Chapter III

Real functions of one real variable

Dr. Harizi

Contents

3. Real functions of one real variable 3
3.1 Notions of Function 3
3.1.1 General Definitions 3
3.1.2 Bounded functions, monotonic function 3
3.1.3 Even, odd, periodic function 4
3.1.4 Algebraic operations on functions 4
3.2 Limit of a function 5
3.2.1 Finite limit of a function at a point $x0$ نهايـة منتهيـة عنـد نقطـة 5
3.2.3 Finite limit of a function at infinity $(x \to \pm\infty)$ نهايـة منتهيـة عنـد المـالانهايـة 6
3.3 Continuity of a Function 9
3.4 Differentiability of a function 10
3.4.1 Derivation Operations 11
3.4.2 Derivatives of Standard Functions 12
3.4.3 L'hopital's rule 12

3. Real functions of one real variable

3.1 Notions of function

3.1.1 General definitions

Definition 3.1.1 We call digital function on a set D_f any process which, at all element $x\ of\ D_f$, allows to associate at most one element of the set R, then called image of x and denoted $f(x)$. The elements of D_f which have an image by f form the definition set of f, noted D_f.

Example 3.1.1 The function $f: x \rightarrow \sqrt{x - 1}$ is defined for all x ∈ R such that x − 1 ≥ 0. So $Df = [1, +\infty[$

Definition 3.1.2. We call a graph, or representative curve, of a function f defined on an interval $D_f \subset R$, the set $\Gamma f = \{(x, f(x)): x \in D_f\}$ formed from the points $(x, f(x)) \in R^2$ of the plan provided with an orthonormal reference $(o, \vec{\imath}, \vec{\jmath})$

3.1.2 Bounded functions, monotonic function

Definition 3.1.3 Let $f : D_f \rightarrow R$ be a function. We say that:

a. f is **bounded from above** if there is a number M such that for all x from D_f: $f(x) \leq K$.
we write: $\exists M \in R, \forall x \in D_f: f(x) \leq M$

b. f is **bounded from below** if there is a number m such that for all x from D_f: $f(x)$.
we write: $\exists m \in R, \forall x \in D_f: m \leq f(x)$

c. f is **bounded** if it is bounded both from above and below.
It is to say: $\exists M > 0, \forall x \in D_f: |f(x)| \leq M$

d. Function that is not bounded is called unbounded function

Definition 3.1.4. Let $f : D_f \rightarrow R$ be a function. We say that:

a. f is <u>increasing</u> on D_f if: $\forall x, y \in D_f, x < y \Rightarrow f(x) \leq f(y)$.

b. f is <u>strictly increasing</u> on D_f if: $\forall x, y \in D, x < y \Rightarrow f(x) < f(y)$.

c. f is <u>decreasing</u> on D_f if: $\forall x, y \in D, x < y \Rightarrow f(x) \geq f(y)$.

d. f is <u>strictly decreasing</u> on D_f if: $\forall x, y \in D, x < y \Rightarrow f(x) > f(y)$.

e. f is <u>monotonic</u> (strictly monotonic, respectively) on D_f if f is <u>increasing **or** decreasing</u> (strictly increasing **or** strictly decreasing, resp) on D_f.

Example 3.1.2.

a. Exponential functions exp: R → R is strictly increasing.
b. The absolute value function x → |x| defined on R is not monotonic.

3.1.3 Even, odd, periodic function

Definition 3.1.5. Let I be an interval of R symmetric with respect to 0.
Let $f : I \to R$ be a function. We say that:

a. f is even (زوجية) if $\forall x \in I: f(-x) = f(x)$.
b. f is odd (فردية) if $\forall x \in I: f(-x) = -f(x)$.

Example 3.1.3.

a. The function defined on R by $x \to x^{2n}$ $(n \in N)$ is even.
b. The function defined on R by $x \to x^{2n+1}$ $(n \in N)$ is odd.

