Chapter I

Methods of Mathematical Reasoning

Contents

1 Methods of Mathematical Reasoning

1.1 Mathematical Logic

تمهيد : يهدف هذا الدرس إلى تقديم قواعد الاستدلال الرياضي، وهي أنماط البرهان المستعملة لحل مسائل رياضية. إن ذلك يتطلب عرض المفاهيم الأساسية للمنطق الرياضي، وهي ضرورية لبناء أنماط البرهان.

1.1.1 Assertions (Statement) القضية او العبارة

An assertion is a sentence that can be true or false and cannot be both at the same time.

القضية المنطقية:نسمي قضية منطقية كل نص يمكن الحكم عليه ودون غموض صحيح أو خاطئ. ترميز:نرمز للقضايا بالحروف P،Q، R، ...
قيمة صدق القضية: نرفق بكل قضية قيمة صدق هي **1** إذا كانت صادقة (صحيحة) و **0** إذا كانت خاطئة.ونرمز لذلك ب

Example 1.1.1.

a) 2 + 2 = 4 is a true statement.
b) 3 × 2 = 7 is a false statement.
c) For all x ∈ R we have $x^2 \geq 0$ is a true statement.
d) For all x ∈ R we have $|x| = 1$ is a false assertion.

1 -المربع هو مستطيل (<u>قضية</u>)
2-ما ثمن هذا الكتاب ؟ (<u>ليست قضية</u>)
3- المربع ليس مستطيلا (<u>قضية وهي نفي القضية 1</u>)
4 - كل المثلثات قائمة (<u>قضية</u>)
5-$2<3$ (<u>قضية</u>)
6- $2 \geq 3$(<u>قضية وهي نفي القضية 5</u>)

1.1.2 Mathematical Logical Operators (الروابط المنطقية)

If P is an assertion and Q is another assertion, we will define new assertions constructed from P and Q

➢ *The logical operator of Conjunction "and " ∩* (رابط التقاطع او الوصل "و")

The conjunction of the statements P and Q is the statement "P and Q" and it's denoted by $P \cap Q$. The statement $P \cap Q$ is true only when both P and Q are true (P is true and Q is true).

1-الوصل "و" : لتكن P و Q قضيتين. نسمي الوصل بين P و Q، القضية الجديدة والتي نرمز لها بـ $P \wedge Q$ (نقرأ P و Q) والتي تكون صادقة في الحالة الوحيدة: لما يكون: P **و** Q صادقتين معا.
مثلا: $(4=2^2) \wedge (\sqrt{2} \in \mathbb{Q})$ قضية خاطئة.

We summarize this in a truth table: جدول قيمة صدق القضية

P	Q	$P \cap Q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 1.1.2.

a) $(3 + 5 = 8) \wedge (3 \times 6 = 18)$ is a true statement.
b) $(2 + 2 = 4) \wedge (2 \times 3 = 7)$ is a false statement.

- ***The logical operator of Disjunction "or"*** $(\cup)$ **("أو" عامـل الاتحـاد او الفصل)**

The disjunction of the statements P and Q is the statement "P or Q" and it's denoted by $P \cup Q$. The statement $P \cup Q$ is true only when at least one of P or Q is true.

2- الفصل " أو" : لتكن P و Q قضيتين. نسمي **الفصل** بين P و Q، القضية الجديدة والتي نرمز لها بـ $P \vee Q$ (نقرأ P أو Q) والتي تكون خاطئة في الحالة الوحيدة: لما يكون Q و P خاطئتين معا.
مثلا: $(4 = 2^2) \vee (\sqrt{2} \in \mathbb{Q})$ قضية صادقة.

P	Q	$P \cup Q$
T	T	T
T	F	T
F	T	T
F	F	F

Example 1.1.3

a. $(2 + 2 = 4) \vee (3 \times 2 = 6)$ is a true statement.
b. $(2 = 4) \vee (4 \times 3 = 7)$ is a false assertion.

- ***The implication or conditional "⇒"*** **(الإستلـزام او الشـرط)**

The implication or conditional is the statement "If P then Q" .... فإن .... اذا كان and is denoted by $P \Rightarrow Q$. The statement $P \Rightarrow Q$ is often read as "P implies Q,

٠ **الاستلزام :** **تعريف 1:** نسمي استلزاما كل قضية من الشكل: إذا كان P فإن Q.

