Chapter II

Sets, Relations, and Maps

Contents

2 Sets, Relations, and Applications

2.1 Set theory نظرية المجموعات

Definition 2.1.1 A set is a collection of objects; any one of the objects in a set is called a **member** or an **element** of the set. If A is a set and a is an element of it, we write a ∈ A. The fact that 'a is not an element of A' is written as: $a \notin A$.

Example 2.1.1

- $\{0, 1\}, \{red, black\}, and\ N = \{0, 1, 2, \dots\}\ are\ sets.$
- if A is the set $\{1, 4, 9, 2\}$, then $1 \in A, 4 \in A, 2 \in A\ and\ 9 \in A. But\ 7 \notin A,$

المجموعة هي وحدة تضم أشياء أو عناصر. نظرية المجموعات هو فرع من علم المنطق الرياضي .تهتم تلك النظرية بدراسة المجموعات والعلاقة بينها

يمكن للمجموعة أن تكون خالية ولكن لا يمكن لها أن تحتوي على نفس العنصر أكثر من مرة.

Definition 2.1.2 The set S, that contains no element, is called the empty set or the null set and is denoted by { } $or\ \emptyset$. A set that has only one element is called a singleton set.

Important sets in math

- ✓ Natural numbers: $N = \{0,1,2,3, \dots\}$
- ✓ Integers: $Z = \{\dots, -2, -1,0,1,2, \dots\}$
- ✓ Positive integers: $Z^+ = \{1,2, 3. \dots\}$
- ✓ Rational numbers: $Q = \{p/q \mid p \in Z, q \in Z, q \neq 0\}$
- ✓ Real numbers: R

There are different ways for defining a set.

1. Statement form في شكل عبارة
   For example: X is the set of even integers between 0 and 12.

2. Listing all its elements (list notation) قائمة كل العناصر
   X = {2, 4, 6, 8, 10}:

3. Stating a property with notation ذكر الخاصية المشتركة بين جميع عناصر المجموعة
   The set X = {2, 4, 6, 8, 10} can be written as: **X = {x: 0 < x ≤ 10, x is an even integer}**

Example 2.1.2 E is the set of even integers between 50 and 63.

1) E = {50, 52, 54, 56, 58, 60, 62}
2) $E = \{x \in \mathbb{N}: 50 \leq x < 63,\ x\ is\ an\ even\ integer\}$

1.1.1 Operations on sets: Inclusion, union, intersection, complementary

- Inclusion الاحتواء

Definition 2.1.3 A set E is included in a set F, if every element of E **is also** an element of F, E is said to be a **subset or a part of F**. We write E ⊂ F.

In other words: $E \subset F \Leftrightarrow \forall x, x \in E \Rightarrow x \in F$.

تعريف: تكون المجموعة E محتواه في المجموعة F ، إذا كان كل عنصر من عناصر E هو أيضًا عنصر من عناصر F و نقول أن E هي **مجموعة فرعية (او جزئية)** من المجموعة F

نكتب $E \subset F$ بمعنى آخر $\forall x, x \in E \Rightarrow x \in F$.

Example 2.1.3. We have N ⊂ Z ⊂ Q ⊂ R.

Notes:

- **When there exists $x \in E$** such that $x \notin F$, then we say that E **is not a subset** of F; and we write $E \not\subseteq F$
- Let X = {a, b, c}. Then a ∈ X but {a} ⊆ X.

ملاحظة:

- عندما يكون هناك $x \in E$ بحيث يكون $x \notin F$ ، إذن نقول أن E ليست مجموعة فرعية من F؛ ونكتب $E \not\subseteq F$.

Definition 2.1.4 Two sets E and F are **equal if and only if** each set is included in the other, that is to say: $E = F \Leftrightarrow E \subset F \text{ and } E \subset F$.