Definition 3.1.6. Let $f: R \to R$ be a function and T a real number, T > 0. The function f is called periodic of period T if:

$$\forall x \in R: f(x + T) = f(x).$$

Example 3.1.4. The functions sin and cos are 2π $periodic$. The tangent function is π periodic.

3.1.4 Algebraic operations on functions

The set of functions from $D \subset R$ to R, is denoted F (D, R).

Definition 3.1.7 Let f and $g \in F(D, R)$ and λ ∈ R. We define:

- Sum of two functions $f + g: x \to (f + g)(x) = f(x) + g(x)$.
- For λ ∈ R, we define $\lambda f: x \to (\lambda f)(x) = \lambda f(x)$.
- Product of two functions $fg: x \to (fg)(x) = f(x)\, g(x)$.

The functions $f + g, \lambda f$ and fg are $functions$ $belonging$ to $F(D, R)$.

Definition 3.1.8 Let f and $g \in F(D, R)$ and $\lambda \in R$. We say that:

- $f \leq g$ $if: \forall x \in D, f(x) \leq g(x)$.
- $f < g$ $if: \forall x \in D, f(x) < g(x)$.

Example 3.1.5 Let f and g be two functions defined on $]0, 1[$ by:
$f(x) = x, g(x) = x^2$. We have $g < f$, because $\forall x \in]0, 1[, x^2 < x$.

3.2 Limit of a function

3.2.1 Finite limit of a function at a point x_0 نهاية منتهية عند نقطة

Let $f: D_f \rightarrow R$ be a function. Let $x_0 \in R$ be a point of D_f or an extremity of D_f.

Definition 3.2.1 Let $\ell \in R$. We say that the function f has a limit ℓ at x_0 if we have:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in D_f, |x - x_0| < \delta \Rightarrow |f(x) - \ell| < \varepsilon$$

We write in this case: $\lim_{x \to x_0} f(x) = \ell$.

Explanation of the definition: We say that f has a (finite) limit ℓ in x_0 if, when $x \rightarrow x_0$, $f(x) \rightarrow \ell$

Example 3.2.1 Consider the function $f(x) = 2x - 1$ which is defined on R. At the point $x = 1$,

Indeed, for all $\varepsilon > 0$, we have $|f(x) - 1| = 2|x - 1| < \varepsilon$, if we have, a fortiori, |x

$$|x - 1| < \frac{\varepsilon}{2}$$

The right choice will then be to take $\delta = \frac{\varepsilon}{2}$

- **Limit to the right \ to the left x_0 النهاية عن يمين وعن شمال النقطة**

Definition 3.2.2

a. We say that the function f admits ℓ as a limit to the right of x_0, or when x tends towards x_0^+, if $\forall \varepsilon > 0, \exists \delta > 0$, such that: $x_0 < x < x_0 + \delta, \rightarrow |f(x) - \ell| \leq \varepsilon$

We will write, in this case:

$$\lim_{x \to x_0^+} f(x) = \ell$$

b. We say that the function f admits ℓ as a limit to the left of x_0, or when x tends towards x_0^-, if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that: $x_0 - \delta < x < x_0$, results $|f(x) - \ell| \leq \varepsilon$. We will write, in this case:

$$\lim_{x \to x_0^-} f(x) = \ell$$

Example 3.2.2. The function $x \in R^+ \rightarrow \sqrt{x}$ tends to 0 when $x \rightarrow 0^+$

Noticed. If the function f admits a limit ℓ to the left of the point x_0 and a limit ℓ`

to the right of x_0, for f to admit a limit at the point x_0 it is necessary and sufficient that.

$$\boldsymbol{\ell} = \boldsymbol{\ell}^{`}$$

$$\lim_{x \to x_0} f(x) = \ell$$

Example 3.2.3. Consider the function defined by

$$f(x) = \begin{cases} 1, si\ x \geq 0 \\ -1, si\ x < 0 \end{cases}$$

It admits 1 as the limit to the right of 0 and -1 as the limit to the left of 0. But it admits **no limit at the point 0.**

Uniqueness of the limit

Proposition If f admits a limit at the point x_0, this limit is unique.

