Chapter 2

Kinematics of material point

2.1 Introduction

- Kinematics is a branch of mechanics that studies the movement of particles without taking into account the causes of this movement.
- On the other hand, kinematics does not care about forces that cause particle movement.

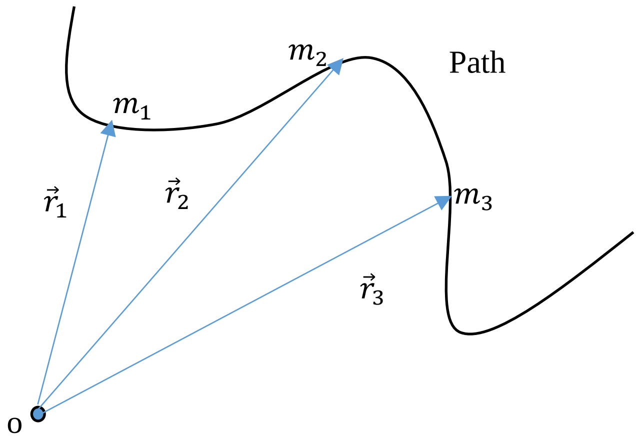
2. 2 Variables of Motion

Kinematics aims to provide a description of variables of motion (motion elements):

- the spatial position of particles,
- velocity, and
- acceleration.

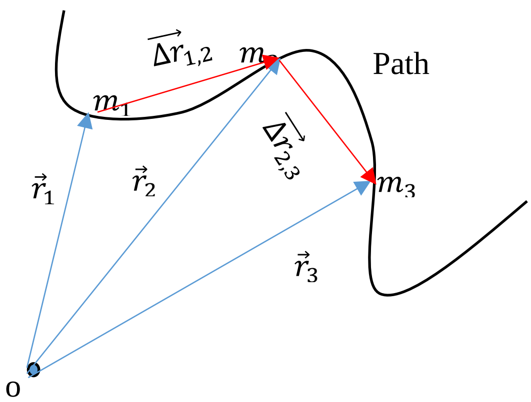
2. 2. 1 Vector position

- Vector position is a vector, which connect the origin of reference to the position of the material point in a given time.



2. 2. 2 Displacement vector

We define the displacement vector from the point $m_1$to the point m2 as the difference between the position vector m_2 and the position vector $\vec{r}_1$. We can expect that the displacement vector will always be less than or equal to the real distance traveled by the material path.



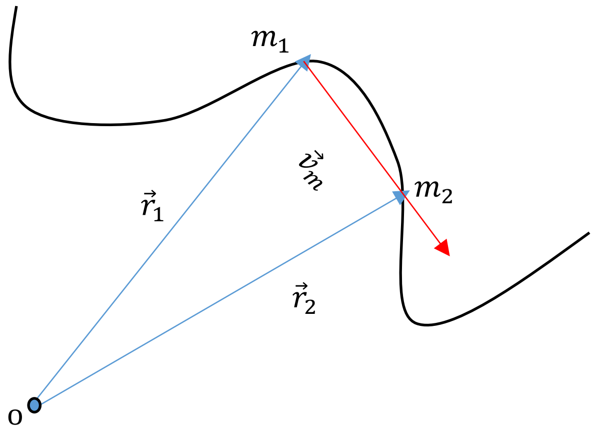
2.2. 3 Velocity

- **Average velocity :** We define the average velocity of material point during the time interval Δt as the displacement vector of the particle divided by that time interval,

  $\vec{v}_m = \frac{\Delta \vec{r}}{\Delta t}$

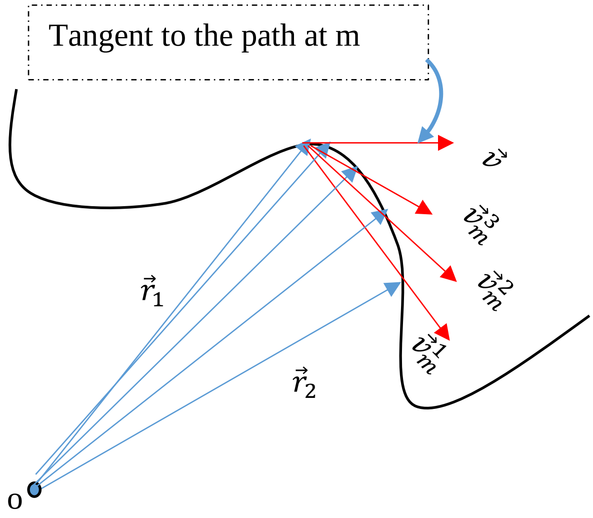
Note that:

- The direction of average velocity is same as the direction vector displacement.
- The magnitude of the average velocity is called **speed**.



- **Instantaneous velocity :**

Instantaneous velocity is the speed at a defined point. We can start from the definition of the average velocity between two points and each time we bring the point m_2 closer to the point m1 until m_2 coincides with m1. This means that the traveling time from point m1 to point m2 is too small. It is easy to see that the instantaneous velocity vector is tangent to the trajectory at this point. So,



$\vec{v} = \lim_{t\to 0} \vec{v}_m$, $\vec{v} = \lim_{t\to 0} \frac{\Delta \vec{r}}{\Delta t}$

$\vec{v} = \frac{\overrightarrow{dr}}{dt}$

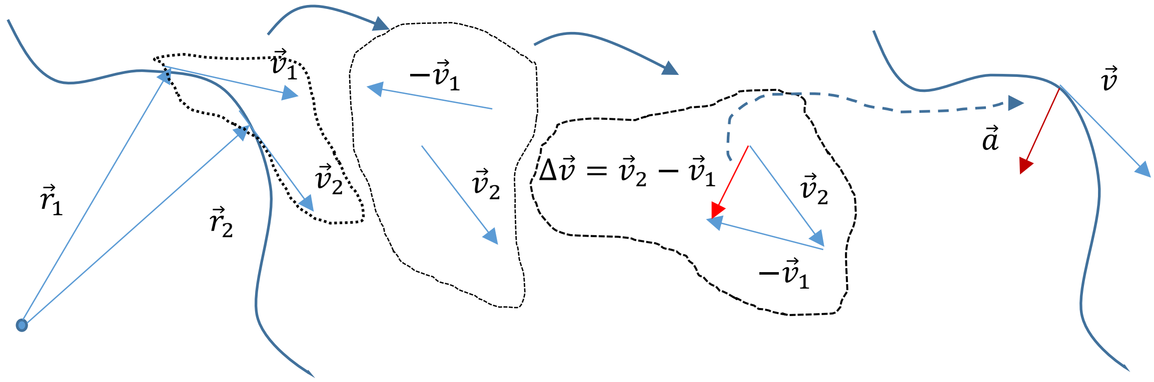
2.2. 4 Acceleration

- **Average Acceleration:** We define the average acceleration of material point during the time interval Δt as the difference between the velocity vectors of the particle divided by that time interval,

  $\vec{\gamma}_m = \frac{\Delta \vec{v}}{\Delta t}$

Note that:

- The average acceleration is a vector quantity directed along $\Delta \vec{v}$.

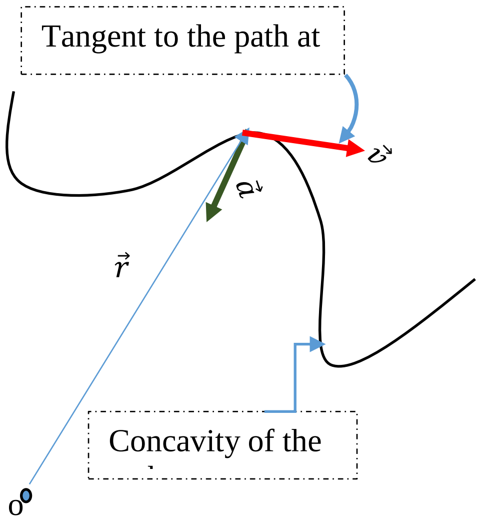


- **Instantaneous acceleration :**

Instantaneous acceleration is the acceleration at a defined point. The instantaneous acceleration $\vec{a}$ is defined as the limiting value of the ratio $\frac{\Delta\vec{v}}{\Delta t}$ when Δt approaches to the zero. So,

$\vec{a} = \lim_{t\to 0}\vec{\gamma}_m$, $\vec{a} = \lim_{t\to 0}\frac{\Delta\vec{v}}{\Delta t}$

$\vec{a} = \frac{d\vec{v}}{dt}$



2. 3. Variables (Elements) of motion in the different references.