تعريف تكون المجموعتانE و F متساويتان **إذا وفقط إذا** كانت كل منهما محتواه في الأخرى،

أي : $(E = F) \Leftrightarrow (E \subset F$ و $E \subset F)$

Example 2.1.4

A = {x ∈ R: |x − 1| ≤ 1} = [0, 2]

Demonstration:

A = {x ∈ R: |x − 1| ≤ 1} = {x ∈ R: −1 ≤ x − 1 ≤ 1} = {x ∈ R: 0 ≤ x ≤ 2} = [0, 2]

{1,2,3} = {3,1,2}

Definition 2.1.5 Let E be a set, the power set of E is the set of all subsets of E. The power set is denoted by P(A). which is characterized by the following relation:

$$P(E) = \{A : A \subset E\}$$

تعريف لنفترض أن E مجموعة، نشكل مجموعة تسمى **مجموعة أجزاء** E أو مجموعة مجموعات E ، نرمز لها بـ $P(E)$ وتعرف بالعلاقة التالية $P(E) = \{A : A \subset E\}$

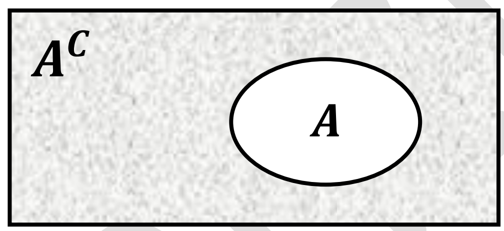
If E is a set with n element, then Card $P(E) = 2^n$

Example 2.1.5 If E = {1, 2, 3}, then P (E) = {∅, {1},{2},{3},{1,2},{1,3}, {2, 3}, {1, 2, 3}}. Card $P(E) = 2^3 = 8$

Definition 2.1.6 Let E be a set and A ⊂ E. Then, the complement of A, denoted by $C_E A \ or \ A^C$, is the set of elements of E which do not belong to A

$\boldsymbol{A^C}$ **is defined by** $\boldsymbol{A^C = \{x \in E : x \notin A\}}$**.**

$$\boldsymbol{A \cup A^C = E}$$



تعريف لتكن E مجموعة و A ⊂ E. إذن، مكمل A، المشار إليه بـ $C_E A \ or \ A^C$ هو مجموعة عناصر E التي لا تنتمي إلى A.

- $E^c = \emptyset \ and \ \emptyset^c = E$
- $A \cup A^c = E \ and \ A \cap A^c = \emptyset.$

➢ **The union** الاتحاد

Definition 2.1.7 The union of A and B, denoted by $A \cup B$, is the set that consists of **all elements of A** and also **all elements of B**. That is to say:

$$\boldsymbol{A \cup B = \{x |\ x \in A \ or \ x \in B\}}.$$

تعريف اتحـــاد A وB، الذي يُشار إليه بالرمز A ∪ B ، هو المجموعة التي تتكون من جميع عناصر A وأيضًا جميع عناصر B أي : $\boldsymbol{A \cup B = \{x |\ x \in A \ or \ x \in B\}}$.

➢ **The intersection** التقاطع

Definition 2.1.8 The intersection of A and B, denoted by $A \cap B$, is the set of all **common elements of A and B**. More specifically,

$$\boldsymbol{A \cap B = \{x |\ x \in A \ and \ x \in B\}}.$$

Note: The sets A and B are said to be **disjoint** if $A \cap B = \emptyset$

Example 2.1.6

$If\ A = \{1,2,3\} and\ B = \{2,3,4,5\}, so\ A \cup B = \{1,2,3,4,5\}\ and\ A \cap B\ = \{2,3\}.$

Proposition

Let A, B, C be parts of a set E, then it is clear that:

• Commutativity: $A \cap B = B \cap A$ and $A \cup B = B \cup A$.

• Associativity: $A \cap (B \cap C) = (A \cap B) \cap C$
$A \cup (B \cup C) = (A \cup B) \cup C$.