3.2.2 Infinite limit of a function at a point x_0 نهاية غير منتهية عند نقطة

Let $x_0 \in R$. We will put by definition:

a. $\lim_{x \to x_0} f(x) = +\infty$ if:

$\forall A > 0, \exists \delta > 0, such\ as\ |x - x_0| < \delta \Rightarrow f(x) > A.$

b. $\lim_{x \to x_0} f(x) = -\infty$ if:

$$\forall A > 0, \exists \delta > 0, such\ as\ |x - x_0| < \delta \Rightarrow f(x) < -A.$$

3.2.3 Finite limit of a function at infinity ($x \to \pm\infty$) نهاية منتهية عند المالانهاية

a. $\lim_{x \to +\infty} f(x) = \ell, if$

$$\forall \varepsilon > 0, \exists A > 0, tel\ que\ x > A \Rightarrow |f(x) - \ell| < \varepsilon.$$

We say that f has a (finite) limit ℓ at $+\infty$ if, when x becomes very large, $f(x)$ becomes very close to ℓ

b. $\lim_{x \to -\infty} f(x) = \ell, if$

$$\forall \varepsilon > 0, \exists A > 0, tel\ que\ x < -A \Rightarrow |f(x) - \ell| < \varepsilon.$$

we say that f has a (finite) limit at $-\infty$ if, when x becomes very large in negative value, $f(x)$ becomes very close to ℓ

3.2.4 Infinite limit at infinity نهاية غير منتهية عند المالانهاية

Let $x_0 \in R$. We will put by definition:

a. $\lim_{x \to +\infty} f(x) = +\infty$ if:

$$\forall A > 0, \exists\, B > 0, such\ as\ x > B \Rightarrow f(x) > A.$$

b. $\lim_{x\to+\infty} f(x) = -\infty$ if:

$$\forall A > 0, \exists\, B > 0, such\ as\ x > B \Rightarrow f(x) < -A.$$

c. $\lim_{x\to-\infty} f(x) = +\infty$ if:

$$\forall A > 0, \exists\, B > 0, such\ as\ x < -B \Rightarrow f(x) > A.$$

d. $\lim_{x\to-\infty} f(x) = -\infty$ if:

$$\forall A > 0, \exists\, B > 0, such\ as\ x < -B \Rightarrow f(x) < -A.$$

Examples:

Prove the following limits using the definition

$$\lim_{x\to1} 2x - 3 = -1$$

$$\lim_{x\to4} \sqrt{x} = 2$$

$$\lim_{x\to1} \frac{1}{(1-x)^2} = +\infty$$

$$\lim_{x\to1} 2x^2 + 3x - 1 = +\infty$$

Indeterminate form

Some forms of limits are called indeterminate. An **indeterminate form** is an expression whose limit cannot be determined solely from the limits of the individual functions.

Example of indeterminate forms: 0/0, ∞/∞, +∞ −∞, 0×∞, ∞^0, 0^0

3.2.5 Properties of function limits

Let $f, g: [a, b] \to R$ and $x_0 \in]\,a, b[$, we have:

a. If $\lim_{x\to x_0} f(x) = \pm\infty \Longrightarrow \lim_{x\to x_0} \frac{1}{f(x)} = 0$
b. If $f \le g\ and\ \lim_{x\to x_0} f(x) = \ell\ \lim_{x\to x_0} g(x) = \ell' \Longrightarrow \ell \le \ell'$
c. If $f \le g\ and\ \lim_{x\to x_0} f(x) = +\infty\ so{:}\ \lim_{x\to x_0} g(x) = +\infty$

d. If $f \leq g$ *and* $\lim_{x \to x_0} g(x) = -\infty$ *so*: $\lim_{x \to x_0} f(x) = -\infty$

Theorem Let $f, g, h: [a, b] \to R$ and $x_0 \in]a, b[$, if we have:

1. $f(x) \leq g(x) \leq h(x), pour\ tout\ x \in]a, b[$,
2. $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = \ell \in R.$

$\Longrightarrow$ So $\lim_{x \to x_0} g(x) = \ell$

3.2.6 Operations on the limits

Theorem Let $f, g: [a, b] \to R$ and $x_0 \in]a, b[$, such that: $\begin{cases} \lim_{x \to x_0} f(x) = \ell \\ and \\ \lim_{x \to x_0} g(x) = \ell' \end{cases}$, so:

a. $\lim_{x \to x_0} (f(x) + g(x)) = \ell + \ell'$

b. $\lim_{x \to x_0} \left(\lambda f(x)\right) = \lambda \ell$

c. $\lim_{x \to x_0} f(x)g(x) = \ell.\ell'$

d. $\lim_{x \to x_0} |f(x)| = |\ell|$

e. $\lim_{x \to x_0} |f(x) - \ell| = 0$

f. $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{\ell}{\ell'}\ if\ \ell' \neq 0$

Let $f : [a, b] \to [c, d], g : [c, d] \to R\ and\ x_0 \in]a, b[, y_0 \in [c, d]$, such that: $\lim_{x \to x_0} f(x) = y_0, and \lim_{y \to y_0} g(y) = \ell, so$:

$$Alors \lim_{x \to x_0} (g \circ f)(x) = \ell$$

Proposition Let $f, g: [a, b] \to R\ and\ x_0 \in]a, b[$, we have:

a. $if \lim_{x \to x_0} f(x) = +\infty, so \lim_{x \to x_0} \frac{1}{f(x)} = 0$

b. $if \lim_{x \to x_0} f(x) = -\infty, so \lim_{x \to x_0} \frac{1}{f(x)} = 0$

c. $if\ f \leq g \lim_{x \to x_0} f(x) = \ell, and \lim_{x \to x_0} g(x) = \ell', so: \ell \leq \ell'$

d. $if\ f \leq g \lim_{x \to x_0} f(x) = +\infty, so \lim_{x \to x_0} g(x) = +\infty,$

3.3 Continuity of a function

3.3.1 General definitions

Definition 3.3.1 Consider a function $f: D_f \to R$, D_f being an interval of R. We say that f is continuous at the point $x_0 \in D_f$ if:

$$\lim_{x \to x_0} f(x) = f(x_0)$$

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in I, |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

To remember:

f continuous $\leftrightarrow$

1. $f(x_0)$ exists: f is defined at x_0

   الدالة معرفة عند النقطة x_0

2. $\lim_{x \to x_0^+} f(x) = \ell$ and $\lim_{x \to x_0^-} f(x) = \ell'$: $\boldsymbol{\ell}$ **and** $\boldsymbol{\ell'}$ **exist**

   الدالة تقبل نهاية عن يمين وعن شمال النقطة x_0

3. $\boldsymbol{\ell = \ell'}$

   النهايتان عن يمين وعن شمال النقطة x_0 متساويتان

Examples

1. Is the function $f(x) = \frac{x^2-4}{x-2}$ continuous?

f is discontinuous at 2 because $f(2)$ is undefined.

2. Determine whether the function $f(x) = \begin{cases} -x^2 - 4 & if\ x \leq 3 \\ 4x - 8 & if\ x > 3 \end{cases}$ is continuous at $x = 3$

- ✓ We calculate $f(3)$: $f(3) = -3^2 - 4 = -5$, Thus, <u>$f(3)$ is defined.</u>
- ✓ We calculate : $\lim_{x \to >3} f(x)$ *and* $\lim_{x \to <3} f(x)$

$$\lim_{x \to <3} f(x) = -3^2 - 4 = -5$$

$$\lim_{x \to >3} f(x) = \lim_{x \to >3} 4(3) - 8 = 4$$

- ✓ $-5 \neq 4 \Leftrightarrow not\ continuous\ at\ 3$

Definition 3.3.2. A function defined on an interval I is continuous on I if it is continuous at every point of I. The set of continuous functions on I is denoted by C (I).