2. 3. 1 Cartesian coordinate system.

2. 3. 2. Vector position

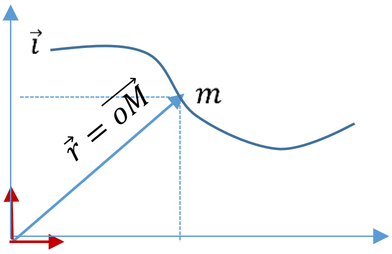
If the particle moves on one axis (ox)

$$\overrightarrow{OM} = \|\overrightarrow{OM}\| \times \vec{\imath}$$

$$\vec{r} = x(t)\vec{\imath}$$

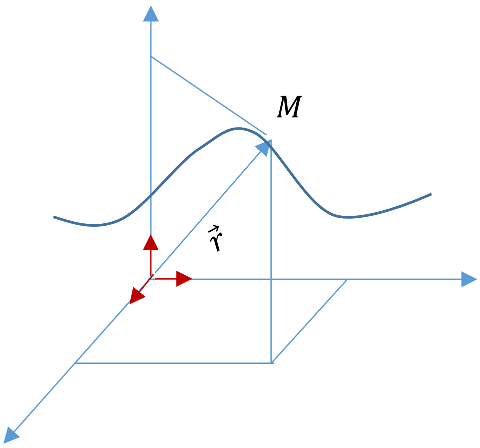
❖ If the particle moves on the plane, the vector position is given by its two coordinates,

$\vec{r} = x(t)\vec{\imath} + y(t)\vec{\imath}$



❖ Vector in space can be written with respect to its three coordinates as;

$$\overrightarrow{OM} = \quad x(t)\vec{\imath} + y(t)\vec{\jmath} + z(t)\vec{k}$$



❖ The coordinates $x(t), y(t), and\ z(t)$ are called the time equations of the movement.

❖ The relation between the coordinates $x, y\ and\ z$ is called the trajectory equation.

2. 3. 3. Velocity

The vector velocity is the derivative of the position vector with respect to time,

$$\frac{d\vec{r}}{dt} = \frac{dx(t)}{dt}\vec{\imath} + \frac{dy(t)}{dt}\vec{\jmath} + \frac{dz(t)}{dt}\vec{k}$$

We represent the derivative with respect to time by dot (point) upon the function $\frac{dx(t)}{dt} = \dot{x}$ and then we can write

$\vec{v} = \frac{d\vec{r}}{dt} = \dot{x}\vec{\imath} + \dot{y}\vec{\jmath} + \dot{z}\vec{k}$

$$\|\vec{v}\| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$$

2. 3. 4. Acceleration

$\vec{a} = \frac{d\vec{v}}{dt} = \ddot{x}\vec{\imath} + \ddot{y}\vec{\jmath} + \ddot{z}\vec{k}$, the magnitude of the acceleration is $\|\vec{a}\| = \sqrt{\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2}$.

Notes:

$$If \begin{cases} \vec{v}.\vec{a} > 0, & the\ movement\ is\ accelerated \\ \vec{v}.\vec{a} < 0, & the\ movement\ is\ deccelerated. \\ \vec{v}.\vec{a} = 0, & the\ movement\ is\ uniforme \end{cases}$$

Movement with a constant acceleration

We have $a = constante$,

$$\frac{dv}{dt} = a \rightarrow dv = adt \rightarrow \int_{v_0}^{v} dv = a \int_{t_0}^{t} dt$$

$$v = a(t - t_0) + v_0$$

$$\frac{dx}{dt} = v \rightarrow dx = vdt \rightarrow \int_{x_0}^{x} dx = \int_{t_0}^{t} vdt$$

$$\int_{x_0}^{x} dx = \int_{t_0}^{t} (a(t - t_0) + v_0)dt$$

$$x = \frac{1}{2} a(t - t_0)^2 + v_0(t - t_0) + x_0$$

Useful relation

We have

$$dv = adt$$

Multiplying the two terms by v we obtain,

$$vdv = avdt$$

Also, we have

$$v = \frac{dx}{dt}$$

By replacement,

$$vdv = a\frac{dx}{dt}dt$$

$$vdv = adx$$

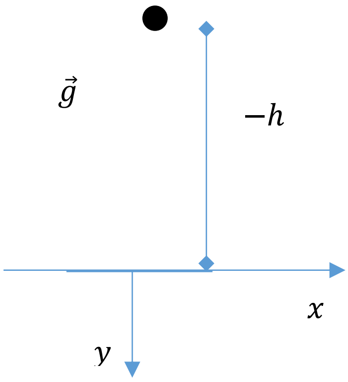
$$\int_{v_0}^{v} vdv = a\int_{x_0}^{x} dx$$

$$(v^2 - v_0^2) = 2a(x - x_0)$$

Example 1:

Let we study an object which falling freely from a high h,

The movement is one-dimensional under the acceleration of the gravity of earth



$a = g = constante,\ g = \frac{dv}{dt} \rightarrow dv = gdt;\ \int_{v_0}^{v} dv = g\int_{t_0}^{t} dt$

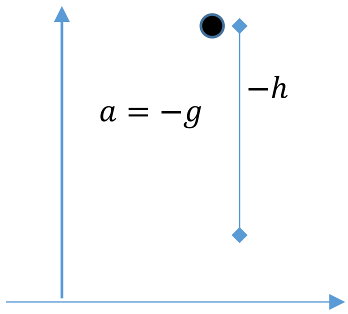
$(v - v_0) = g(t - t_o)$, At the initial time $t = t_0 \begin{cases} t_0 = 0 \\ v_0 = 0 \end{cases}$,

$$v = gt$$

$(y - y_0) = \frac{1}{2}a(t^2 - t_0^2).\ t = t_0 \begin{cases} t_0 = 0 \\ y_0 = -h \end{cases}$'

$y - (-h) = \frac{1}{2}at^2,\qquad \rightarrow y = \frac{1}{2}at^2 - h$

--------------------- Case when the positive direction is up ------------------------------------



$a = -g = constante,\ a = \frac{dv}{dt} \rightarrow dv = adt;\ \int_{v_0}^{v} dv = -g\int_{t_0}^{t} dt$

$(v - v_0) = -g(t - t_o)$, At the initial time $t = t_0 \begin{cases} t_0 = 0 \\ v_0 = 0 \end{cases}$'

$v = at$…………

$(y - y_0) = -\frac{1}{2}g(t^2 - t_0^2).\ t = t_0 \begin{cases} t_0 = 0 \\ y_0 = h \end{cases}$'

$y - h = -\frac{1}{2}gt^2,\qquad \rightarrow y = -\frac{1}{2}gt^2 + h$

Example 2:

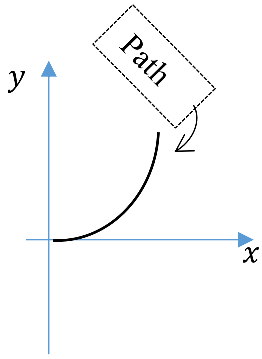
Consider the time equation of a material point,

$x(t) = \frac{1}{2}t^2,\ y(t) = t^2 - 1,\ z(t) = 0.$

$\vec{r} = \frac{1}{2}t^2\vec{\imath} + (t^2 - 1)\vec{\jmath}.$ Then $\vec{v} = t\vec{\imath} + 2t\vec{\jmath} \rightarrow \|\vec{v}\| = \sqrt{5}t$

$\vec{a} = \vec{\imath} + \vec{\jmath}$, then $\|\vec{a}\| = \sqrt{1+1} = 2$

$\vec{v}.\vec{a} = (\vec{\imath} + t\vec{\jmath}).(+\vec{\jmath}) = t > 0$, the movement is accelerated.



To find the trajectory (path) we replace t by x in the y time equation;

$t = x(t), \rightarrow\ y(t) = \frac{1}{2}x^2$. The path is a parabola.