•Distributivity: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$A \cap (B \cup C) = (A \cap B) \cup (A \cup C)$

Demonstration of $A \cap (B \cup C) = (A \cap B) \cup (A \cup C)$

Let $x \in A \cap (B \cup C)) \Leftrightarrow (x \in A\ and\ x \in B \cup C)$
$\Leftrightarrow \big(x \in A\ and\ (x \in B\ or\ x \in C)\big)$
$\Leftrightarrow \big((x \in A\ and\ x \in B) or\ (x \in A\ and\ x \in C)\big)$
$\Leftrightarrow (x \in A \cap B\ or\ x \in A \cap C)$
$\Leftrightarrow x \in (A \cap B) \cup (A \cap C)$

this means that $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$

the equivalence ($\Leftrightarrow$) means that demonstration is correct in both directions so :

$$(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$$

$$\begin{cases} A \cap (B \cup C) \subset (A \cap B) \cup (A \cup C) \\ \qquad\qquad \boldsymbol{and} \\ (A \cap B) \cup (A \cup C) \subset A \cap (B \cup C) \end{cases} \rightarrow A \cap (B \cup C) = (A \cap B) \cup (A \cup C)$$

Definition 2.1.9 Let E be a set, we say that the set **E is finite** if the **number of elements of E is finite**. The number of elements of E is called the **cardinality of E**, it is denoted by $\boldsymbol{Card}\ (\boldsymbol{E})\ or\ |\boldsymbol{E}|$. A set which is **not finite** is called an **infinite** set.

تعريف لتكن E مجموعة، نقول أن المجموعة E محدودة إذا كان عدد عناصرها محدودا. يُرمز لعدد عناصر المجموعة E ب $|\boldsymbol{E}|$ او $\boldsymbol{Card}\ (\boldsymbol{E})$

Example 2.1.9

- If E = {0, 1, 2, 3}, then Card (E) = 4.
- The set N is infinite, Card (∅) = 0.

➢ **The difference** الفـرق بيـن مجموعتيـن

Definition 2.1.10 Let A and B be two sets. we call **difference of A and B**, denoted $A\backslash B$, the set formed of the elements x which belong to A and do not belong to B, that is to say: $A\backslash B = \{x \in A : x \notin B\}$

نسمي الفرق بين A وB، المجموعة المكونة من العناصر x التي تنتمي إلى A ولا تنتمي إلى B ونرمز لها بـ A\B

Definition 2.1.11 We call the **symmetric difference** of A and B, denoted $A\Delta B$, the set formed by the elements x which belong to $A \cup B$ and do not belong to $A \cap B$, that is to say:

$$\boldsymbol{A\Delta B = (A \cup B)\backslash (A \cap B) = (A\backslash B) \cup (B\backslash A)}$$

نسمي الفرق التناظري بين A و B، المجموعة المكونة من العناصر x آلتي تنتمي إلى $A \cup B$ ولا تنتمي الى $A \cap B$ ب، ونرمز له بـ: $A\Delta B$

Example 2.1.10 If $E = R, A = [0,1]$ and $B =]0,+\infty[$, then:

$$A\backslash B = \{0\}, B\backslash A =]1,+\infty[\text{ and } A\Delta B = \{0\} \cup]1,+\infty[$$

Example 2.1.11

- Let $A = \{1, 2, 4, 18\}$ *and* $B = \{x \in Z: 0 < x \leq 5\}$.

  $$Then, A\backslash B = \{18\}, B\backslash A = \{3,5\} \text{ and } A\Delta B = \{3,5,18\}.$$

- Let $S = \{x \in R : 0 \leq x \leq 1\}$ and T = $\{x \in R : 0.5 \leq x < 7\}$.

  Then, $S \backslash T = \{x \in R : 0 \leq x < 0.5\}$ and $T \backslash S = \{x \in R : 1 < x < 7\}$.

We state more properties of sets.