تعريف 2: نسمي استلزاما بين P و Q أو نقول P يستلزم Q ونكتب $P \Rightarrow Q$، القضية $\bar{P} \vee Q$.

For this conditional statement, **P** is called the **hypothesis (الفرضية او المقدمة)** and **Q** is called the **conclusion (الاستنتاج او الخلاصة)** .

The conditional statement $P \Rightarrow Q$ means that Q is true whenever P is true.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Example 1.1.4.

Suppose that I say "$\underbrace{If\ it\ is\ not\ raining}_{P}, \underbrace{then}_{\Rightarrow} \underbrace{Daisy\ is\ riding\ her\ bike}_{Q}$."

We can represent this conditional statement a $P \Rightarrow Q$ s where :
P = "It is not raining" and **Q** = "Daisy is riding her bike."

We will now check the truth value of $P \Rightarrow Q$ based on the truth values of P and Q.

1. Suppose that both P and Q are true. That is, it is not raining and Daisy is riding her bike. In this case, it seems reasonable to say that I told the truth and that $P \Rightarrow Q$ is true.

2. Suppose that P is true and Q is false: it is not raining and Daisy is not riding her bike. It would appear that by making the statement, "If it is not raining, then Daisy is riding her bike," that I have not told the truth. So in this case, the statement $P \Rightarrow Q$ is false.

3. Now suppose that P is false and Q is true or that it is raining and Daisy is riding her bike. Did I make a false statement? The key is that I did not make any statement about what would happen if it was raining, and so I did not tell a lie. So we consider the conditional statement, "If it is not raining, then Daisy is riding her bike," to be true in the case where it is raining and Daisy is riding her bike.

4. Finally, suppose that both P and Q are false. That is, it is raining and Daisy is not riding her bike. As in the previous situation, since my statement was $P \Rightarrow Q$, I made no claim about what would happen if it was raining, and so I did not tell a lie. So, the statement $P \Rightarrow Q$ cannot be false in this case and so we consider it to be true.

- ***Equivalence (~ or ⇔)*** *(التكافـؤ المنطقي)*

Equivalence is defined by the assertion (P ⇒ Q) and (Q ⇒ P), it is denoted P ⇔ Q. We read (P is equivalent to Q) or (P if and only if Q). This assertion is true when P and Q are true simultaneously or when P and Q are false simultaneously. Its truth table is a following.

- التكافؤ المنطقي : نقول أن القضيتين P و Q متكافئتان ونكتب $Q \Leftrightarrow P$ للتعبير عن القضية: $(P \Leftarrow Q) \wedge (Q \Leftarrow P)$
منه يكون التكافؤ صادقا إذا كانت للقضيتين P و Q نفس قيمة الصدق.

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Example 1.1.5.

For x, y ∈ R, the equivalence "xy = 0 ⇔ x = 0 or y = 0" is true

- ***The negation*** **Not ($\overline{\ \ }$)** النفـــي

The negation of the statement P is the statement "not P" and is denoted by $\overline{P}$. The negation of P is true only when P is false, and $\overline{P}$ is false only when P is true.

P	$\overline{P}$
T	F
F	T

Example 1.1.6.

The negation of the assertion 3 ≥ 0 is the assertion 3 < 0

Theorem (De Morgan's Laws)

For statements P and Q,

- The statement: $\overline{P \cap Q}$ is logically equivalent to $\overline{P} \cup \overline{Q}$. This can be written as: $\overline{P \cap Q} \Leftrightarrow \overline{P} \cup \overline{Q}$

- The statement: $\overline{P \cup Q}$ is logically equivalent to $\overline{P} \cap \overline{Q}$. This can be written as: $\overline{P \cup Q} \Leftrightarrow \overline{P} \cap \overline{Q}$

P	Q	$P \vee Q$	$\neg(P \vee Q)$	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Table 2.3: Truth Table for One of De Morgan's Laws

Proposition 1.1.1.