Example 3

Let Consider a particle m moves on xy plane with a constant acceleration $\vec{a}\binom{0}{a}$ and at the initial time $t_0 = 0$ the vector velocity is given by $\vec{v}\begin{pmatrix} v_{0x} \\ v_{0y} \end{pmatrix}$ and the vector position is given by $\vec{r}\binom{0}{0}$.

$$\vec{a} = \frac{d\vec{v}}{dt} \rightarrow d\vec{v} = \vec{a}dt$$

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At time $t_0 \begin{cases} \vec{a} = a\vec{j} \\ \vec{v}_0 = v_{0x}\vec{i} + v_{0y}\vec{j} \end{cases}$"

$$\vec{a} = v_{0x}\vec{i} + v_{0y}\vec{j}$$

$$\vec{v}\begin{pmatrix} v_{0x} \\ at + v_{0y} \end{pmatrix} \rightarrow \vec{r}\begin{pmatrix} v_{0x}t \\ \frac{1}{2}at^2 + v_{0y}t \end{pmatrix}$$

If α is the angle that the velocity vector $\vec{v}_0$ makes with the x axis and v_0 is the norm of this velocity vector, we can still write $\vec{v}_0 = v_0\cos(\theta)\vec{i} + v_0\sin(\theta)\vec{j}$

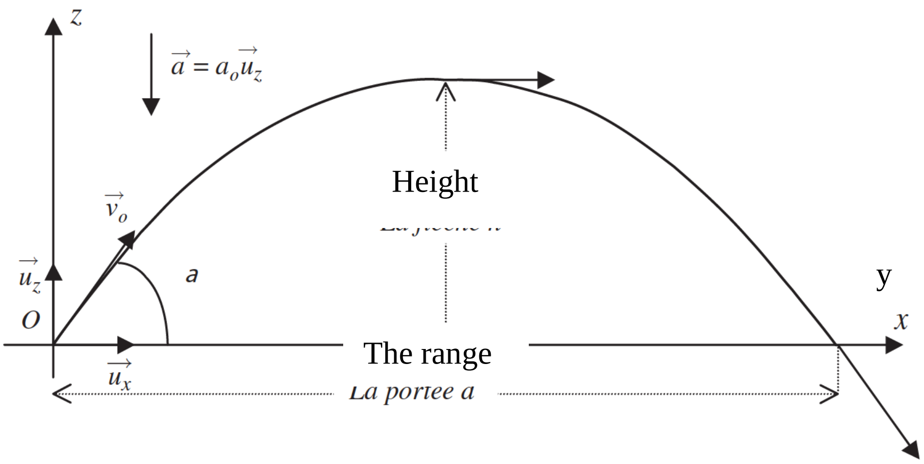
We can find the trajectory by writing the time expression from x(t) and replacing it in y(t) equation.

$$t = \frac{x}{v_0\cos(\theta)}$$

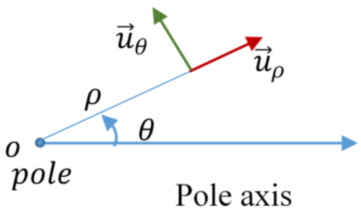
And then

$$y = \frac{1}{2}a\frac{x^2}{v_{0x}^2\cos^2(\alpha)} + tang(\alpha)x$$

If $\vec{a} = -g\vec{j}$, we find ourselves in the projectile case.



2. 1 Polar coordinate system



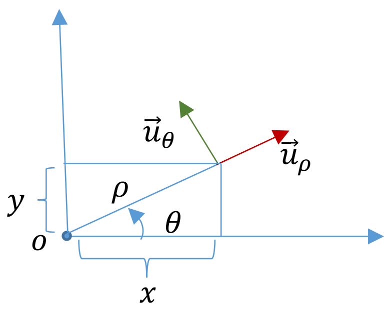
Polar coordinates are used to study the motion of a material point in two dimensions. The location of a point in this coordinates system is determined by its distance from a fixed point at the center of the coordinate space (called the pole) and by the measurement of the angle formed by a fixed line (the polar axis, corresponding to the x-axis in Cartesian coordinates) and a line from the pole through the given point.

The base of this coordinates system is $(\vec{u}_\rho, \vec{u}_\theta)$:

$\vec{u}_\rho$: is directed form the origin towards the material point position.

$\vec{u}_\theta$: is directed in the direction of increase of ɵ.

3. 1. 2. Vector position



The vector position is given by $\vec{r} = \overrightarrow{oM} = \rho\vec{u}_\rho$.

Relation between Cartesian and polar coordinates.

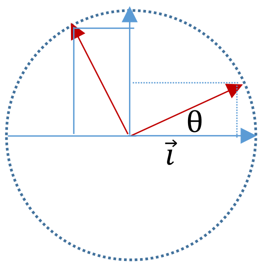
$\vec{r} = \rho\vec{u}_\rho = \rho\cos(\theta)\,\vec{\imath} + \rho\sin(\theta))\,\vec{\jmath}.$

$$\begin{cases} x = \rho\cos(\theta) \\ y = \rho\cos(\theta) \end{cases} \rightarrow \begin{cases} \rho = \sqrt{x^2+y^2} \\ tang(\theta) = \frac{y}{x} \end{cases}.$$

Relation between Cartesian and polar bases

$$\vec{u}_\rho = \cos(\theta)\,\vec{\imath} + \sin(\theta)\,\vec{\jmath}$$

$$\vec{u}_\theta = -\,\mathrm{sins}(\theta)\,\vec{\imath} + \cos(\theta)\vec{\jmath}$$



Also we can write

$$\vec{\imath} = \cos(\theta)\,\vec{u}_\rho - \sin(\theta)\,\vec{u}_\theta$$

$$\vec{\jmath} = \sin(\theta)\,\vec{u}_\rho + \cos(\theta)\,\vec{u}_\theta$$

3. 1. 3. Velocity

$$\vec{v} = \frac{d}{dt}\overrightarrow{OM} = \dot{\vec{r}} = \dot{\rho}\vec{u}_\rho + \rho\dot{\vec{u}}_\rho$$

$\vec{u}_\rho$ changes its direction with time, so it derivative with respect to time is not equal to zero.

$$\vec{u}_\rho = \cos(\theta)\,\vec{\imath} + \sin(\theta)\,\vec{\jmath} \Rightarrow \dot{\vec{u}}_\rho = \dot{\theta}\,(-\sin(\theta)\,\vec{\imath} + \cos(\theta)\vec{\jmath}) \;:\; \dot{\vec{u}}_\rho = \dot{\theta}\vec{u}_\theta$$

$$\vec{u}_\theta = -\sin(\theta)\,\vec{\imath} + \cos(\theta)\,\vec{\jmath} \rightrightarrows \dot{\vec{u}}_\theta = \dot{\theta}\,(-\cos(\theta)\,\vec{\imath} - \sin(\theta)\vec{\jmath}) \;:\; \dot{\vec{u}}_\theta = -\dot{\theta}\vec{u}_\rho$$

$$\vec{v} = \dot{\rho}\vec{u}_\rho - \rho\dot{\theta}\vec{u}_\theta$$

$$\vec{v} = v_\rho\vec{u}_\rho + v_\theta\vec{u}_\theta$$

$$\|\vec{v}\| = \sqrt{{v_\rho}^2 + {v_\theta}^2}$$

Acceleration

$$\vec{a} = \frac{d\vec{v}}{dt} = \ddot{\rho}\vec{u}_\rho + \dot{\rho}\dot{\theta}\vec{u}_\theta + \dot{\rho}\dot{\theta}\vec{u}_\theta + \rho\ddot{\theta}\vec{u}_\theta - \rho\dot{\theta}^2\vec{u}_\rho$$

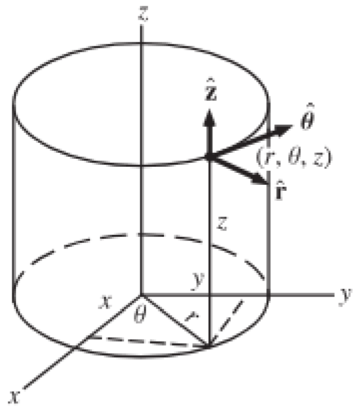
$$\vec{a} = \frac{d\vec{v}}{dt} = (\ddot{\rho} - \rho\dot{\theta}^2)\vec{u}_\rho + (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta})\vec{u}_\theta$$

$$\begin{cases} a_\rho = (\ddot{\rho} - \rho\dot{\theta}^2) \\ a_\theta = (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta}) \end{cases}$$

$$\|\vec{a}\| = \sqrt{a_\rho^2 + a_\theta^2}$$

Cylindrical coordinates

Cylindrical coordinate are three dimensions coordinates system used for motion which have cylindrical symmetry.



