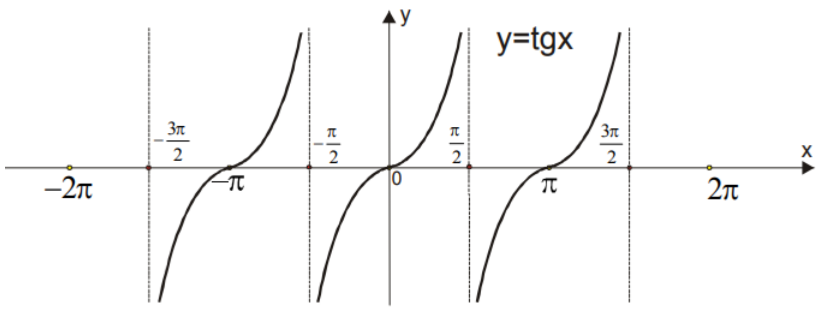

Figure 1.8: Tangent function

1.3.4 *Cotangent function*

Definition 1.3.4. *The cotangent function $y = \cot x$ is defined as follows*

$$\begin{array}{llll} \tan: & \mathbb{R}|\{k\pi\} & \longrightarrow \mathbb{R} & \\ & x & \longrightarrow \tan x = \dfrac{\cos x}{\sin x} & , k \in \mathbb{Z} \end{array}$$

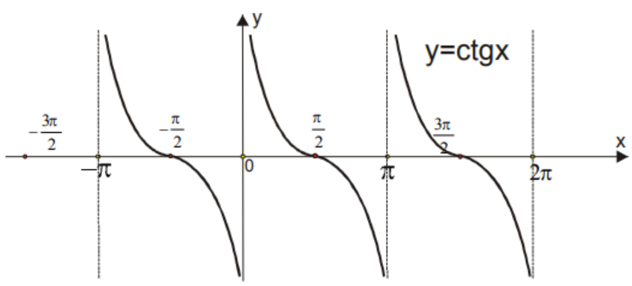

Figure 1.9: Cotangent function

1.4 Inverse Trigonometric Functions

1.4.1 *Arcsine*

Beware: the function $y = \sin x$ has no inverse function, because it is not a bijection. But if we look at its restriction on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ then: $f^{-1} : [-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and

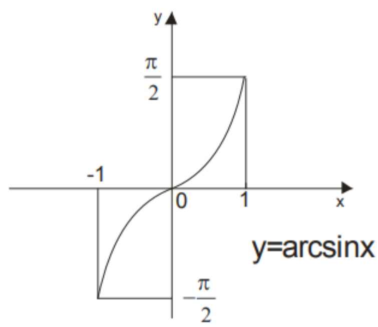

Figure 1.10: Arcsine function

Still remember that:

- $\arcsin(\sin x) = x, \quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$

- $\sin(\arcsin(x)) = x, \quad x \in [-1, 1].$

1.4.2 *Arccosine*

And here we will observe the same reason a restriction of $y = \cos x$ on the interval $[0, \pi]$, then $g^{-1} : [-1, 1] \longrightarrow [0, \pi]$ and

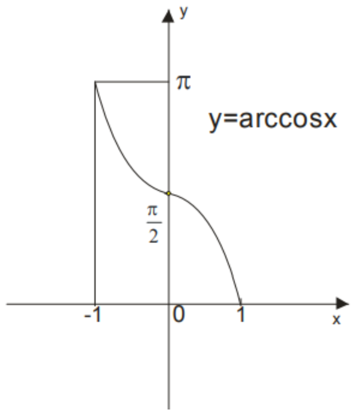

Figure 1.11: Arccosine function

Valid:

- $\arccos(\cos x) = x, \quad x \in [0, \pi].$
- $\cos(\arccos(x)) = x, \quad x \in [-1, 1].$

1.4.3 *Arctan*

Looking at the restriction of $y = \tan x$ on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and $h^{-1} : \mathbb{R} \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

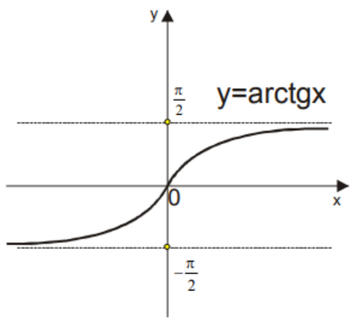

Figure 1.12: Arctan function

Valid:

- $\arctan(\tan x) = x, \quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
- $\tan(\arctan(x)) = x, \quad x \in \mathbb{R}$.