Let P, Q and R be three assertions. We have the following equivalences:

1. $P \Leftrightarrow \overline{(\overline{P})}$
2. $(P \cap Q) \Leftrightarrow (Q \cap P)$
3. $(P \cup Q) \Leftrightarrow (Q \cup P)$
4. $\overline{P \cap Q} \Leftrightarrow \overline{P} \cup \overline{Q}$
5. $\overline{P \cup Q} \Leftrightarrow \overline{P} \cap \overline{Q}$
6. $P \cap (Q \cup R) \Leftrightarrow (P \cap Q) \cup (P \cap R)$
7. $P \cup (Q \cap R) \Leftrightarrow (P \cup Q) \cap (P \cup R)$
8. $(P \Rightarrow Q) \Leftrightarrow (\overline{Q} \Rightarrow \overline{P})$

Operator	Symbolic Form	Summary of Truth Values
Conjunction	$P \cap Q$	True only when both P and Q are true
Disjunction	$P \cup Q$	False only when both P and Q are false
Negation :P	$\overline{P}$	Opposite truth value of P
Conditional (implication)	$P \rightarrow Q$	False only when P is true and Q is false
Equivalence	$P \leftrightarrow Q$	True when: both P and Q are true Or both are false

1.1.3 Quantifiers

Are operators used to express the quantity in math: such us, all and some.

➢ ***The Universal Quantifier*** $\forall$ (المكمم الكلي)

The phrase "for every x" (sometimes "for all x") is called a **universal quantifier** and is denoted by **∀x**.

The assertion ∀x ∈ E, P (x) is a true assertion when the assertions P(x) are true for all elements x of the set E.

We read: for all x belonging to E, P (x) is.

∀	For all x, for every x, for each

A sentence ∀x P(x) is true if and only if P(x) is true no matter what value (from the universe of discourse) is substituted for x.

Example 1.1.7

- $\forall x\ (x^2 \geq 0)$, i.e., the square of any number is not negative.'
- $\forall x, \forall y\ (x + y = y + x)$, i.e., the commutative law of addition.
- $\forall x, \forall y, \forall z((x + y) + z = x + (y + z))$, i.e., the associative law of addition.
- $\forall x\ \in\ R, x^2\ \geq\ 1$ is a false assertion.

➢ *The Existential Quantifier* (∃) (المكمم الوجودي)

The phrase "there exists an x such that" is called an **existential quantifier** and is denoted by ∃x.

A sentence '$\exists x\ P(x)'$ is true if and only if there is at least one value of x (from the universe of discourse) that makes $P(x)$ true.

We read there exists x belonging to E such that P (x) (be true)

∃	For some x, there exists, there is a, there is at least one, there exists an x such that

Example 1.1.8

- $\exists x\ (x \geq x^2)$ is true since x = 0 is a solution. There are many others.
- $\exists x,\ \exists y(x^2 + y^2\ = 2xy)$ is true since x = y = 1 is one of many solutions.
- ∃x ∈ R, $x^2\ \leq\ 0$ is true since x = 0.
- ∃x ∈ R, $x^2\ <\ 0$ est false

➢ *The negation of quantifiers*

Consider the statement: 'All students in this class have red hair'.

What is required to show that the statement is false?
To show that the statement is false we need to show that there is at least one who does not have red hair.

Negation: There exists a student in this class that does not have red hair.

To negate a universal quantification: $\forall x\, P(x)$

1. Negate the proposition P(x): $\overline{P(x)}$
2. Change the universal quantifier to an existential quantifier $\overline{\forall x} = \exists x$

$$negation\ of\ \left(\overline{\forall x \in E, P(x)}\right) = (\exists x \in E, \overline{P(x)})$$

Consider the statement: ‘there is a student in this class with red hair’.

What is required to show that the statement is false?
To show that the statement is false we need to show that all the students in the class don’t have red hair.

Negation: All student in the class don’t have red hair

To negate an existential quantification

1. Negate the proposition (statement) P(x): $\overline{P(x)}$
2. Change to a universal quantification. $\overline{\exists x} = \forall x$

$$negation\ of\ \left(\overline{\exists x \in E, P(x)}\right) = (\forall x \in E, \overline{P(x)})$$

Example 1.1.9

a. $(\forall x \in R, \exists y \in R: 2x + y = 2)$ its negation is:
$$\exists x \in R, \forall y \in R: 2x + y \neq 2.$$

b. $(\exists x \in R, \forall y \in R: (x + y = 1)$ and $(2xy \leq 1))$ its negation is:
$$\forall x \in R, \exists y \in R: (x + y \neq 1)\ or\ (2xy > 1)$$

c. $(\forall x \in R, \exists y \in R, \forall z \in R: x + y \geq z^2)$ its negation is:
$$\exists x \in R, \forall y \in R, \exists z \in R: x + y < z^2$$