$$\overrightarrow{OM} = \vec{r} = \overrightarrow{Om} + \overrightarrow{mM}$$

$$\overrightarrow{OM} = \vec{r} = \rho\vec{u}_\rho + z\vec{k}$$

Velocity

$$\vec{v} = \dot{\rho}\vec{u}_\rho - \rho\dot{\theta}\vec{u}_\theta + \dot{z}\vec{k}$$

$$\vec{v} = v_\rho\vec{u}_\rho + v_\theta\vec{u}_\theta + v_z\vec{k}$$

$$\|\vec{v}\| = \sqrt{{v_\rho}^2 + {v_\theta}^2 + v_z^2}$$

Acceleration

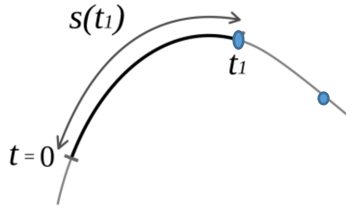
$$\vec{a} = \frac{d\vec{v}}{dt} = (\ddot{\rho} - \rho\dot{\theta}^2)\vec{u}_\rho + \left(2\rho\dot{\theta}\vec{u}_\theta + \rho\ddot{\theta}\right)\vec{u}_\theta + \ddot{z}\vec{k}$$

$$\begin{cases} a_\rho = \left(\ddot{\rho} - \rho\dot{\theta}^2\right) \\ a_\theta = \left(2\rho\dot{\theta}\vec{u}_\theta + \rho\ddot{\theta}\right) \\ a_z = \ddot{z} \end{cases}$$

$$\|\vec{a}\| = \sqrt{{a_\rho}^2 + {a_\theta}^2 + a_z^2}$$

Curvilinear coordinates and curvilinear abscissa

Consider a point m moves on a curvilinear path (fig). The distance between two point on the path s(t) (the length of arc between traveled between t_1 and t_2) is called curvilinear abscissa. s gives the position of the particle as measured by the displacement along the curve. As in the rectilinear case, s may be positive or negative, depending on which side of O the particle is.



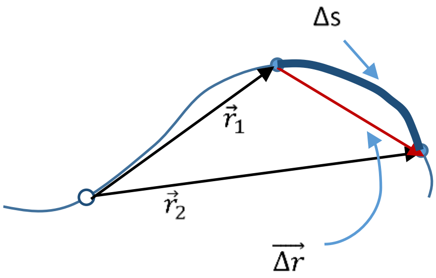
The vector position is $\vec{r} = \overrightarrow{oM}$

The displacement vector of point m is characterized by $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1 = \overrightarrow{OM}_2 - \overrightarrow{OM}_1$

Velocity vector

When the particle moves from M_1 to M_2, The length of arc AB represents the displacement Δs along the path. As it is mentioned before, the average vector velocity is given by $\vec{v}_m = \frac{\Delta\vec{r}}{dt}$.

Multiplying and dividing the last equation by Δs (the length of the arc $\widehat{M_2M_1}$), we obtain



$$\vec{v}_m = \frac{\Delta\vec{r}}{\Delta t}\frac{\Delta s}{\Delta s} \to \left(\frac{\Delta s}{\Delta t}\right)\left(\frac{\Delta\vec{r}}{\Delta s}\right)$$

The instantaneous velocity is obtained when $\begin{cases}\Delta t \to 0 \\ \Delta s \to 0\end{cases}$, so

$$\vec{v} = \left(\frac{ds}{dt}\right)\left(\frac{d\vec{r}}{ds}\right)$$

- $\left(\frac{ds}{dt}\right)$: Is the variation of the length of the curvilinear abscissa with respect to the time. Then $\left(\frac{ds}{dt}\right) = \dot{s}$ represents the speed of M (magnitude of velocity).
- $\left(\frac{d\vec{r}}{ds}\right)$: When ds is too small the magnitude of $\|d\vec{r}\| = ds$. Then, $\left(\frac{d\vec{r}}{ds}\right)$ represents the vector unity tangent to the path at the point m.
- We can write $\vec{u}_t = \left(\frac{d\vec{r}}{ds}\right)$

$$\vec{v} = \dot{s}\vec{u}_t$$

Acceleration

The acceleration of the particle at time t is defined as,

$$\vec{a} = \frac{d\vec{v}}{dt} \to \vec{a} = \frac{d}{dt}(\dot{s}\vec{u}_t)$$

Because the unit vector of the tangent to the path changes its direction with respect to time, its derivative with respect to time is not equal to zero.

$$\vec{a} = \ddot{s}\vec{u}_t + \dot{s}\dot{\vec{u}}_t$$

$$\dot{\vec{u}}_t = ?$$

Let us introduce the unit vector $\vec{u}_n$, normal to the curve and directed toward the concave side. Letting θ be the angle that the tangent to the curve at M makes with the X-axis, we may write

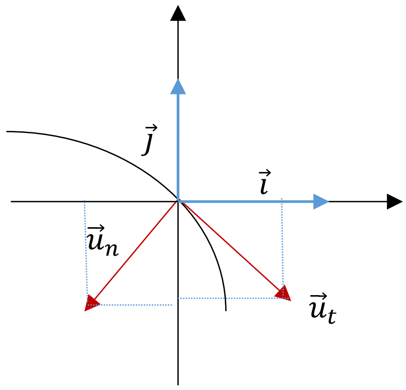
$\vec{u}_t = \cos(\theta)\,\vec{\imath} - \sin(\theta)\,\vec{\jmath}.$

$\vec{u}_n = -\,\mathrm{sins}(\theta)\,\vec{\imath} - \cos(\theta)\vec{\jmath}.$

$$\dot{\vec{u}}_t = -\dot{\theta}\sin(\theta)\,\vec{\imath} - \dot{\theta}\cos(\theta)\,\vec{\jmath} \rightarrow$$

$$\dot{\vec{u}}_t = \dot{\theta}(-\sin(\theta)\,\vec{\imath} - \cos(\theta)\,\vec{\jmath}) \rightarrow$$

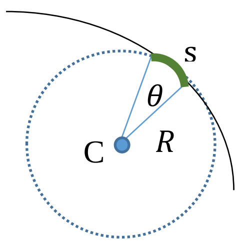
$\dot{\vec{u}}_t = \dot{\theta}\,\vec{u}_n.$



Finely, we can write,

$$\vec{a} = \ddot{s}\vec{u}_t + \dot{s}\,\dot{\theta}\,\vec{u}_n.$$

From the curve, we can deduce that $s = R\theta \rightarrow \dot{s} = R\dot{\theta}$



$$\dot{\theta} = \frac{\dot{s}}{R}$$

$$\vec{a} = \ddot{s}\vec{u}_t + \frac{\dot{s}^2}{R}\vec{u}_n$$

Note that, $v = \dot{s}$

$$\vec{a} = \dot{v}\vec{u}_t + \frac{v^2}{R}\vec{u}_n$$

$$\vec{a} = a_t\vec{u}_t + a_n\vec{u}_n$$

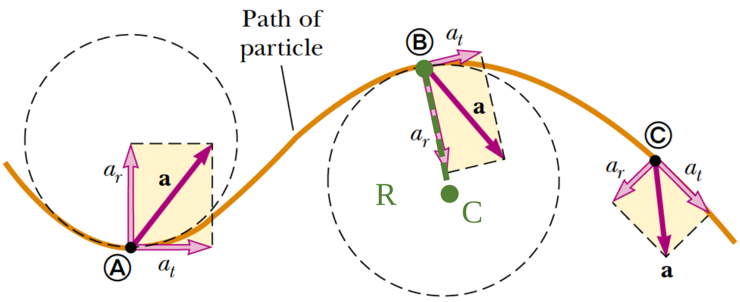
$$\begin{cases} a_t = \dfrac{d\|\vec{v}\|}{dt} \text{ the tangent acceleration component} \\ a_n = \dfrac{v^2}{R} \text{ the normal acceleration component} \end{cases}$$

$$\text{Note that}: \ v = \|\vec{v}\| \rightarrow \begin{cases} v^2 = \|\vec{v}\|^2 \\ \dot{v} = \dfrac{d\|\vec{v}\|}{dt} \end{cases}$$

The magnitude of the acceleration is given by,

$$\|\vec{a}\| = \sqrt{a_t^2 + a_n^2}$$

- a_t: is proportional to the time rate of change of the magnitude of the velocity; so the tangent component gives how the magnitude of velocity is changing with respect to the time.
- a_n: is associated with the change in direction of the velocity, because it corresponds to $\frac{d\vec{u}_t}{dt}$ and it is always positive.