Continuity on the left \ on the right

Definition 3.3.3 Consider a function $f: D_f \to R$ and I, being an interval of R.

1. The function is continuous to the right at x_0 if

$$\lim_{x \to > x_0} f(x) = f(x_0) \Leftrightarrow$$

$$\forall \varepsilon > 0, \exists \delta > 0, such\ that: \boldsymbol{x_0} < \boldsymbol{x} < \boldsymbol{x_0} + \boldsymbol{\delta}, \to |f(x) - f(x_0))| \leq \varepsilon.$$

2. The function f is said to be left continuous at x_0 if

$$\lim_{x \to < x_0} f(x) = f(x_0) \Leftrightarrow$$

$$\forall \varepsilon > 0, \exists \delta > 0, such\ that: \boldsymbol{x_0} - \boldsymbol{\delta} < x < x_0, \to |f(x) - f(x_0))| \leq \varepsilon.$$

3.3.2 Operations on continuous functions

Theorem 3.3.1. Let D_f be an interval, and $f\ and\ g$ be functions defined on D_f and continuous at $x_0 \in D_f$ Then

1. $\lambda f\ is\ continuous\ at\ x_0, (\lambda \in R)$.
2. $f + g\ is\ continuous\ at\ x_0$.
3. $f.g\ is\ continuous\ at\ x_0$.
4. $\frac{f}{g}\ (if\ g\ (x0)\ 6 =\ 0)\ is\ continuous\ at\ x_0$.

3.4 Differentiability of a function

Definition and properties

Definition 3.4.1. Let Df be an interval of R, x_0 a point of $Df, and\ f$ a function $(f: Df \to R)$

We say that f is **differentiable (derivable)** at the point x_0 if the limit $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ **exists** **finitely**

تعـريف نقـول ان الدالــة f **قابلة للاشتقــاق** في النقطة x_0 ادا كانـــت النهـــاية $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ **مــوجودة ومنتهيــة**

This limit is called the derivative of f at x_0 and is noted $f'(x_0)$

تسمـــى هــذه النهـــاية مشتــق الدالـــة f عند النقطـــة x_0 ونرمـــز لهـــا بــ $f'(x_0)$

If we put $x - x_0 = h$ we obtain:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0)$$

Graphically (بيانيا)

The derivative of f at x_0 is the **slope of the tangent line** at this point ($x_0, f(x_0)$), which is the limit as h → 0 of the slopes of the lines through

Example Let f be the real function defined on R by $f\ (x) = x^2$. The derivative of f at a point $x^0 \in R$ is:

$$\lim_{h \to 0} \frac{f(h + x_0) - f(x_0)}{h} = \lim_{h \to 0} \frac{(h + x_0)^2 - {x_0}^2}{h} = \lim_{h \to 0} \frac{{x_0}^2 + h^2 + 2x_0.h - {x_0}^2}{h}$$
$$= \lim_{h \to 0} h + 2x_0 = 2x_0$$

For $x_0 = 1, f'(x_0) = 2$

The function $y = f\ (x)$ is said to be differentiable in an open interval I if it is differentiable at every point of I

Proposition 3.4.1. Every differentiable function is continuous, but the converse is not true. (differentiable $\Rightarrow$ continuous)

كـل دالــة قابلــة للاشتقـــاق عند نقطة مـــا فهي مستمرة عند تلــك النقطــة. العكـــس غيـــر صحيـــح

3.4.1 Derivation operations

Theorem 3.4.1 Let f and g be two functions defined on the interval $I \subset R$ with and x0 ∈ I. If the functions f and g are differentiable at x_0, then