Let E be the universal set and A, B ⊆ U. Then,

- $A \cup E = E \text{ and } A \cap E = A$
- $(A^C)^C = A$
- $A \subseteq A^c \text{ if and only if } A = \emptyset.$
- $A \subseteq B \text{ if and only if } B^c \subseteq A^c$
- $A = B^c \text{ if and only if } A \cap B = \emptyset \text{ and } A \cup B = E$
- $A\backslash B = A \cap B^c \text{ and } A \; B = B \cap A^c$
- $A\Delta B = (A \cup B) \; (A \cap B)$
- $De - Morgan's\ Laws$:

$$(A \cup B)^c = A^c \cap B^C$$
$$(A \cap B)^c = A^c \cup B^c$$

Exercise 2.1 (Demonstration of Morgan's Law).

Let A and B be parts of a set E, demonstrate that:

$$1.\ (A \cap B)^c = A^c \cup B^c \quad and \quad 2.\ (A \cup B)^c = A^c \cap B^c$$

Demonstration

1. Let $x \in (A \cap B)^c \leftrightarrow x \notin (A \cap B)$

$$x \notin (A \cap B)\ is\ the\ negation\ of\ x \in (A \cap B)$$

So $x \notin (A \cap B) \leftrightarrow \overline{x \in (A \cap B)} \leftrightarrow \overline{x \in A\ and\ x \in B} \leftrightarrow x \notin A\ ou\ x \notin B \leftrightarrow$ $x \in A^c\ ou\ x \in B^c \leftrightarrow x \in A^c \cup B^c$

2. Let $x \in (A \cup B)^c \leftrightarrow x \notin (A \cup B)$

$$x \notin (A \cup B)\ is\ the\ negation\ of\ x \in (A \cup B)$$

So $x \notin (A \cup B) \leftrightarrow \overline{x \in (A \cup B)} \leftrightarrow \overline{x \in A\ \mathbf{or}\ x \in B} \leftrightarrow x \notin A\ \mathbf{and}\ x \notin B \leftrightarrow x \in A^c\ and\ x \in B^c \leftrightarrow x \in A^c \cap B^c$

1.1.2 Cartesian Product الجداء الديكارتي

Definition 2.1.12 We call the Cartesian product of two sets E and F the set of couples (x, y) where x ∈ E and y ∈ F. it denoted by E ×F

$$\boldsymbol{E \times F = \{(x, y): x \in E\ and\ y \in F\}}.$$

الجداء الديكارتي لمجموعتين F و E هو مجموعة الأزواج (x,y) حيث $x \in E$ و $y \in F$. ويرمز له بـ $E \times F$

Example 2.1.12 If E = {1, 2} and F = {3, 5}, then:

- $E \times F = \{(1,3),(1,5),(2,3),(2,5)\}$
- $F \times E = \{(3,1),(3,2),(5,1),(5,2)\}6 = E \times F$
- Note: E x F ≠ E x F !

The elements of $E \times F\ \{(x,y): x \in E\ and\ y \in F\}$ are also called ordered pairs. Thus, $(x_1, y_1) = (x_2, y_2)$ if and only if $x_1 = x_2\ and\ y_1 = y_2$

- The Euclidean plane, denoted by $R^2 = R \times R = \{(x,y) : x, y \in R\}$
- ∅ × F = E × ∅ = ∅
- If card E = n and card F= m we have card $E \times F = n \times m$

Exercise

Let: $X = \{a, c\}\ and\ Y = \{a, b, e, f\}$.
Write down the elements of:

- X × Y
$$X \times Y = \{(a,a),(a,b),(a,e),(a,f),(c,a),(c,b),(c,e),(c,f)\}$$
- $Y \times X$
$$Y \times X = \{(a,a),(a,c),(b,a),(b,c),(e,a),(e,c),(f,a),(f,c)\}$$
- X^2
$$X^2 = X \times X = \{(a,a),(a,c),(c,a),(c,c)\}$$
- What could you say about two sets A and B if A × B = B × A?