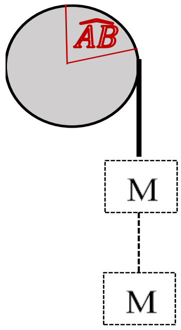
As we said before ;

$$If \begin{cases} \vec{v}.\vec{a}_t > 0, & the\ movement\ is\ accelerated \\ \vec{v}.\vec{a}_t < 0, & the\ movement\ is\ deccelerated \\ \vec{v}.\vec{a}_t = 0, & the\ movement\ is\ uniforme \end{cases}$$

- If the curvilinear motion is uniform $\vec{v}.\vec{a}_t = 0 \rightarrow a_t = 0$ (the magnitude of the velocity remains constant), v constant, so that no tangential acceleration. In this case, we say that the acceleration is central (centripetal).
- The normal acceleration disappears ($a_n = 0$) only in the case when the radius of curvature is infinite ($R \rightarrow +\infty$). For that, the motion is rectilinear (one dimension motion).
- If the radius of curvature is constant with respect to time ($R = c$), the motion is called circular.

Example

A disk D (Fig.) is rotating freely about its horizontal axis. A cord is wrapped around the outer circumference of the disk, and a body M, attached to the cord, falls under the action of gravity. The motion of M is uniformly accelerated but its acceleration is less than that due to gravity. At t_0 the velocity of body A is 0.04 m s^{-1} , and 2 s later M has fallen 0.2 m. Find the tangential and normal accelerations, at any instant, of any point on the circumference (border) of the disk.



Solution:

Given that the origin of coordinates is at the position t = 0, the equation of the uniformly accelerated motion of M is $x = \frac{1}{2}at^2 + v_0 t$. But we know that $v_0 = 0.04m^{-1}$. Thus

$$x = \frac{1}{2}at^2 + 0.04t$$

We know that at t = 2 s, the body a distance 0.2 m. by replacing the values of t and x in equation, we deduce the acceleration, $a = 0.06\,ms^{-2}$. The time equation will be written as,

$$x = 0.03t^2 + 0.04t$$

Therefore the velocity of M is $v = \frac{dx}{dt} = 0.06t + 0.04$

The distance makes by the body when it moving down equal to the arc made by any point on the circumference of the disk. Then, the tangential acceleration of the point A is thus the same as the acceleration of the body M.

$$v = 0.06t + 0.04 \rightarrow a = \frac{dv}{dt}$$

$$a(M) \equiv a_t = 0.06ms^{-2}$$

We have $a_n = \frac{v^2}{R}$ and R = 0.1m.

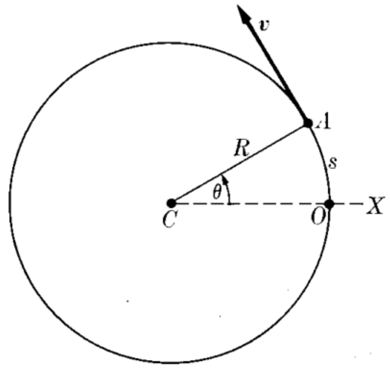
$$a_n = \frac{(0.06t + 0.04)^2}{0.1}$$

$$a_n = 0.036t^2 + 0.048t + 0.016ms^{-2}$$

The total acceleration of point A is thus $\|\vec{a}\| = \sqrt{a_t^2 + a_n^2}$

Circular motion

Uniform Circular motion and Angular Velocity



Let us now consider the special case in which the path is a circle (circular motion). The velocity v, being tangent to the circle and is perpendicular to the radius R = CA. When we measure distances along the circumference of the circle from the origin 0, we have,

$$s = R\theta$$

$$v = \frac{ds}{dt} \to \dot{s} = \frac{d(R\theta)}{dt}$$

$$v = R\frac{d\theta}{dt} \to v = R\dot{\theta}$$

$\dot{\theta} = w$: represent the angular velocity, which measured in $rads^{-1}$, the relation between the tangent and the angular velocities is $v = wR$

Instead of studding the curvilinear abscissa s with respect to time, we can only study θ with respect to time because R is constant $s(t) = R\theta(t)$.

$$w = \frac{d\theta}{dt} \to d\theta = wdt$$

$$\int_{\theta_0}^{\theta} d\theta = w\int_{t_0}^{t} dt \quad \to \quad \theta(t) = w(t - t_0) + \theta_0$$

In this case ($w = constant$), the motion is periodic and the particle passes through each point of the circle at regular intervals of time.

Period: The period T(s) is the time required for a complete one turn (one revolution), the complete turn is $s = 2\pi R$. Because the movement is uniform, the distance is equal to speed multiplied by time.

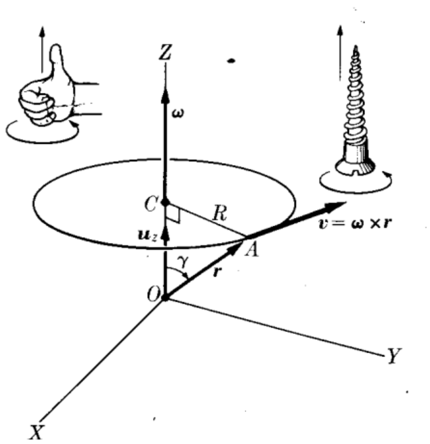
$$s = vT \to T = \frac{2\pi R}{v} = \frac{2\pi R}{wR}$$

$$T = \frac{2\pi}{w}, \quad w : is\ the\ angular\ velocity.$$

Frequency: The frequency f is the number of revolutions per unit time. Let's assume that during time t, the particle makes an N turn, then we can write $t = NT$. In unit time t=1s the number of turn is $f = \frac{1}{T}$. Then the frequency is written as,

$$f = \frac{1}{T} s^{-1}.$$

The angular velocity can be expressed as a vector (Fig.) whose direction is perpendicular to the plane of motion in the sense of advance of a right-handed screw rotated in the same sense as the particle moves. So, $\vec{w} = w\vec{k}$



From the figure, we can deduce, $R = rsin(\gamma)$.

Because $v = Rw$, We can write $v = wrsin(\gamma)$,

$$v = wrsin(\gamma)$$

$$wrsin(\gamma)\ is\ writen\ as\ \|\vec{w}\|\|\vec{r}\|sin(\alpha)$$

$$\begin{cases} \vec{w} = w\vec{k} \\ \vec{r} = -R\vec{u}_n \\ \alpha = 0, \quad because\ \vec{v} \perp \vec{r} \end{cases}$$

Finely we can write, $\vec{v} = \vec{w} \wedge \vec{r}$

> This result can be generalized on any vector in rotational movement with an angular velocity $\vec{w}$; the derivative of this vector with respect to time is expressed by
>
> $$\dot{\vec{V}} = \vec{w} \wedge \vec{V}$$

Accelerated Circular Motion: Angular Acceleration

When the angular velocity of a particle changes with time, the angular acceleration is defined by the vector $\vec{\varphi} = \frac{d\vec{w}}{dt} \rightarrow \vec{\varphi} = \dot{w}\vec{k} + w\frac{d\vec{k}}{dt}$, $\vec{k}$ doesn't change with respect to time.

$$\vec{\varphi} = \dot{w}\vec{k} \rightarrow \varphi = \dot{w} = \frac{d^2\theta}{dt^2}$$

$$\int_{w_0}^{w} dw = \varphi \int_{t_0}^{t} dt \quad \rightarrow \quad w(t) = \varphi(t - t_0) + w_0$$

$$\int_{\theta_0}^{\theta} d\theta = w \int_{t_0}^{t} (\varphi(t - t_0) + w_0)dt$$

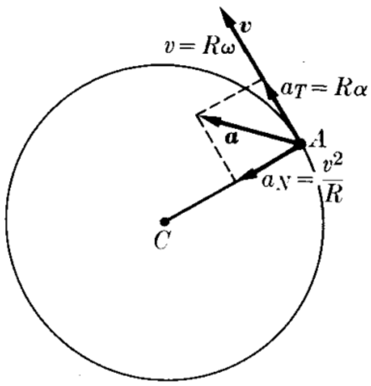
$$\theta(t) = \frac{1}{2}\varphi(t - t_0)^2 + w_0(t - t_0) + \theta_0$$

As we say before, the acceleration is written by

$\vec{a} = \dot{v}\vec{u}_t + \frac{v^2}{R}\vec{u}_n = a_t\vec{u}_t + a_n\vec{u}_n.$

$$\begin{cases} a_t = \dfrac{dv}{dt} = R\dfrac{dw}{dt} \\ a_n = \dfrac{v^2}{R} = \dfrac{(wR)^2}{R} \end{cases}$$

$$\begin{cases} a_t = R\varphi \\ a_n = Rw^2 \end{cases}$$



Representation of velocity, Tangential and normal acceleration.