1. ∀α ∈ R, αf is differentiable at x0 and we have:
$$(\alpha f)'(x_0) = \alpha f'(x_0)$$
2. f + g is differentiable at x0 and we have
$$(f + g)'(x_0) = f'(x_0) + g'(x_0)$$
3. fg is differentiable at x0 and we have
$$(f.g)'(x_0) = f'(x_0).g(x_0) + f(x_0).g'(x_0)$$

4. If $g'(x_0) \neq 0$ the function $\frac{f}{g}$ is differentiable

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}$$

$$(1\backslash g)'(x_0) = \frac{-g'(x_0)}{g^2(x_0)}$$

Theorem Let J be an interval of R, $f: I \to J$ and $g: J \to R$. If f is differentiable in $x_0 \in I$ and g differentiable at $f(x_0) \in J$, the composite function $g \circ f : I \to R$ is differentiable at x_0 and

$$(g \circ f)'(x) = f'(x).g'(f(x))$$

3.4.2 Derivatives of standard functions

Fonction f	Dérivée f'	Fonction f	Dérivée f'
x^n	nx^{n-1}	u^n	$nu'u^{n-1}$, $n \in \mathbb{N}^*$
$\frac{1}{x}$	$-\frac{1}{x^2}$	$\frac{1}{u}$	$-\frac{u'}{u^2}$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$	$\sqrt{u}$	$\frac{u'}{2\sqrt{u}}$
$\ln x$	$\frac{1}{x}$	$\ln u$	$\frac{u'}{u}$
e^x	e^x	e^u	$u'e^u$
$\sin(x)$	$\cos(x)$	$\sin(u)$	$u'\cos(u)$
$\cos(x)$	$-\sin(x)$	$\cos(u)$	$-u'\sin(u)$

Proposition Let $f : I \to R$ be a differentiable function on an Intervale I we have:

(1) $\forall x \in I, f'(x) = 0$ if and only if f is constant on I

(2) $If\ \forall x \in I, f'(x) \geq 0$ ($resp\ f'(x) > 0$), then f is increasing (resp strictly increasing).

(3) if $\forall x \in I, f'(x) \leq 0$ ($resp\ f'(x) < 0$), then f is decreasing (resp strictly decreasing)

3.4.3 L'hopital's rule

Theorem 3.4.7 Let f and g be two functions differentiable on I and both tending towards 0 (or both tend to ∞) $for\ x \to a$ ($or\ \infty$). We assume that $g(x) \neq 0$ does not vanish in a neighborhood of a and that: $\lim_{x\to x_0} \frac{f'(x)}{g'(x)} = l$ ($\lim_{x\to x_0} \frac{f'(x)}{g'(x)}$ exists)

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{f'(x)}{g'(x)}=l$$

In other words, if the limit: $lim\frac{f(x)}{g(x)}$ is of indeterminate type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ for : $x\to a\ or\ \infty$ then we have: $lim\frac{f(x)}{g(x)}=lim\frac{f\prime(x)}{g\prime(x)}=l$

Example

Find $\lim_{x\to 0}\frac{x^2}{sinx}$ and $\lim_{x\to 0}\frac{3x^2+x+4}{5x^2+8x}$

Solution As observed above, this limit is of indeterminate type $\frac{0}{0}$, so l'Hôpital's rule applies. We have

$$\lim_{x\to 0}\frac{x^2}{\sin x}\ \left(\frac{0}{0}\right)\ \stackrel{\text{l'H}}{=}\ \lim_{x\to 0}\frac{2x}{\cos x}=\frac{0}{1}=0,$$

where we have first used l'Hôpital's rule and then the substitution rule. □

$$\lim_{x\to\infty}\frac{3x^2+x+4}{5x^2+8x}\ \left(\frac{\infty}{\infty}\right)\ \stackrel{\text{l'H}}{=}\ \lim_{x\to\infty}\frac{6x+1}{10x+8}\ \left(\frac{\infty}{\infty}\right)\ \stackrel{\text{l'H}}{=}\ \lim_{x\to\infty}\frac{6}{10}=\frac{6}{10}=\frac{3}{5}.$$