They are equal: A = B

2.2 Relations

In this section, we introduce the set theoretic concepts of relations and functions. We will use these concepts to relate different sets. This method also helps in constructing new sets from existing ones.

في هذا القسم، نقدم المفاهيم النظرية للعلاقات والوظائف. سوف نستخدم هذه المفاهيم لربط مجموعات مختلفة. تساعد هذه الطريقة أيضًا في إنشاء مجموعات جديدة من المجموعات الموجودة.

2.2.1 Relations العلاقات

Definition 2.2.1 Let E and F be two nonempty sets. A relation $\mathcal{R}$ from E to F is a subset of E × F which contains a collection of certain ordered pairs $(x \in E)$ and $(y \in F)$. We write $x\mathcal{R}y$ to $mean\ (x,y) \in \mathcal{R} \subseteq E \times F$. A relation on a set E ($E \times E$) is called a binary relation.

لتكن E و F مجموعتان غير خاليتين. نسمي علاقة $\mathcal{R}$ بين المجموعتين E و F كل خاصية تسمح بان نرفق عناصر من E $(x \in E)$ بعناصر من F $(y \in F)$ ونكتب xRy وهي المجموعة الفرعية من E × F التي تضم مجموعة معينة من الثنائيات المرتبة

$$x\mathcal{R}y : (x,y) \in \mathcal{R} \subseteq E \times F$$

Definition 2.2.2 Let $\mathcal{R}$ be a binary relation on a set E. For all $x,y,z \in E$, we say that $\mathcal{R}$ is:

1. Reflexive (الانعكاس), if each element is in relation to itself, that is to say:
$$x\mathcal{R}x, \forall x \in E.$$
2. Symmetric (التنـــاظر), if for all x, y ∈ E, if x is in relation with y then y is in relation with x, i.e. :

$$x\mathcal{R}y \Rightarrow y\mathcal{R}x, \forall x, y \in E.$$

3. Transitive (التعـدي), if for all $x, y, z \in E$, if x is in relation with y and y in relation with z then x is in relation with z, i.e:
$$(x\mathcal{R}y \; and \; y\mathcal{R}z) \Rightarrow x\mathcal{R}z, \forall x, y, z \in E.$$

4. Anti-symmetric (ضد تناظرية) , if two elements are related to each other, then they are equal, that's to say:
$$(x\mathcal{R}y \; and \; y\mathcal{R}x) \Rightarrow x = y, \forall x, y \in E$$

Example 2.2.1 consider the relations below. Tell whether the following relations are reflexive, symmetric, transitive and anti-symmetric

a. $\mathcal{R}_1 = \{(x, y)|x = y\}$
$x\mathcal{R}_1 x ? \;\; x = x \; \mathcal{R}_1 \; is \; reflexive$
$x\mathcal{R}_1 y \Rightarrow yRx? \;\; x = y \Rightarrow y = x$ is symmetric
$x\mathcal{R}_1 y \; and \; y\mathcal{R}_1 z \Rightarrow x\mathcal{R}_1 z? \, x = y \; and \; y = z \Rightarrow x = z \;\; \mathcal{R}_1 \; is \; transitive$