Note in uniform circular motion (no angular acceleration $\varphi = 0$) there is no tangential acceleration, but there is still a normal or centripetal acceleration due to the change in the direction of the velocity. We have $\vec{w} = \vec{C}^{st}$

$$\vec{a}_t = \frac{d\vec{v}}{dt} = \frac{d(\vec{w}\wedge\vec{r})}{dt} = \vec{w}\wedge\frac{d\vec{r}}{dt} = \vec{w}\wedge\vec{v}$$

We have $\vec{v} = \vec{w}\wedge\vec{v}$, then we can write

$$\vec{a}_t = \vec{w}\wedge(\vec{w}\wedge\vec{v})$$

II. Relative motion

Motion is a relative concept, it must be always referred to a specific frame of reference chosen by the observer.

An observer on the earth considered himself in rest. But someone who is on the moon look him in circular motion.

Notion of Absolute and relative frame

The concept of absolute or relative frame is changing from one observer to another.

Since different observers may use different frames of references, it is important to know how observations made by different observers are related.

For the example below, we consider that the observer on the street is the absolute frame of reference, and then the two frames related to the cars are relative to the observer on the street and are relative to each other.

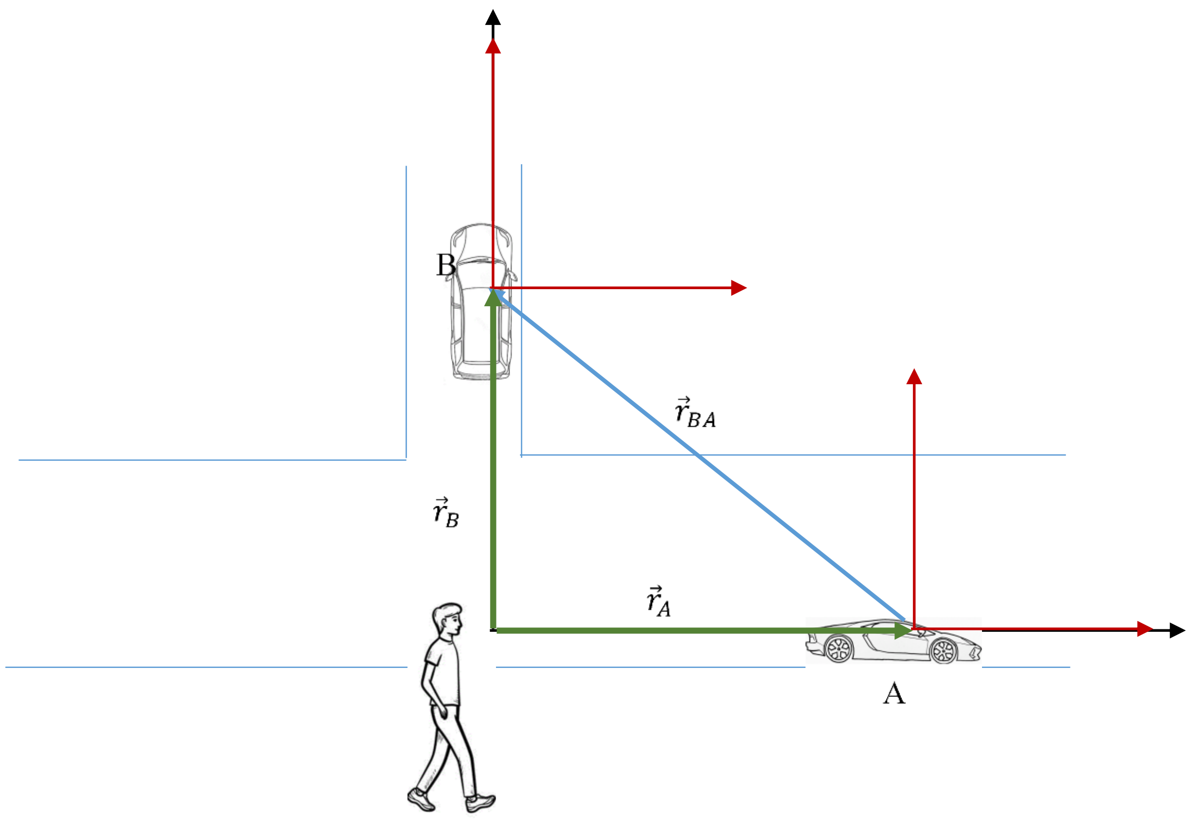


Fig . Notion of relative motion

$\vec{r}_A$: is the vector position of car A for the absolute frame.

$\vec{r}_B$: is the vector position of car B for the absolute frame.

$\vec{r}_{BA} = \vec{r}_B - \vec{r}_A$: is the vector position of car A for the A frame.

$\vec{r}_{AB} = \vec{r}_A - \vec{r}_B$: is the vector position of car A for the A frame.

Let consider that the two cars move with constant velocities with respect to the Absolut frame (observer on the street), we ca write $\vec{r}_A = V_A\vec{\imath}$, and $\vec{r}_B = V_B\vec{\jmath}$.

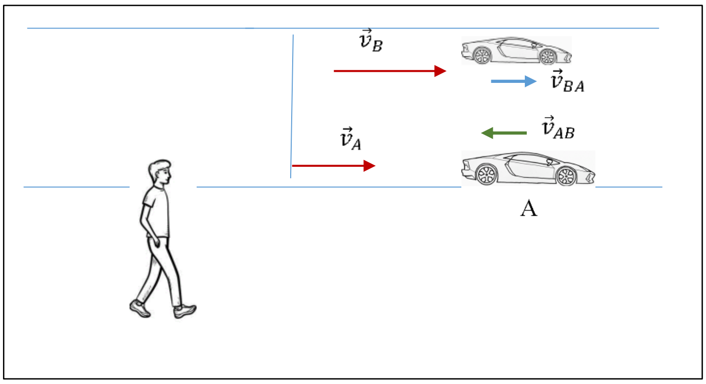
The vector position of B with respect to the car A is $\vec{r}_{BA} = \vec{r}_B - \vec{r}_A$.

$$\vec{r}_{BA} = V_A t\vec{\imath} - V_B tj$$

$$\begin{cases} x_{BA} = V_A t \\ y_{BA} = -V_B t \end{cases}$$

We can find of $t = \frac{1}{V_A} x_{BA}$ so the motion of B with respect to the A is linear given $y_{BA} = -\frac{V_A}{V_B} t$.

In the same way, the motion of A with respect to B is linear.$y_{AB} = -\frac{V_B}{V_A} t$.



Special cases: (cars move on same way)

- **On the same direction**

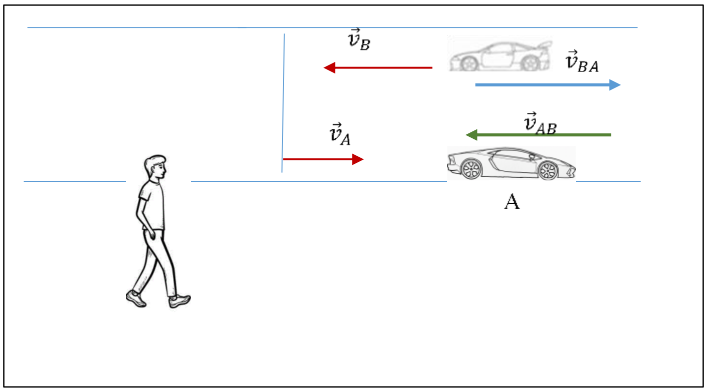
$$\begin{cases} \vec{r}_A = V_A t\vec{\imath} \\ \vec{r}_B = V_B t\vec{\imath} \end{cases} \to \vec{r}_{BA} = (V_B - V_A)t\vec{\imath}.$$

$\vec{v}_{BA} = (V_B - V_A)\vec{\imath} \;\to\; \|\vec{v}_{BA}\| = |V_B - V_A|$, the speed of B with respect to A is equals to the difference between the two velocities with respect to the absolute observer.