Example : 1.1.10

$$the\ negation\ of\ \left(\forall x \in R:\ \underbrace{x^2 \geq 0}_{P(x)}\right)\ is\ \exists x \in R:\ \underbrace{x^2 < 0}_{\overline{P(x)}}$$

$$The\ negation\ of\ (\exists x \in R:\ \underbrace{x < 0}_{P(x)}\ \ is\ \forall x \in R:\ \underbrace{x \geq 0}_{\overline{P(x)}}$$

- *Mixed quantifiers* توظيف العديد من الفئات الكمية

We can combine several quantifiers in a statement. Their **arrangements must not be changed** if they are of a different nature.

يمكننا الجمع بين عدة محددات كمية في عبارة واحدة لكن يجب عدم تغيير ترتيباتها إذا كانت ذات طبيعة مختلفة.

Example 1.1.11

a. $(\forall \boldsymbol{x} \in \boldsymbol{R}, \exists \boldsymbol{y} \in R: 2x + y = 2)$ and $(\exists \boldsymbol{y} \in \boldsymbol{R}, \forall \boldsymbol{x} \in R: 2x + y = 2)$ are two different propositions.

b. $(\exists x \in R, \exists y \in R: 2x + y = 2)$ and $(\exists y \in R, \exists x \in R: 2x + y = 2)$ are two equivalent quantified propositions.

1.2 Reasoning (proofs) قواعد الاستدلال الرياضي

في الرياضيات، البرهان أو الإثبات هي حجة استدلالية لتحديد صحة عبارة رياضية تستند على مُسلَّمات (Axioms) ومبرهنات (Theorems).

نفرض ان P وQ قضيتان منطقيتان توجد العديد من طرق الاستدلال الرياضي لاثبات صحة او خطأ القضية $P \rightarrow Q$

1.2.1 Direct Proof (الاستدلال المباشر)

This is the most straightforward type of proof. We want to proof that the assertation: $P \Rightarrow Q$ is true. We Assume P is true, and then use the rules of inference, axioms, definitions, and logical equivalences to prove Q. In a direct proof, each step follows logically from the previous one, ultimately demonstrating that the theorem or statement is true.

هذا هو ابسط انواع الإستدلالات. نريد أن نثبت أن $P \Rightarrow Q$ صحيح. نحن نفترض أن P صحيح، ثم نستخدم قواعد الاستدلال والبديهيات والتعريفات والمعادلات المنطقية لإثبات Q.

Example 1.2.1.

Let a, b ∈ R. Show that $a = b \Rightarrow \frac{a+b}{2} = b$.

Let's take $a = b$, then $a + b = b + b$, so $a + b = 2b$. So $\frac{a+b}{2} = b$.

Example 1.2.2

The sum of two odd numbers is even.

Assume m and n are odd numbers. Because m and n are odd there are integers j and k such that: $m = 2j - 1$ and $n = 2k - 1$).

Now $m + n = (2j - 1) + (2k - 1) = 2(j + k - 1)$. $Let\ i = j + k - 1$ then $m + n = 2i$ is even (by definition).

It is sometimes difficult to construct a direct proof of a conditional statement.

إن الاتجاه الطبيعي لإثبات هذا الاستلزام أي الانطلاق من الفرضيات والوصول إلى المطلوب أو ما يسمى البرهان المباشر ليس سهلا دائما لأنه يتطلب بناء نتائج وسطى للوصول إلى المطلوب، إن هذه الصعوبة، دفعت الإنسان إلى التفكير في أنماط أخرى للبرهان. إن التكافؤ الأول في النتيجة السابقة يقدم أولى هذه القواعد في الاستدلال الرياضي وهي:

1.2.2 Proof by Contrapositive البرهان بالعكس النقيض

It is based on the following equivalence:

$$\underbrace{(P \Rightarrow Q)}_{statment} \Leftrightarrow \underbrace{\overline{Q} \Rightarrow \overline{P}}_{contrapositive}$$

$\underbrace{\overline{Q} \Rightarrow \overline{P}}_{contrapositive}$ (Non Q) ⇒ (Non P) is called the contrapositive of P ⇒ Q.