1.4.4 *Arcctan*

$k^{-1} : \mathbb{R} \longrightarrow [0, \pi]$

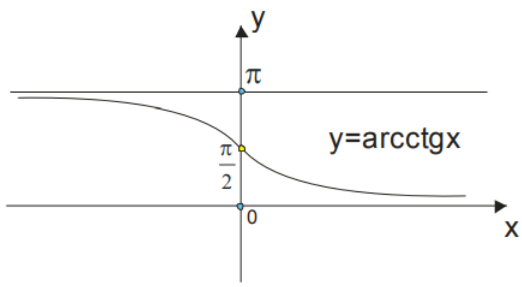

Figure 1.13: Arcctan function

Valid:

- $\text{arccot}(\cot x) = x, \quad x \in [0, \pi]$.
- $\cot(\text{arccot}(x)) = x, \quad x \in \mathbb{R}$.

1.5 Hyperbolic Functions

These are the functions:

- **hyperbolic sine**: $sh\ x = \frac{e^x - e^{-x}}{2}$,
- **hyperbolic cosine**: $ch\ x = \frac{e^x + e^{-x}}{2}$,
- **hyperbolic tangent**: $th\ x = \frac{sh\ x}{ch\ x}$,

- **hyperbolic cotangent**: $cth\ x = \dfrac{ch\ x}{sh\ x}$.

Graphs of these functions are obtained from graphics: $y = e^x$ and $y = e^{-x}$, $\left(y = \dfrac{1}{2}e^x,\ and\ y = \dfrac{1}{2}e^{-x}\right)$.

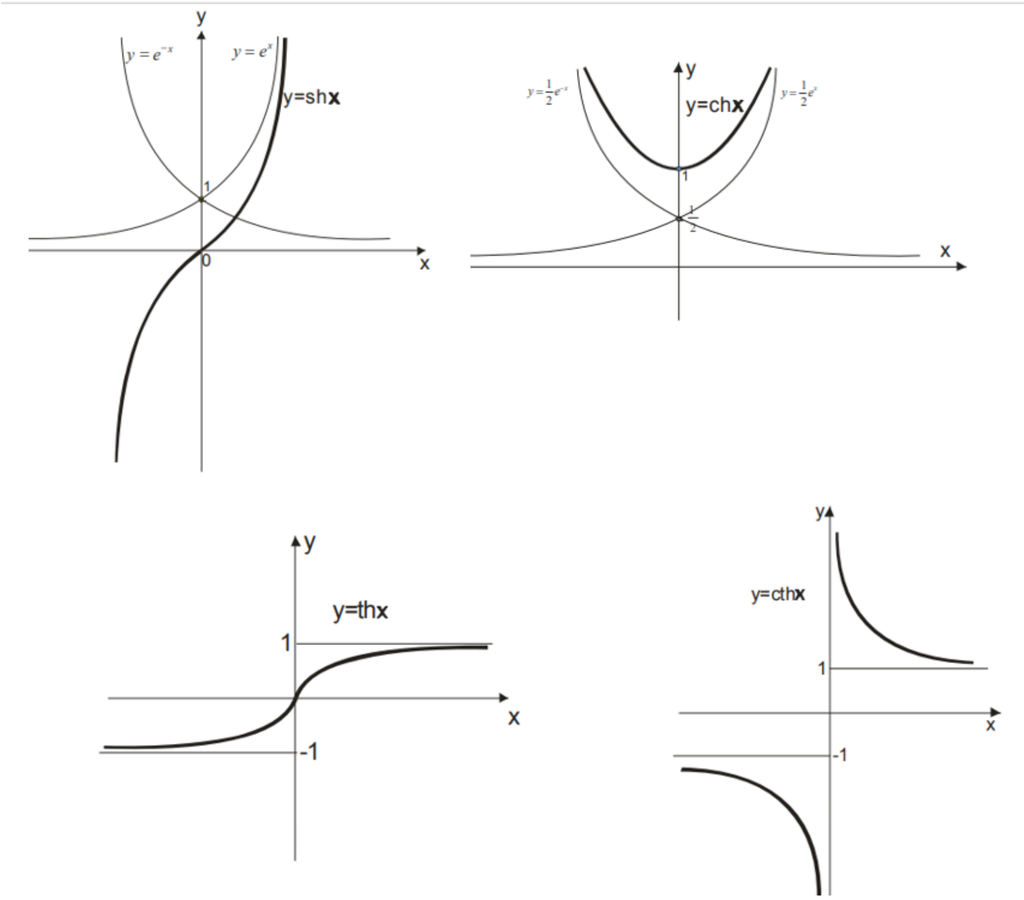

Figure 1.14: Hyperbolic functions

There are valid identities:

1. $ch^2x - sh^2x = 1$,
2. $sh(x + y) = sh\ x.ch\ y + ch\ x.sh\ y$,
3. $ch(x + y) = ch\ x.ch\ y + sh\ x.sh\ y$,
4. $sh\ 2x = 2.sh\ x.ch\ x$,
5. $ch\ 2x = ch^2x + sh^2x$.