- **On the opposite direction**

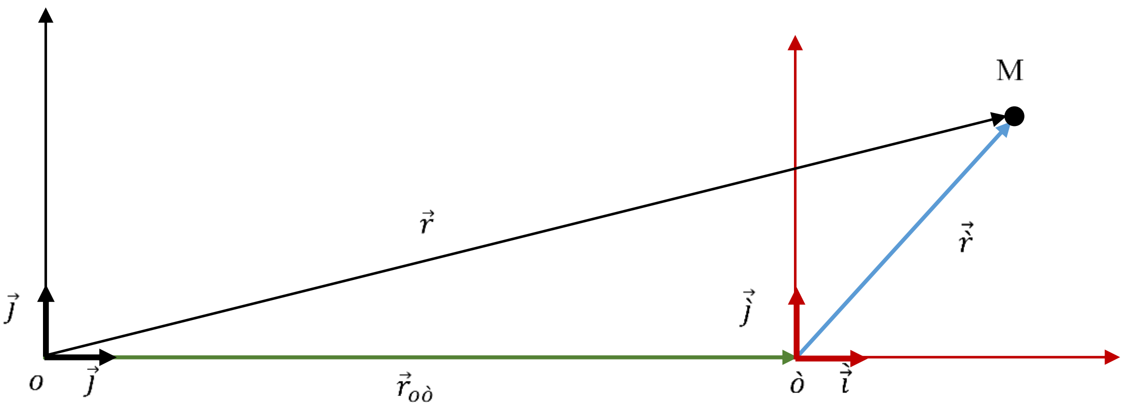
$$\begin{cases} \vec{r}_A = V_A t\vec{\imath} \\ \vec{r}_B = -V_B t\vec{\imath} \end{cases} \to \vec{r}_{BA} = (V_B + V_A)t\vec{\imath}$$

$\vec{v}_{BA} = (V_B + V_A)\vec{\imath} \;\to\; \|\vec{v}_{BA}\| = V_B + V_A$, the speed of B with respect to A is equals to the sum of the two velocities with respect to the absolute observer.



Relative frame in translation movement

Let consider a situation looks like the previous situation and consider that an observer stand on the street and a car travel with a speed V. Inside this car there something falls down. We want to study the movement of this object with respect to the passenger of car and with respect to observer on the street. We can represent this situation by two frames one is considered **absolute** which attached to the observer on the street and the other one is considered **relative** attached to the car.



Figure

- The car moves with a constant velocity $\vec{v}_{o\grave{o}}$ we can write $\vec{r}_{o\grave{o}} = v_{o\grave{o}}\vec{\imath}$.
- The vector position of the point M with respect to $\grave{R}$ is $\grave{\vec{r}}$.
- The vector position of point M with respect to R is the sum of $\vec{r}_{o\grave{o}}$ and $\grave{\vec{r}}$,

$$\vec{r} = \vec{r}_{o\grave{o}} + \grave{\vec{r}}$$

The velocity of M with respect to R is $\frac{d\vec{r}}{dt}$

$$\left(\frac{d\vec{r}}{dt}\right)_R = \left(\frac{d\vec{r}_{o\grave{o}}}{dt}\right)_R + \left(\frac{d\grave{\vec{r}}}{dt}\right)_R$$

$$\vec{v}_a = \left(\frac{d\vec{r}_{o\grave{o}}}{dt}\right)_R + \left(\frac{d\grave{\vec{r}}}{dt}\right)_R$$

$$\left(\frac{d\vec{r}_{o\grave{o}}}{dt}\right)_R = \frac{dx_{o\grave{o}}}{dt}\vec{\imath} + \frac{dy_{o\grave{o}}}{dt}\vec{\jmath} + \frac{dz_{o\grave{o}}}{dt}\vec{k} + x_{o\grave{o}}\frac{d\vec{\imath}}{dt} + y_{o\grave{o}}\frac{d\vec{\jmath}}{dt} + z_{o\grave{o}}\frac{d\vec{k}}{dt}$$

Because the frame $(o, \vec{\imath}, \vec{\jmath}, \vec{k})$, is in rest, the derivative of their unit vectors with respect to time is equal to zero, $\frac{d\vec{\imath}}{dt} = \frac{d\vec{\jmath}}{dt} = \frac{d\vec{k}}{dt} = \vec{0}$

$$\vec{v}_{o\grave{o}} = \left(\frac{d\vec{r}_{o\grave{o}}}{dt}\right)_R = \frac{dx_{o\grave{o}}}{dt}\vec{\imath} + \frac{dy_{o\grave{o}}}{dt}\vec{\jmath} + \frac{dz_{o\grave{o}}}{dt}\vec{k}$$

$$\left(\frac{d\grave{\vec{r}}}{dt}\right)_R = \frac{d\grave{x}}{dt}\grave{\vec{\imath}} + \frac{d\grave{y}}{dt}\grave{\vec{\jmath}} + \frac{d\grave{z}}{dt}\grave{\vec{k}} + x_{o\grave{o}}\frac{d\grave{\vec{\imath}}}{dt} + y_{o\grave{o}}\frac{d\grave{\vec{\jmath}}}{dt} + z_{o\grave{o}}\frac{d\grave{\vec{k}}}{dt}$$

Because the frame, $(o, \grave{\vec{\imath}}, \grave{\vec{\jmath}}, \grave{\vec{k}})$ moving but does not changing their directions, then the derivative of theirs units vectors with respect to time is equal to zero, , $\frac{d\grave{\vec{\imath}}}{dt} = \frac{d\grave{\vec{\jmath}}}{dt} = \frac{d\grave{\vec{k}}}{dt} = \vec{0}$.

$$\vec{v}_r = \left(\frac{d\vec{r}}{dt}\right)_R = \frac{d\grave{x}}{dt}\vec{\grave{i}} + \frac{d\grave{y}}{dt}\vec{\grave{j}} + \frac{d\grave{z}}{dt}\vec{\grave{k}} = \left(\frac{d\vec{r}}{dt}\right)_{\grave{R}}$$

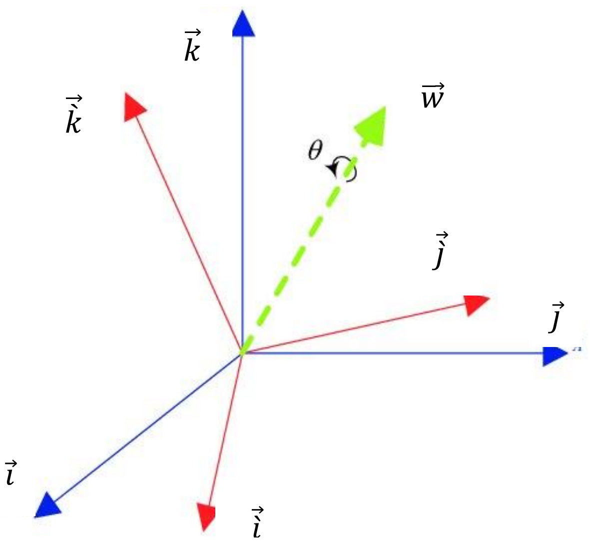
The absolute velocity is the sum of the velocity of relative frame respecting to the absolute frame and the velocity of the point respecting to the relative frame.

$$\vec{v}_a = \vec{v}_{o\grave{o}} + \vec{v}_r$$

$$\begin{cases} \vec{v}_a: \textbf{absolut velocity}, \textit{the velocity of M recorded by the observer O} \\ \vec{v}_{o\grave{o}}: \textbf{training velocity}, \ \textit{the velocity of } \grave{O} \textit{ recorded by the observer O} \\ \vec{v}_r: \textbf{relative velocity}, \ \textit{the velocity of M recorded by the observer } \grave{O} \end{cases}$$

Relative frame in rotational movement

Let us now consider two observers O and O' rotating relative to each other but with no relative translational motion. For simplicity, let us consider that the two frames are attached to itself with a common origin.



- The frame $(o, \vec{i}, \vec{j}, \vec{k})$ does not move.
- The frame $(o, \vec{\grave{i}}, \vec{\grave{j}}, \vec{\grave{k}})$ rotates with an angular velocity $\vec{w}$. That means that the observer on $(o, \vec{i}, \vec{j}, \vec{k})$ sees $(o, \vec{\grave{i}}, \vec{\grave{j}}, \vec{\grave{k}})$ in rotational motion with an angular velocity $\vec{w}$.
- The observer on $(o, \vec{\grave{i}}, \vec{\grave{j}}, \vec{\grave{k}})$ see himself in rest.