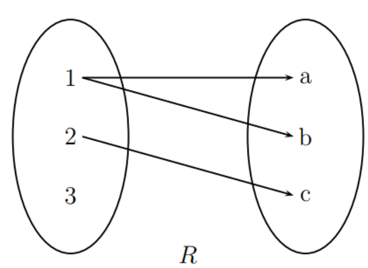
b. $\mathcal{R}_2 = \{(x, y)|x \leq y\}. \{(1,1); (1,3); (2,4)\} \subset \mathcal{R}_2$
- $x\mathcal{R}_2 x ? \;\; x \leq x \; \mathcal{R}_2 \; is \; reflexive$
- $x\mathcal{R}_2 y \Rightarrow y\mathcal{R}_2 x?$ $x \leq y \not\Rightarrow y \leq x$ is not symmetric
- $\underbrace{x\mathcal{R}_2 y \; and \; y\mathcal{R}_2 x}_{P} \Rightarrow \underbrace{x = y}_{Q}$? $x \leq y$ and $y \leq x \Rightarrow x = y$.
  $\mathcal{R}_2 \;\; anti-symmetric$
- $x\mathcal{R}_2 y \; and \; y\mathcal{R}_2 z \Rightarrow x\mathcal{R}_2 z$? $x \leq y$ and $y \leq z \Rightarrow x \leq z$
  $\mathcal{R}_2 \; is \; transitive$

Example 2.2.2 we define the relation $\mathcal{R}$ on A by:
$$x \, \mathcal{R} \, y \Leftrightarrow x - y \geq 0 \; or \; we \; can \; write: \mathcal{R} = \{(x, y)| \, x - y \geq 0\}$$

Let A = {a, b, c, d}. Some binary relations $\mathcal{R}$ on A are:
- $\mathcal{R} = A \times A.$
- $\mathcal{R} = \{(a, a), (b, b), (c, c), (d, d), (a, b), (a, c), (b, c)\}.$
- $\mathcal{R} = \{(a, a), (b, b), (c, c)\}.$

Sometimes, we draw pictures to have a better understanding of different relations.
في بعض الأحيان، نستعين برسم للحصول على فهم أفضل للعلاقات المختلفة

Example 2.2.3 Let A = {1, 2, 3}, B = {a, b, c} and let $\mathcal{R}$ = {(1, a),(1, b),(2, c)}. The following figure represents the relation $\mathcal{R}$.



1.2.2 Equivalence relation علاقة التكافؤ

Definition 2.2.3 Let E be a nonempty set. A relation on E is called an *equivalence relation* if it is reflexive, symmetric, and transitive. Tow elements related by equivalence relation are said to be equivalent, denoted as a∼b.

علاقة التكافؤ : لتكن $\mathcal{R}$ علاقة ثنائية معرفة على المجموعة E
تعريف: نقول أن $\mathcal{R}$ علاقة **تكافؤ** إذا كانت انعكاسية وتناظرية ومتعدية في آن واحد

Example 2.2.4 Let R be a relation on the set of real numbers ℝ such that $a\,\mathcal{R}\,b$ if and only if a-b is an integer $\mathcal{R} = \{(a,b) \in R^2 | (a-b) \in \mathbb{Z}\}$. Is this an equivalence relation?

Reflexive? $\forall a \in \mathbb{R}\; a\,\mathcal{R}\,a?\;\; a - a = 0 \in \mathbb{Z}$ so $\mathcal{R}$ is reflexive

Symmetric? $\forall a, b \in \mathbb{R}\;\; a\,\mathcal{R}\,b \Rightarrow b\mathcal{R}a?\;\; \forall a, b \in \mathbb{R} : a - b \in \mathbb{Z}\; -(a-b) \in \mathbb{Z}$ so $\mathcal{R}$ is symmetric

Transitive? $a\mathcal{R}b \text{ and } b\mathcal{R}c \rightarrow a\mathcal{R}c$? $\forall a, b \in \mathbb{R}: a - b \in \mathbb{Z} \text{ and } b - c \in \mathbb{Z} \rightarrow a - c \text{ is it an integer}?$
$a - c = a - b + b - c = (a-b) + (b-c) \in \mathbb{Z}$ so, $\mathcal{R}$ is transitive

R is: Reflexive, symmetric and transitive so $\mathcal{R}$ is an equivalence relation

- **Equivalence classes فئة التكافؤ**

We call equivalence classes of a, the set of elements equivalent to an element through the equivalent relation R, is denoted $[a]_R$