Any sentence and its contrapositive are logically equivalent. In a contrapositive proof, you prove the contrapositive of a given statement instead of the statement itself. *If the contrapositive is true, then the original statement must also be true*.

أي جملة ومضادها متكافئان منطقيا. في البرهان بالعكس النقيض، نتثبت العكس النقيض لعبارة معينة بدلاً من العبارة نفسها. إذا كانت صحيحة، فيجب أن يكون البيان الأصلي صحيحا أيضا.

This type of proof is often used when both the hypothesis and the conclusion are stated in the form of negations.

البرهان بالعكس النقيض: لإثبات صحة الاستلزام $P \Rightarrow Q$ يكفي إثبات صحة الاستلزام $\overline{Q} \Rightarrow \overline{P}$.

Example 1.2.2

Let x ∈ R. show that $\underbrace{(x \neq 2 \text{ et } x \neq -2)}_{P} \Rightarrow \underbrace{(x^2 \neq 4)}_{Q}$

By contraposition this is equivalent to

$$\underbrace{(x^2 = 4)}_{Q} \Rightarrow \underbrace{(x = 2 \text{ ou } x = -2)}_{P}$$

Indeed, let $x^2 = 4$, then $(x - 2)(x + 2) = 0$, $therefore\ x = 2\ or\ x = -2$

Example 1.2.3

$$If\ \underbrace{n > 0\ and\ 4^n - 1 is\ prime}_{P}, then\ \underbrace{n\ is\ odd}_{Q}.$$

Assume that $\bar{Q}$: n is even. $n = 2k$. Then $4^n - 1 = 4^{2k} - 1 = (4^k - 1)(4^k + 1)$
Therefore, $4^n - 1$ factors (are both factors bigger than 1?) and hence <u>is not prime</u> ($\bar{P}$).

So, the statement: $(If\ n > 0\ and\ 4^n - 1 is\ prime\ , then\ n\ is\ odd)$ is true.

1.2.3 Proof by Contradiction بالنقيـــض او بالخلف

Assume $P\ and\ \overline{Q}$ (you assume the statement is false) and work forward from these two assumptions until a contradiction is obtained. Since a false assumption leads to a contradiction, the original statement must be true. This type of proof is often used when the conclusion is stated in the form of a negation, but the hypothesis is not.

غالبًا ما يستخدم هذا النوع من الإثبات عندما يتم ذكر النتيجة في شكل نفي، ولكن الفرضية ليست كذلك. نفترض ان P صحيحة و Q خاطئة أي $\overline{Q}$ صحيحة وننطلق من هاتين الفرضيتين حتى يتم الحصول على التناقض قد يكون تناقضا مع قواعد الرياضيات او مع الفرضيات.

Example 1.2.4

Exemple 1.2.3. *Soient* $a, b > 0$. *Montrer que si* $\frac{a}{1+b} = \frac{b}{1+a} \Rightarrow a = b$.
Nous raisonnons par l'absurde en supposant que

$$\frac{a}{1+b} = \frac{b}{1+a} \quad et \quad a \neq b.$$

On a

$$\left(\frac{a}{1+b} = \frac{b}{1+a}\right) \Leftrightarrow a(a+1) = b(b+1)$$
$$\Leftrightarrow a^2 - b^2 = -(a-b)$$
$$\Leftrightarrow (a-b)(a+b) = -(a-b)$$

Ceci est équivalent

$$(a-b)(a+b) = -(a-b) \quad et \quad a - b \neq 0.$$

donc en divisant par $a - b$ *on obtient*

$$a + b = -1.$$

La somme de deux nombres positifs ne peut être négative. Nous obtenons une contradiction.

1.2.4 Reasoning by Producing a Counterexample المثال المضاد

If we want to show that an assertion of the type $(\forall x \in E : P(x))$ is true then for each x of E we must show that P (x) is true. On the other hand, to show that this assertion is false then it is sufficient to find x ∈ E such that P (x) is false.