The position vector $\vec{r}$ of the particle M referred to $(o, \vec{i}, \vec{j}, \vec{k})$ is

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

Therefore the velocity of particle referred to R is,

$$\vec{v}_a = \left(\frac{d\vec{r}}{dt}\right)_R = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$$

The position vector $\vec{\grave{r}}$ of the particle M referred to $(o, \vec{\grave{i}}, \vec{\grave{j}}, \vec{\grave{k}})$ is

$$\vec{\grave{r}} = \grave{x}\vec{\grave{i}} + \grave{y}\vec{\grave{j}} + \grave{z}\vec{\grave{k}}$$

Therefore the velocity of particle referred to $(o, \vec{\grave{i}}, \vec{\grave{j}}, \vec{\grave{k}})$ is,

$$\vec{v}_r = \left(\frac{d\vec{\grave{r}}}{dt}\right)_{\grave{R}} = \frac{d\grave{x}}{dt}\vec{\grave{i}} + \frac{d\grave{y}}{dt}\vec{\grave{j}} + \frac{d\grave{z}}{dt}\vec{\grave{k}}$$

$$\left(\frac{d\vec{r}}{dt}\right)_R = \left(\frac{d\vec{r}_{o\grave{o}}}{dt}\right)_R + \left(\frac{d\vec{\grave{r}}}{dt}\right)_R$$

$\left(\frac{d\vec{r}_{o\grave{o}}}{dt}\right)_R = \vec{v}_{o\grave{o}} = \vec{0}$, then

$$\left(\frac{d\vec{r}}{dt}\right)_R = \left(\frac{d\vec{\grave{r}}}{dt}\right)_R$$

$$\left(\frac{d\vec{r}}{dt}\right)_R = \left(\frac{d\grave{x}}{dt}\vec{\grave{i}} + \frac{d\grave{y}}{dt}\vec{\grave{j}} + \frac{d\grave{z}}{dt}\vec{\grave{k}}\right)_R$$

$$\left(\frac{d\vec{r}}{dt}\right)_R = \frac{d\grave{x}}{dt}\vec{\grave{i}} + \frac{d\grave{y}}{dt}\vec{\grave{j}} + \frac{d\grave{z}}{dt}\vec{\grave{k}} + \grave{x}\frac{d\vec{\grave{i}}}{dt} + \grave{y}\ \frac{d\vec{\grave{j}}}{dt} + \grave{z}\ \frac{d\vec{\grave{k}}}{dt}$$

$$\left(\frac{d\vec{r}}{dt}\right)_R = \vec{v}_r + \grave{x}\frac{d\vec{\grave{i}}}{dt} + \grave{y}\ \frac{d\vec{\grave{j}}}{dt} + \grave{z}\ \frac{d\vec{\grave{k}}}{dt}$$

From the chapter 1, we know:

$\left(\frac{d\vec{\grave{i}}}{dt}\right)_R = \vec{w}\wedge\vec{\grave{i}}$, $\left(\frac{d\vec{\grave{j}}}{dt}\right)_R = \vec{w}\wedge\vec{\grave{j}}$ and $\left(\frac{d\vec{\grave{k}}}{dt}\right)_R = \vec{w}\wedge\vec{\grave{k}}$, where w is rotating velocity of the relative frame.

$$\left(\frac{d\vec{r}}{dt}\right)_R = \vec{v}_a = \vec{v}_r + \grave{x}(\overrightarrow{w}\wedge\vec{\grave{i}}) + \grave{y}\ (\vec{w}\wedge\vec{\grave{j}}) + \grave{z}\ (\vec{w}\wedge\vec{\grave{k}})$$

$$\vec{v}_a = \vec{v}_r + (\vec{w}\wedge\overrightarrow{\grave{x}\grave{i}}) + (\vec{w}\wedge\overrightarrow{\grave{y}\grave{j}}) + (\vec{w}\wedge\overrightarrow{\grave{z}\grave{k}})$$

$$\vec{v}_a = \vec{v}_r + \vec{w}\wedge(\grave{x}\vec{\grave{i}} + \grave{y}\vec{\grave{j}} + \grave{z}\vec{\grave{k}})$$

$$\vec{v}_a = \vec{v}_r + \vec{w}\wedge\vec{\grave{r}}$$

This expression gives the relation between the velocities $\vec{v}_a$ and $\vec{v}_r$ of M, as recorded by observers 0 and 0' in relative rotational motion of.

Acceleration

The acceleration recorded by observers 0 is called absolute acceleration,

$$\left(\frac{d\vec{v}_a}{dt}\right)_R = \left(\frac{d}{dt}\left(\vec{v}_r + \vec{w}\wedge\vec{\grave{r}}\right)\right)_R = \left(\frac{d\vec{v}_r}{dt}\right)_R + \left(\frac{d(\vec{w}\wedge\vec{\grave{r}})}{dt}\right)_R$$

Because

$$\frac{d\vec{w}}{dt} = \vec{0}$$

$$\left(\frac{d\vec{v}_a}{dt}\right)_R = \left(\frac{d\vec{v}_r}{dt}\right)_R + \vec{w}\wedge\left(\frac{d\vec{\grave{r}}}{dt}\right)_R$$

From the result above;

$$\left(\frac{d\vec{v}_r}{dt}\right)_R = \left(\frac{d\vec{v}_r}{dt}\right)_{\grave{R}} + \vec{w}\wedge\vec{v}_r \ , \ \ and \ \left(\frac{d\vec{r}}{dt}\right)_R = \left(\frac{d\vec{r}}{dt}\right)_{\grave{R}} + \vec{w}\wedge\vec{r}$$

$\left(\frac{d\vec{v}_r}{dt}\right)_{\grave{R}} = \vec{a}_r,\ \left(\frac{d\vec{r}}{dt}\right)_{\grave{R}} = \vec{v}_r$

$$\left(\frac{d\vec{v}_r}{dt}\right)_R = \vec{a}_r + \vec{w}\wedge\vec{v}_r$$

$$\left(\frac{d\vec{r}}{dt}\right)_R = \vec{v}_r + \vec{w}\wedge\vec{r}$$

By replacing,

$$\left(\frac{d\vec{v}_a}{dt}\right)_R = \vec{a}_r + \vec{w}\wedge\vec{v}_r + \vec{w}\wedge(\vec{v}_r + \vec{w}\wedge\vec{r})$$

$$\vec{a}_a = \vec{a}_r + 2(\vec{w}\wedge\vec{v}_r) + \vec{w}\wedge(\vec{w}\wedge\vec{r})$$

- The term, $2(\vec{w}\wedge\vec{v}_r)$, is called the Coriolis acceleration.
- The term, $\vec{w}\wedge(\vec{w}\wedge\vec{r})$, is corresponding to a centripetal acceleration.

General motion of the relative frame

Let us now consider the relative frame makes translation and rotational motion referring to the absolute frame.

Velocity

$$\left(\frac{d\vec{r}}{dt}\right)_R = \left(\frac{d\vec{r}_{o\grave{o}}}{dt}\right)_R + \left(\frac{d\vec{r}}{dt}\right)_R$$

$$\vec{v}_a = \vec{v}_{o\grave{o}} + \vec{v}_r + \vec{w}\wedge\vec{r}$$

$$\vec{v}_a = \vec{v}_{o\grave{o}} + \vec{w}\wedge\vec{r} + \vec{v}_r$$

$$\vec{v}_{o\grave{o}} + \vec{w}\wedge\vec{r} = \vec{v}_t : training\ speed$$

$$\vec{v}_a = \vec{v}_t + \vec{v}_r$$

Acceleration

$$\vec{a}_a = \left(\frac{d\vec{r}}{dt}\right)_R = \left(\frac{d\vec{v}_{o\grave{o}}}{dt}\right)_R + \left(\frac{d\vec{v}_r}{dt}\right)_R + \left(\frac{d(\vec{w}\wedge\vec{r})}{dt}\right)_R$$

$$\begin{cases} \left(\dfrac{d\vec{v}_{oò}}{dt}\right)_R = \vec{a}_{oò} \\ \left(\dfrac{d\vec{v}_r}{dt}\right)_R = \vec{a}_r + (\vec{w}\wedge\vec{v}_r) \\ \left(\dfrac{d(\vec{w}\wedge\vec{\dot{r}})}{dt}\right)_R = \vec{w}\wedge(\vec{v}_r + \vec{w}\wedge\vec{\dot{r}}) \end{cases}$$

$$\vec{a}_a = \vec{a}_{oò} + \vec{a}_r + 2(\vec{w}\wedge\vec{v}_r) + \vec{w}\wedge(\vec{w}\wedge\vec{\dot{r}})$$

$$\begin{cases} \vec{a}_t = \vec{a}_{oò} + \vec{w}\wedge(\vec{w}\wedge\vec{\dot{r}}) \\ \vec{a}_c = 2(\vec{w}\wedge\vec{v}_r) \end{cases}$$

$$\vec{a}_a = \vec{a}_t + \vec{a}_r + \vec{a}_c$$