نحن نطلق على فئات التكافؤ لa ، مجموعة العناصر المكافئة لعنصر ما من خلال علاقة التكافؤ R ، يُشار إليها بالرمز$[a]_R$

Let R be the equivalence relation on Z such that: $a\mathcal{R}b \Leftrightarrow (a,b) \in \mathbb{Z}^2 : a = b \text{ or } a = -b$
The equivalence classes of $a \in \mathcal{R}$:

$$[1] = [1,-1], [2] = [2,-2] \ldots . [a] = [a,-a]\; a \in Z$$

1.2.3 Partial Order relation علاقة الترتيب الجزئي

Definition 2.2.4 A binary relation $\mathcal{R}$ on E is called a partial order relation (an order relation) if it is antisymmetric, transitive and reflexive.

Example 2.2.5 Let $\mathcal{R}$ be the relation defined on ℝ by the relation $x \leq y$, that is to say

$$x\mathcal{R}y \Leftrightarrow (x,y) \in \mathbb{R} \times \mathbb{R}: x \leq y$$

1.2.4 Total order relation علاقة الترتيب الكلي

Definition 2.2.5 Let $\mathcal{R}$ be an order relation defined on a set E, we say that $\mathcal{R}$ is total order, if $for\ all\ x, y\ \in\ E,$

1. a partial order, and
2. for any pair of elements x and y of $\mathcal{R}$: $x\mathcal{R}y\ only\ \textbf{or}\ yRx\ only.$

2.3 Functions (maps) التطبيقــات

2.3.1 Definitions

Definition 2.3.1. Let X and Y be two sets, we call function of X in Y, any relation f between the elements of X and those of Y which associates with every element of X at most one element of Y.

A function from a set X to another set Y which to every element x ∈ E assigns a unique element y ∈ F is called a map or mapping (application).

Functions are usually denoted using letters f, g ... If f is a function from the set X to the set Y, we write.

$$\boldsymbol{f}: \boldsymbol{X} \to \boldsymbol{Y}$$
$$\boldsymbol{x} \to \boldsymbol{f}(\boldsymbol{x})$$

The **starting set X** is called the domain of the function f
The **arrival set** F is called Images**.**
The **element x** is said to be the **antecedent** and **y** is said to be the image of x by f.

Definition 2.3.2 Let $f\ :\ E\ \to\ F$ be a map. We call the domain of f, denoted D_f the set of elements $x\ \in\ E$ to which there exists a unique element $y\ \in\ F$, such that $y\ =\ f\ (x)$

Definition 2.3.3 (Graph) Let E and F be given sets. The graph of a function

$$f: E\ \to\ F\ is\ \Gamma_f\{(x, f\ (x)): x\ \in\ E\}\ \subset\ E\ \times\ F$$

Definition 2.3.4 (Equality). Let $f, g\ :\ E\ \to\ F$ be tow maps. We say that f, g are equal if and only if for all $x\ \in\ E: f\ (x)\ =\ g\ (x)$. We then write $f\ =\ g$.

Definition 2.3.5 The function f : X → X defined by f(x) = x, for all x ∈ X, is called the identity function.

The function $f : X \to R \; with \; f(x) = 0, for \; all \; x \in X$, is called <u>the zero function</u>.

Example 2.3.1 Let $f : R \to R$ defined by $f(x) = \sqrt{x+1}$, then

$$D_f = \{x \in R : x + 1 \geq 0\} = [-1, +\infty[$$

Definition 2.3.6 Let $f : E \to F$ be a function. Let A ⊆ E and $A \neq \emptyset$. The restriction of $f \; to \; A$, denoted by f_A, is the function

$$f_A : A \to F$$

$$f_A = f(x) \; \forall \; x \in A.$$

Composition of functions

Definition 2.3.7 Let E, F and G be three sets and f, g be two functions such that:

$f : E \to F \; and \; g : F \to G$ (condition: $img_f \subseteq dom_g$) Then, the composition of f and g, denoted by $g \circ f$, is defined as:

$$(g \circ f)(x) = g(f(x)), for \; all \; x \in E$$

Example 2.3.2
Let f, g be two functions so that: $f : R \to R^+ \; et \; g : R^+ \to [1, +\infty[$

$$f(x) = x^2, \forall x \in R \qquad and \qquad g(x) = 2x + 1, \forall x \in R^+$$

Then $g \circ f : R \to [1, +\infty[$ is given by:

$$(g \circ f)(x) = g(f(x)) = g(x^2) = 2x^2 + 1 \; \forall x \in R$$

2.3.2 Direct and inverse image

Definition (direct image) Let A ⊂ E and $f : E \to F$. The direct image of A by f is the set $f(A) = \{f(x) : x \in A\} \subset F$.

Example 2.3.3 Let $f : R \to R$ defined by $f(x) = 2x + 1, \forall x \in R$.

$If \; A = [0,1]$, then $f([0,1]) = \{f(x) : x \in [0,1]\}$

We have $x \in [0,1] \Leftrightarrow 0 \leq x \leq 1 \Leftrightarrow 1 \leq 2x + 1 \leq 3$,

$$so \; f([0,1]) = [1,3]$$

Example 2.3.4 Let f be the map defined by $f(x) = x^2 \; of \; R \to R^+$, then $f^{-1}([0,1]) = \{x \in R : 0 \leq x^2 \leq 1\} = \{x \in R : 0 \leq |x| \leq 1\} = [-1, 1]$

2.3.3 Injective, surjective, bijective function

Definition (Injection) A function $f : E \to F$ is said to be injective (also called one-one or an injection) if for $all\ x, y \in E$, $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$. Equivalently, f is one-one if for all $x_1, x_2 \in$ F, $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

$$\forall x_1, x_2 \in E : f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

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Example 2.3.5

1. Let X be a nonempty proper subset of Y. Then f(x) = x is a one-one map from X to Y.
2. The function f: Z → Z defined by $f(x) = x^2$ is not one-one as $f(-1) = f(1) = 1$.
3. The function f : {1, 2, 3} → {a, b, c, d} defined by f(1) = c, f(2) = b and f(3) = a, is injective.

Definition (surjection) A function $f : X \to Y$ is said to be surjective (also called onto or a surjection) if $f^{-1}(\{b\}) \neq \emptyset$ for each b ∈ Y. Equivalently, $f : X \to Y$ is onto if there exists a pre-image under f, for each b ∈ Y.

$$\forall y \in F, \exists x \in E: y = f(x).$$

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Examples

1. Let X be a nonempty set. Then the identity map on X is surjective.
2. <u>$f: R^+ \to R^+$</u> defined by $f(x) = x^2$ <u>is surjective</u>
3. <u>$f: R \to R$</u> defined by $f(x) = x^2$ is <u>not surjective</u>: for $y = -1$ there is no x

Definition (bijection) Let X and Y be sets. A function $f : X \to Y$ is said to be bijective (also call a bijection) if f is both injective and surjective. This is equivalent to: for all y ∈ F there exists a unique x ∈ E such that y = f (x). In other words:

$$\forall y \in F, \exists!\, x \in E\ y = f(x).$$

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Examples

1. The function $f : \{1, 2, 3\} \rightarrow \{a, b, c\}$ defined by f(1) = c, f(2) = b and f(3) = a, is a bijection. Thus, f −1 : {a, b, c} → {1, 2, 3} is a bijection

3. Let X be a nonempty set. Then the identity map: $f(x) = x$ on X is a bijection.

4. Let f be the map defined by $f(x) = x - 7$ from Z → Z, then f is bijective. Indeed, let y ∈ Z, such that f (x) = y, then x = y + 7, so there exists a unique x in Z such that y = f (x).