Definition1.2.1 Let P be a proposition depending on x in E. To show that the proposition (∀x ∈ E, P(x)) is false, it is enough to find an x of such that P(x) is false. Finding such x means finding a counterexample to the assertion (∀ x ∈ E, P(x))

البرهان بمثال مضاد: إن المثال عادة لا يمكن أن يكون برهانا، إنه يستخدم للتوضيح.
يكون المثال نمطا من أنماط البرهان في الحالة الوحيدة الآتية: وهي لإثبات أن نصا ما ليس صحيحا على العموم أي لا يملك صفة النظرية (أو المبرهنة) يكفي تقديم مثال يبين عدم صحته. يسمى المثال هنا مثالا مضادا.

Example 1.2.5

Show that the assertion ($\forall x \in R,\ x^2 - 1 > 1$) is false.
P: "$\forall x \in R, x^2 - 1 \neq 0$". We have for $x = 1,\ x^2 - 1 = 0$. So P is false.

Example 1.2.6 Show That $\forall x \in N, x + 1 \neq 0$.

We assume that P is false: $\exists x \in N, x + 1 = 0$ therefore $x = -1$ and $x \in \mathbb{N}$. Contradiction with the definition of $\mathbb{N}$.

1.2.5 Case-by-Case Reasoning

refers to a method of argument where you examine different scenarios or cases individually to prove or analyze a more general statement. It involves breaking down a problem into distinct cases and addressing each one separately to establish a conclusion or solution.

يشير إلى طريقة حجة حيث تقوم بفحص سيناريوهات أو حالات مختلفة بشكل فردي لإثبات أو تحليل بيان أكثر عمومية. وهو يتضمن تقسيم المشكلة إلى حالات متميزة ومعالجة كل حالة على حدة للتوصل إلى نتيجة أو حل.

Exemple 1.2.4. *Soit $n \in \mathbb{N}$. Montrer que $a_n = \frac{1}{2}n(n+1) \in \mathbb{N}$.*
Soit $n \in \mathbb{N}$. Nous distinguons deux cas.
Premier cas : *$n = 2k,\ k \in \mathbb{N}$, alors*

$$a_n = \frac{1}{2}n(n+1) = k(2k+1) \in \mathbb{N}.$$

Deuxième cas : *$n = 2k+1,\ k \in \mathbb{N}$, alors*

$$a_n = \frac{1}{2}(2k+1)(2k+2) = (2k+1)(k+1) \in \mathbb{N}.$$

Dans tous les cas $a_n = \frac{1}{2}n(n+1) \in \mathbb{N}$.

1.2.6 Proof by Induction (Recursive Reasoning) البرهـــان بالتراجـــع

It is a way to prove that something is true for a sequence of numbers.

- Step 1: prove that the statement is true for the first number in the sequence n_0.
- Step2: assume that the statement is true for n
- Step 3: it's also true for the next number in the sequence.

Example 1.2.6

Prove that for all positive integers n, the sum of the first n positive integers is given by the formula: $1+2+3+4\ldots+n \ = \frac{(1+n)n}{2}$

We'll use mathematical induction to prove this statement.

Step 1: for n = 1, we have 1=1+1/2 = 1 which is true.

Step 2 : Induction Hypothesis. Assume that the statement is true for n

$$1+2+3+4\ldots+n \ = \frac{(1+n)n}{2}$$

Step 3 prove it is true for n+1: $1+2+3+4\ldots+n+(n+1) \ = \frac{(2+n)(n+1)}{2}$

$$1+2+3+4\ldots+n+(n+1) = \frac{(1+n)n}{2}+n+1 = \frac{(1+n)n+2(n+1)}{2}$$

we conclude, by induction, that for all positive integers n: $1+2+3+4\ldots+n \ = \frac{(1+n)n}{2}$

نصائح وتوجيهات : إن معالجة مسألة رياضية يتطلب بعض القواعد البسيطة فالتقديم الجيد للحل والوضوح هو أساس المعالجة. ومن بين الملاحظات وما أكثرها نقدم الملاحظات الآتية:

أ- توضيح الفرضيات على حده والمطلوب إثباته على حده. **ب-**استعمال المكممات والروابط المنطقية كلما أتيحت الفرصة.

ج- إن اللغة مفتاح لتوضيح ما نقدم ولهذا من الضروري تقديم الحل بلغة واضحة واستعمال جيد للمصطلحات من نوع "بما أن"، "إذن"، "إذا وفقط إذا"، "وعليه" ... إن الحل لا يمكن أن يقدم كطلاسم من رموز وحروف، بل يجب أن يعرض بلغة واضحة.